

## PRELIMINARY REVIEW

### 1.1 Cartesian Co-ordinates.

Let  $X'OX$  and  $Y'OY$  be two fixed straight lines intersecting at a point  $O$ . These lines are called co-ordinates axes and  $O$  is called the origin. The lines  $X'OX$  and  $Y'OY$  are respectively called the  $x$ -axis and  $y$ -axis. The co-ordinate axes divide the plane into four quadrants. The quadrants  $XOY$ ,  $YOX'$ ,  $X'OY'$  and  $Y'OX$  are respectively called first, second, third and fourth quadrants.

If the lines  $X'OX$  and  $Y'OY$  are mutually perpendicular, the co-ordinate axes are rectangular otherwise oblique.

In rectangular co-ordinate axes, let  $PN$  be perpendicular from  $P$  on  $x$ -axis, the distance  $ON=x$  is called abscissa or the  $x$  co-ordinate of the point  $P$  and the distance  $PN=y$  is called ordinate or  $y$  co-ordinate of point  $P$ .

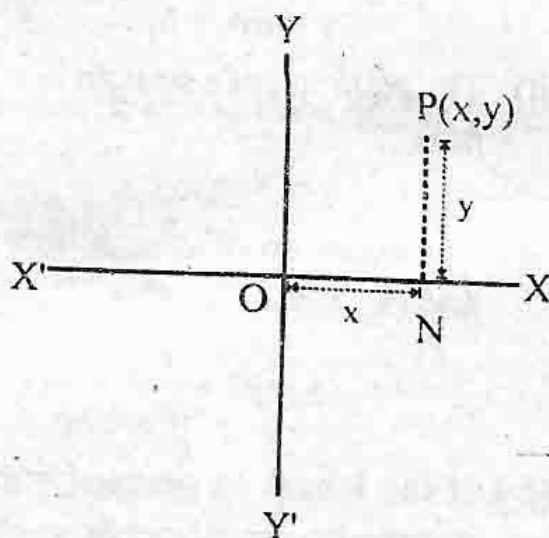


Fig. 1

Abscissa is +ve when the distance  $ON$  is measured in right hand side direction to the  $y$ -axis and -ve when measured to the left. Ordinate is +ve when distance  $NP$  is measured above  $x$ -axis and -ve when the distance is measured below. The ordered pair  $(x, y)$  are called rectangular co-ordinate of a point  $P$  for the rectangular system of axes.

### 1.2 Distance between two given points.

Let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be the two given points. The distance between  $P$  and  $Q$  is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

### 1.3 Co-ordinate of a point dividing in given ratio.

The co-ordinate of a point R dividing the line joining points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  internally or externally in the given ratio  $m:n$  is

$$\left( \frac{mx_2 \pm nx_1}{m \pm n}, \frac{my_2 \pm ny_1}{m \pm n} \right).$$

In particular if R is the mid point of PQ, then  $m = n$  and the co-ordinate is given by

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

### 1.4 Straight lines.

We give below different forms of the straight lines.

- (i) The equation of a straight line having gradient (slope)  $m$  and making intercept of length  $c$  from  $y$ -axis is

$$y = mx + c.$$

- (ii) The equation of a straight line passing through  $(x_1, y_1)$  and gradient  $m$  is

$$y - y_1 = m(x - x_1).$$

- (iii) The equation of straight line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

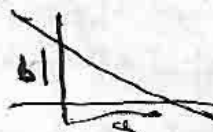
- (iv) If the length of perpendicular from origin on the line is  $p$  and perpendicular makes an angle  $\alpha$  with +ve  $x$ -axis, the equation of the straight line is

$$x \cos \alpha + y \sin \alpha = p.$$



- (v) If the straight line makes intercept of length  $a$  and  $b$  from  $x$  and  $y$  axes respectively, the equation is

$$\frac{x}{a} + \frac{y}{b} = 1.$$



- (vi) A general first degree equation in  $x$  and  $y$  of the form

$$ax + by + c = 0$$

always represents a straight line.

### 1.5 Angle between two given straight line.

If  $m_1$  and  $m_2$  are gradients of the two straight lines, the angle  $\theta$  between them is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

In particular, these lines are.

- (i) Parallel if  $m_1 = m_2$ ,
- (ii) Perpendicular if  $m_1 m_2 = -1$ .

### 1.6 Length of perpendicular from a point.

The length of perpendicular from point  $(x_1, y_1)$  on the straight line  $ax+by+c=0$  is given by

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

### 1.7 Equation of the straight line bisecting the angles between two given lines.

Let the given straight lines be

$$a_1x + b_1y + c_1 = 0,$$

and

$$a_2x + b_2y + c_2 = 0$$

The equation of the straight line bisecting the angle between them is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

### 1.8 Pair of straight lines.

The general equation of second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of straight lines if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

The angle between these lines is given by

$$\theta = \tan^{-1} \frac{2\sqrt{h^2 - ab}}{a + b}.$$

These lines are

- (i) Parallel if  $h^2 = ab$ ,
- (ii) Perpendicular if  $a + b = 0$ .

### 1.9 Change of axes.

Change of axes can be made in the following three ways :

(i) **Change of origin.** Let  $(x, y)$  be the co-ordinate of a point  $P$  with  $Ox$  and  $Oy$  as co-ordinate axes and  $O$  is origin.

Let the origin be changed to the point  $O'$  whose co-ordinate with respect to  $Ox$  and  $Oy$  as axes be  $(h, k)$ . Let  $O'X$  and  $O'Y$  be new coordinate axes which are parallel to  $Ox$  and  $Oy$  axes. Let  $(X, Y)$  be the co-ordinate of the same point  $P$  with  $O'X$  and  $O'Y$  as axes.

$$\begin{aligned}\text{Then } x &= X + h, \\ y &= Y + k.\end{aligned}$$

(ii) **Rotation of axes without change of origin.**

Let the axes be rotated by angle  $\theta$  (as shown in figure) without changing the origin.

$$\begin{aligned}x &= OL = ON - SM \\ &= X \cos \theta - Y \sin \theta \\ y &= PL = PS + SL = PS + MN \\ &= X \sin \theta + Y \cos \theta.\end{aligned}$$

(iii) **Change of origin and rotation of axes.** Let  $Ox$  and  $Oy$  be the original axes. Let origin be shifted to a point  $O'$   $(h, k)$  and then axes be turned through an angle  $\theta$ .

Let  $(x, y)$  and  $(X, Y)$  be the co-ordinate of a point  $P$  with respect to the two system of axes, then

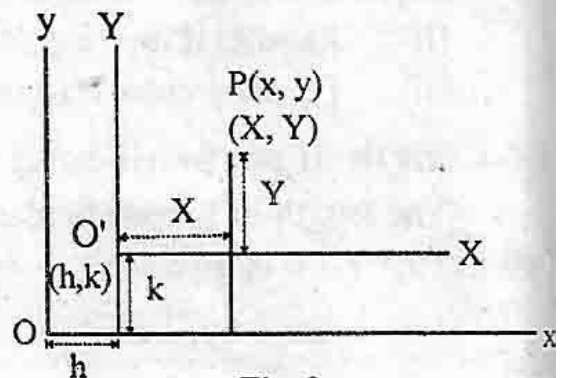


Fig. 2

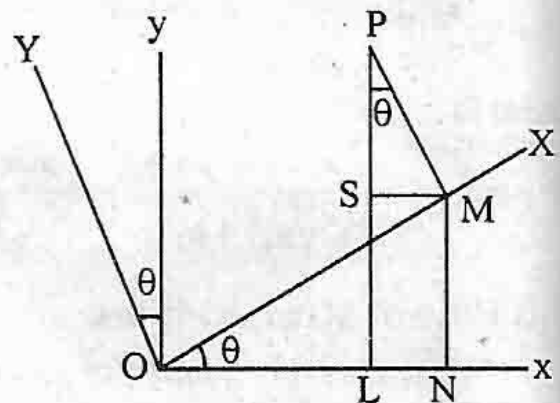


Fig. 3

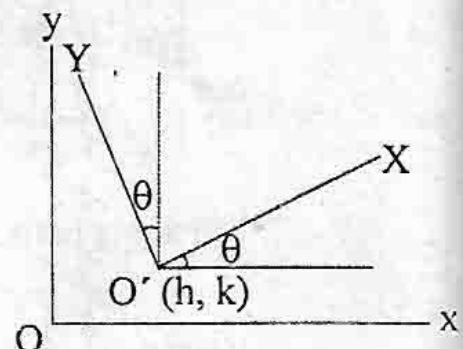


Fig. 4

$$x = h + X \cos \theta - Y \sin \theta$$

$$y = k + X \sin \theta + Y \cos \theta.$$

### 1.10 Invariants.

If by a change of axes without change of origin, the expression  $ax^2 + 2hxy + by^2$  becomes  $a' X^2 + 2h' XY + b' Y^2$ , the axes in both the cases being rectangular, then

$$a + b = a' + b',$$

$$ab - h^2 = a' b' - h'^2.$$

**Proof.** If the axes are rotated through an angle  $\theta$  at the same origin, then

$$x = X \cos \theta - Y \sin \theta$$

$$y = X \sin \theta + Y \cos \theta$$

Then  $ax^2 + 2hxy + by^2$  transform to

$$a(X \cos \theta - Y \sin \theta)^2 + 2h(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta)$$

$$+ b(X \sin \theta + Y \cos \theta)^2$$

$$= X^2(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) + 2XY\{h(\cos^2 \theta - \sin^2 \theta)$$

$$+ (b - a) \sin \theta \cos \theta\} + Y^2(a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta).$$

Since  $ax^2 + 2hxy + by^2$  transforms to  $a' X^2 + 2h' XY + b' Y^2$ . Therefore

$$a' = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta$$

$$= \frac{1}{2}[a(1 + \cos 2\theta) + 2h \sin 2\theta + b(1 - \cos 2\theta)]$$

$$= \frac{1}{2}[(a + b) + (a - b) \cos 2\theta - 2h \sin 2\theta],$$

Similarly

$$b' = \frac{1}{2}[(a + b) - (a - b) \cos 2\theta - 2h \sin 2\theta],$$

$$h' = (b - a) \sin \theta \cos \theta + h(\cos^2 \theta - \sin^2 \theta)$$

$$= \frac{1}{2}[(b - a) \sin 2\theta + 2h \cos 2\theta].$$

Then  $a' + b' = a + b$

$$a' b' = \frac{1}{4}[(a + b) + \{(a - b) \cos 2\theta + 2h \sin 2\theta\}]$$

$$[(a + b) - \{(a - b) \cos 2\theta + 2h \sin 2\theta\}]$$

$$= \frac{1}{4}[(a + b)^2 - \{(a - b) \cos 2\theta + 2h \sin 2\theta\}^2]$$

$$\begin{aligned}
 4(a'b'-h'^2) &= (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2 \\
 &\quad - \{(b-a)\sin 2\theta + 2h\cos 2\theta\}^2 \\
 &= (a+b)^2 - (a-b)^2 - 4h^2 \\
 &= 4(ab-h^2)
 \end{aligned}$$

$$a'b'-h'^2 = ab-h^2.$$

Thus, we get  $a'+b'=a+b$ ,  $a'b'^2-h'^2=ab-h^2$ .

The expressions  $a+b$  and  $ab-h^2$  are called **invarrients**.

## The Conics

**Definition.** The locus of a point which moves such that its distance from a fixed point is in constant ratio to its distance from a fixed straight line. The fixed point is called focus, the fixed line directrix and the constant ratio, the eccentricity, usually denoted by 'e'.

The given conic is an ellipse, a parabola or a hyperbola according as  $e < 1$  or  $e > 1$ . In particular when  $e = 0$  conic is a circle.

### 1.11 Representation of the conic by a general equation of second degree.

Let 'e' be the eccentricity, (h, k) the coordinate of the focus and  $ax+by+c=0$  the equation of directrix, then by the definition of conic

$$(x-h)^2 + (y-k)^2 = e^2 \left( \frac{ax+by+c}{\sqrt{a^2+b^2}} \right)^2$$

which on solving gives an equation of the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.$$

Above is a general equation of second degree

By suitable transformation of axes, above equation can be reduced in the following standard equation of conics :

$$x^2 + y^2 = a^2 \quad (\text{Circle})$$

$$y^2 = 4ax \quad (\text{Parabola})$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{Ellipse})$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{Hyperbola}).$$

Below we give some important properties of the standard equation of conics (without proof).

### 1.12 The Circle.

**Definition.** Circle is the locus of a point which moves such that its distance from the fixed point is constant. The fixed point is called the centre of the circle and the distance of the variable point from the centre is called radius.

The equation of a circle with centre at  $(h, k)$  and radius  $a$  is

$$(x - h)^2 + (y - k)^2 = a^2. \quad \dots(1)$$

In particular if the centre is origin, then the equation of the circle is

$$x^2 + y^2 = a^2. \quad \dots(2)$$

The general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \dots(3)$$

whose centre is  $(-g, -f)$  and radius  $\sqrt{g^2 + f^2 - c}$ .

From equation (1), (2) and (3) we observe that above equations are particular cases of the equation.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Now we shall discuss some important properties of the circles (without proof).

**(1) Equation of tangent.** Equation of tangent at  $(x_1, y_1)$  to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Equation of tangent at point  $(x_1, y_1)$  to the circle

$$x^2 + y^2 = a^2$$

is

$$xx_1 + yy_1 = a^2.$$

**(2) Equation of normal.** Equation of normal at  $(x_1, y_1)$  to the circle (3) is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1).$$

- (3) **Condition of tangency.** The line  $y = mx + a\sqrt{1+m^2}$  always touches the circle  $x^2 + y^2 = a^2$  and the co-ordinate of the point of contact is

$$\left( -\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}} \right).$$

- (4) **Pair of tangents.** If the equation of the circle is

$$S = x^2 + y^2 + 2gx + 2fy + c = 0.$$

Then equation of pair of tangents drawn from point  $(x_1, y_1)$  to the circle (3) is  $SS_1 = T^2$ ,

where  $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$$

- (5) **Chord of contact.** Equation of the chord of contact of tangents drawn to the circle (3) from  $(x_1, y_1)$  is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

- (6) **Polar.**

**Def.** If through a point  $P$  (within or without a circle) there be drawn any straight line to meet the circle in  $Q$  and  $R$ , the locus of the point of intersection of tangents at  $Q$  and  $R$  is called the polar of  $P$ ; also  $P$  is called the pole of the polar.

Polar of the point  $(x_1, y_1)$  with respect to circle (3) is

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

- (7) **Chord with given middle point.** Equation of the chord whose middle point is  $(x_1, y_1)$  for the circle (3) is

$$T = S_1.$$

- (8) **Parametric equation.** The parametric equation of the circle

$$x^2 + y^2 = a^2 \text{ is } x = a \cos \theta : y = a \sin \theta, \text{ where } \theta \text{ is a parameter}$$

- (9) **Orthogonal intersection of two circles.** The two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

and

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{ intersect orthogonally if}$$

$$2gg_1 + 2ff_1 = c + c_1.$$

- (10) **Radical Axis.** The locus of a point which moves such that the length of tangents from it to the circles are equal is called the radical axis of the two circles.

The equations of radical axis of two circles

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

and  $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

is  $S - S_1 = 0 = 2(g - g_1)x + 2(f - f_1)y + c - c_1$ .

The following properties hold for the radical axis.

- (i) The radical axis of two circles is perpendicular to the line joining their centres.
  - (ii) The radical axis of three circles taken in pairs meet in a point.
  - (iii) If two circles cut a third circle orthogonally the radical axis of two circles, passes through the centre of the third circle.
  - (iv) The radical axis of two circles bisects each of their common tangents.
  - (v) The radical centre of three circles described on the three sides of a triangle as diameter, is the orthocentre of the triangle.
- (11) **Equation of the circle through the point of intersection.** Let  $S=0, S_1=0$  be two circles, Then  $S - \lambda S_1 = 0$  is the equation of the circle through the point of intersection, where  $\lambda$  is a constant.
- (12) **Co-axial system of circles.** A system of circles is said to be a co-axial system, when every pair of the system has the same radical axis. If we consider the line of centre as x-axis and the common radical axis as the y-axis then the equation to any member of the system is  $x^2 + y^2 + 2gx + c = 0$  where  $g$  is variable and  $c$  is fixed.

### 1.13 The Parabola

**Definition.** A parabola is the locus of a point which moves so that its distance from focus is equal to the distance from the directrix.

The standard equation of the parabola is  $y^2 = 4ax$ ,

where vertex is at  $A:(0, 0)$  focus  $(a, 0)$  the axis of parabola is x-axis ( $y = 0$ ). Equation of tangent at the vertex  $A:(0, 0)$  is y-axis.

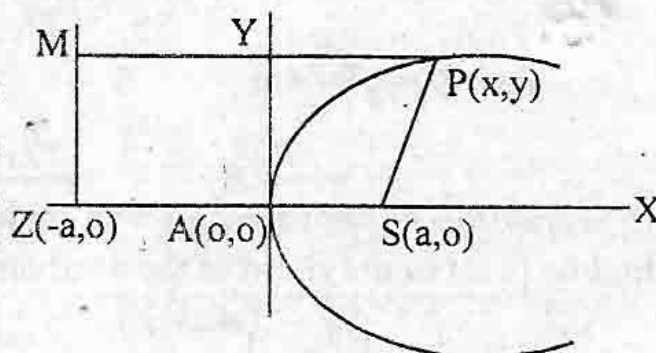


Fig. 5

Length of latus rectum =  $4a$ , equation of the directrix is  $x = -a$ .

Here we shall give some important properties for the standard equation of the parabola.

**(1) Tangent.** (i) The equation of tangent at  $(x_1, y_1)$  to the parabola  $y^2 = 4ax$  is

$$yy_1 = 2a(x + x_1).$$

(ii) The line  $y = mx + \frac{a}{m}$  is the tangent to the parabola  $y^2 = 4ax$

for every  $m$ . The point of contact has co-ordinate  $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ .

(iii) The parametric equation of the parabola  $y^2 = 4ax$  is  $x = at^2$ ,  $y = 2at$ , where  $t$  is parameter. The equation of tangent at  $(at^2, 2at)$  is  $ty = x + at^2$

**(2) Normal.** (i) The equation of normal to the parabola  $y^2 = 4ax$  at the point  $(am^2, -2am)$  is,

$$y = mx - 2am - am^3.$$

It is obvious that from a given point in general three normals can be drawn of which one must be real.

(ii) The equation of normal at  $(at^2, -2at)$  to the parabola  $y^2 = 4ax$  is  $tx + y = 2at + at^3$ .

**(3) Chord of contact and polar.** The equation of the chord of contact of tangents drawn from  $(x_1, y_1)$  to the parabola and also the polar of the point  $(x_1, y_1)$  with respect to parabola is given by  $yy_1 = 2a(x + x_1)$ .

**(4) Pair of tangents.** The equation of pair of tangents to the parabola from  $(x_1, y_1)$  is

$$(y^2 - 4ax)(y_1^2 - 4ax_1) = [yy_1 - 2a(x + x_1)]^2$$

$$SS_1 = T^2,$$

$$\text{where } S = y^2 - 4ax, \quad S_1 = y_1^2 - 4ax_1,$$

$$T = yy_1 - 2a(x + x_1).$$

**(5) Equation of chord with given middle point.** Let  $(x_1, y_1)$  be the middle point of the chord of the parabola then it has equation

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax$$

or

$$T = S_1.$$

**(6) Diameter.** The diameter is the locus of the middle point of a system of parallel chords.

Let  $y = mx + c$  be a given chord and let  $(x_1, y_1)$  be the middle point of the chords drawn parallel to the given chord. Hence it has equation

$$yy_1 = 2a(x + x_1).$$

Therefore  $m = \frac{2a}{y_1}$ . Hence the locus of  $(x_1, y_1)$  i.e. the equation of diameter is  $y = \frac{2a}{m}$ .

**(7) Some results based on tangent to the parabola.**

- (i) The tangent at any point of a parabola bisects the angle between the focal distance of the point and perpendicular on the directrix from the point.
- (ii) The tangents at the extremities of a focal chord of a parabola intersect at right angles on a directrix
- (iii) The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.

### 1.14 The Ellipse.

**Definition.** Ellipse is the locus of a point which moves such that its distance from the fixed point (focus) is  $e (< 1)$  times its distance from a fixed straight line (directrix). The standard equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In ellipse major and minor axes are along x and y-axis.

Length of major axis =  $AA' = 2a$ .

Length of minor axis =  $BB' = 2b$ .

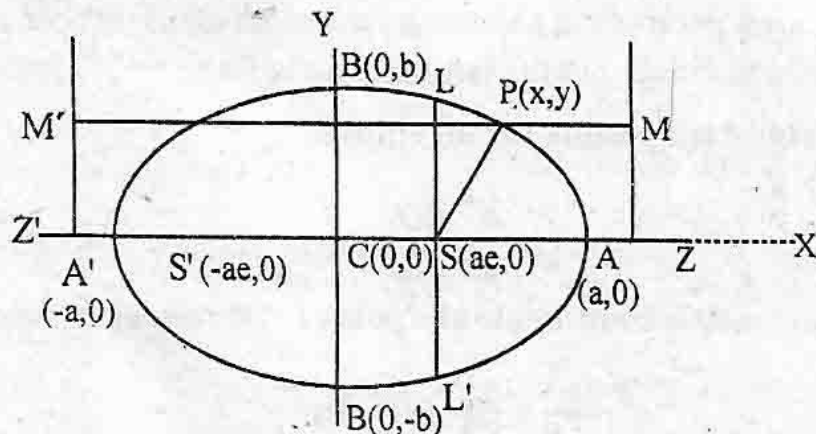


Fig. 6

$$\text{Length of Latus rectum} = LL' = 2 \frac{b^2}{a}$$

Eccentricity  $e$  is given by  $b^2 = a^2(1 - e^2)$ , ( $a > b$ ,  $e < 1$ ).

It has two foci  $S$  and  $S'$  having coordinates  $(ae, 0)$  and  $(-ae, 0)$  respectively.

The equation of two directrices  $ZM$ ,  $Z'M'$  are  $x = \pm \frac{a}{e}$

**Cor. :**

- (i) If  $e = 0$ ,  $b^2 = a^2(1 - e^2)$  gives  $a = b$ . The ellipse becomes circle  $x^2 + y^2 = a^2$ .
- (ii) If the origin is transferred at  $(-a, 0)$  the equation of the ellipse becomes.

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 + \frac{b^2}{a^2}x^2 = 2\frac{b^2}{a}x$$

If  $a$  and  $b$  separately tends to  $\infty$  such that  $\frac{b^2}{a}$  is a finite quantity

Then

$$\lim_{a \rightarrow \infty} \frac{b^2}{a^2} = \lim_{a \rightarrow \infty} \frac{\left(\frac{b^2}{a}\right)}{a} = \lim_{a \rightarrow \infty} \frac{\lambda}{a} = 0$$

$$y^2 = 2\lambda x \text{ where } \frac{b^2}{a} = \lambda$$

which is a parabola.

Thus a parabola is a limiting case of an ellipse whose axes are of infinite length but the latus rectum is finite.

**Some standard results for an ellipse.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

- (1) **Parametric co-ordinate of a point.** Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A circle with major axis of the ellipse as diameter is called auxillary circle, Let P be a point on the ellipse and Q be the point where the ordinate through P meets the auxillary circle. Then  $\angle ACQ =$  Eccentric angle of P and is denoted by  $\phi$ .

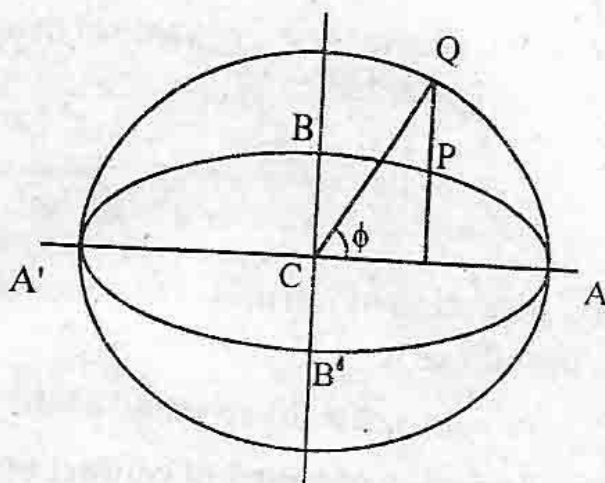


Fig. 7

Then the co-ordinate of the P given by  $x = a \cos \phi$ ,  $y = b \sin \phi$

satisfy the equation of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

Hence the parametric co-ordinate of any point P of the ellipse is  $(a \cos \phi, b \sin \phi)$  where  $\phi$  is the parameter (eccentric angle).

**(2) Equation of the chord joining two points.** The equation of chord joining two points on the ellipse having parametric co-ordinates  $(a \cos \theta, b \sin \theta)$  and  $(a \cos \phi, b \sin \phi)$  is

$$\frac{x}{a} \cos \frac{\theta + \phi}{2} + \frac{y}{b} \sin \frac{\theta + \phi}{2} = c \cdot s \frac{\theta - \phi}{2}.$$

If  $\phi \rightarrow \theta$ , then points becomes coincident, the chord becomes tangent to the ellipse at point  $(a \cos \theta, b \sin \theta)$  having its equation given by

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1.$$

**(3) Equation of tangent.**

(i) Equation of tangent at  $(x_1, y_1)$  to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(ii) The chord  $y = mx + c$  is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad c^2 = a^2 m^2 + b^2.$$

In otherwords, the chord  $y = mx + \sqrt{a^2 m^2 + b^2}$  is tangent to

the ellipse for every value of  $m$  and the co-ordinate of the point of contact is

$$\left( -\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \frac{b^2}{\sqrt{a^2 m^2 + b^2}} \right)$$

(4) **Equation of normal.** The equation of normal at  $(a \cos \phi, b \sin \phi)$  to the ellipse is

$$ax \sec \phi - by \csc \phi = a^2 - b^2$$

(5) **Equation of chord of contact and polar.** The equation of chord of contact of tangents drawn from  $(x_1, y_1)$  to the ellipse and the polar of  $(x_1, y_1)$  with respect to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(6) **Equation of pair of tangents.** The equation of pair of tangents to the ellipse from  $(x_1, y_1)$  is

$$\left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) = \left( \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2$$

$$SS_1 = T^2$$

where symbols have their usual meaning.

(7) **The equation of chord with given middle point.** Equation of chord of the ellipse with  $(x_1, y_1)$  as its middle point is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \quad \text{or} \quad T = S_1$$

(8) **Equation of the director circle.** The locus of a point from which pair of tangents drawn to the ellipse are at right angles, is a circle called director circle.

As in (6) discussed above, the equation of pair of tangents are

$$\left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) = \left( \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2$$

Pair of tangents are at right angles if sum of coefficient of  $x^2$  and  $y^2$  is equal to zero.

$$\frac{1}{a^2} \left( \frac{y_1^2}{b^2} - 1 \right) + \frac{1}{b^2} \left( \frac{x_1^2}{a^2} - 1 \right) = 0.$$

On solving, the equation of locus of  $(x_1, y_1)$  readily gives

$$x^2 + y^2 = a^2 + b^2$$

which is the equation of director circle.

### 1.15 Conjugate diameters.

Two diameters of an ellipse are said to be conjugate if each bisects chord parallel to other.

Let  $y = m_1 x$  and  $y = m_2 x$  be two conjugate diameter of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let  $(x_1, y_1)$  be the middle point of the chord parallel to  $y = m_1 x$

The equation of this chord is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$\text{hence } m_1 = -\frac{b^2 x_1}{a^2 y_1}$$

Therefore locus of  $(x_1, y_1)$  is  $y = -\frac{b^2}{a^2 m_1} x$ .

Since it is same as  $y = m_2 x$ . We have

$$m_2 = -\frac{b^2}{a^2 m_1}$$

$$m_1 m_2 = -\frac{b^2}{a^2}$$

which is required condition for the diameters  $y = m_1 x$ ,  $y = m_2 x$  to be conjugate diameters of the ellipse.

### Properties.

- (1) The eccentric angles of ends of conjugate diameters of an ellipse differs by right angle.

Let  $PCP'$  and  $DCD'$  be two conjugate diameters with  $\theta$  and  $\phi$  as eccentric angles of  $P$  and  $D$ .

$$\text{Gradient of } CP = \frac{b \sin \theta}{a \cos \theta}.$$

$$\text{Gradient of } CD = \frac{b \sin \phi}{a \cos \phi}.$$

Then,

$$\frac{b \sin \theta}{a \cos \theta} \cdot \frac{b \sin \phi}{a \cos \phi} = -\frac{b^2}{a^2}$$

$$\cos(\theta - \phi) = 0$$

$$\theta - \phi = \frac{\pi}{2}.$$

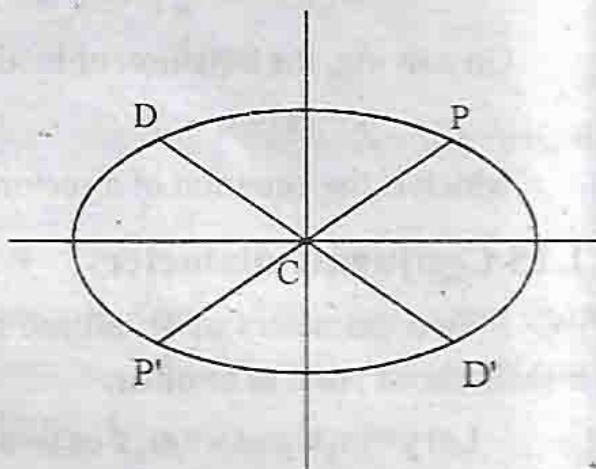


Fig. 8

Therefore ends of conjugate diameters has co-ordinates  $(a \cos \theta, b \sin \theta)$  and  $(-a \sin \theta, b \cos \theta)$  respectively.

## (2) Equi-conjugate diameters.

Two diameters are said to be equiconjugate diameters if they are equal in length.

The diameters  $PCP'$  and  $DCD'$  are equi conjugate if

$$CP^2 = CD^2$$

$$a^2 \cos^2 \theta + b^2 \sin^2 \theta = a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

$$(a^2 - b^2) \cos 2\theta = 0$$

$$\cos 2\theta = 0 \quad (a > b, a^2 - b^2 \neq 0)$$

$$\theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Then  $P$  has co-ordinate  $\left(a \cos \frac{\pi}{4}, b \sin \frac{\pi}{4}\right)$  The equation of  $CP$  and  $CD$  are given by

$$y = \frac{b}{a}x, y = -\frac{b}{a}x$$

## 1.16 The hyperbola

**Definition.** Hyperbola is the locus of a point which moves such that its distance from a fixed point (focus) is  $e$  ( $e > 1$ ) times its distance from a fixed straight line (directrix).

The standard equation of the hyperbola is

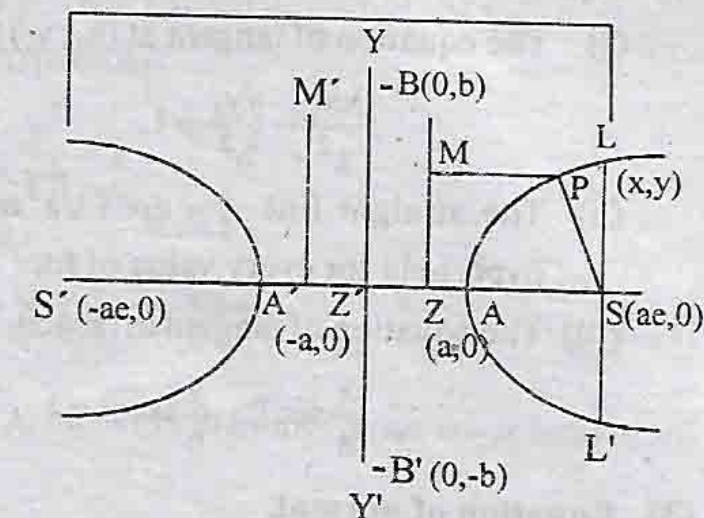


Fig. 9

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The transverse axis and conjugate axis are along  $x$  and  $y$  axis respectively.

The length of transverse axis =  $AA' = 2a$

The length of conjugate axis =  $BB' = 2b$

The length of latus rectum =  $LL' = 2 \frac{b^2}{a}$

The eccentricity ' $e$ ' is given by  $b^2 = a^2(e^2 - 1)$  ( $e > 1$ )

Co-ordinates of the foci  $S$  and  $S'$  are respectively  $(ae, 0)$  and  $(-ae, 0)$ .

The equation of two directrices  $ZM$  and  $Z'M'$  are  $x = \pm \frac{a}{e}$ .

**Some standard result for the hyperbola.**

(1) **Parametric co-ordinates.** The co-ordinate of a point  $P(x, y)$  on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$x = a \sec \theta; \quad y = b \tan \theta.$$

$$x = a \cosh t; \quad y = b \sinh t.$$

**(2) Equation of tangent.**

(i) The equation of tangent at  $(x_1, y_1)$  to the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(ii) The straight line  $y = mx + \sqrt{a^2 m^2 - b^2}$  is tangent to the hyperbola for every value of  $m$ .

(iii) The equation of tangent at  $(a \sec \theta, b \tan \theta)$  to the hyperbola is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1.$$

**(3) Equation of normal.**

(i) The normal at  $(x_1, y_1)$  to the hyperbola is

$$\frac{x - x_1}{\frac{x_1}{a^2}} + \frac{y - y_1}{\frac{y_1}{b^2}} = 0.$$

(ii) The equation of normal at  $(a \sec \theta, b \tan \theta)$  to the hyperbola is by  $\cot \theta + a x \cos \theta = a^2 + b^2$ .

**(4) The equation of the chord whose middle point is known.**

Let  $(x_1, y_1)$  be the middle point of the chord of the hyperbola. then it has the equation

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$

$$T = S_1.$$

**(5) The equation of pair of tangents.** The equation of pair of tangents to the hyperbola from point  $(x_1, y_1)$  are

$$\left( \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \right) \left( \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \right) = \left( \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \right)^2.$$

$$\text{or } SS_1 = T^2.$$

**(6) The equation of director circle.** The director circle of the hyperbola has the equation given by

$$x^2 + y^2 = a^2 - b^2.$$

**(7) Conjugate diameters.** Diameters  $y = m_1 x$  and  $y = m_2 x$  are

conjugate if  $m_1 m_2 = \frac{b^2}{a^2}$ .

(8) **Asymptotes.** A tangent to a curve such that the point of contact of the tangent lies at infinity, is called an asymptote of the curve

The equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be written as

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 1 \quad \dots(1)$$

Let the line  $\frac{x}{a} + \frac{y}{b} = \lambda$  meet (1) in point whose co-ordinates are given by

$$\lambda\left(\lambda - \frac{2y}{b}\right) = 1 \quad \dots(2)$$

Equation (1) is a second degree equation in y, equation (2) suggests that one value of y is infinite and other value of y is also infinite if  $\lambda=0$ . Hence the two values of y will become coincident at infinity and the line.

$$\frac{x}{a} + \frac{y}{b} = 0$$

is tangent at infinity i.e. an asymptote. Hence  $\frac{x}{a} + \frac{y}{b} = 0$  is an

asymptote of the hyperbola, similarly  $\frac{x}{a} - \frac{y}{b} = 0$  will be another asymptote. The combined equation of the asymptotes are

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

(9) **Conjugate hyperbola.** The hyperbola which has for its transverse and conjugate axis, the conjugate and transverse axis of the given hyperbola is called the conjugate hyperbola.

Thus the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \dots(1)$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \dots(2)$$

Equation (1) and (2) have the same asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \dots(3)$$

We infer that equation of asymptotes differs from that of the hyperbola by a constant and the equation of conjugate hyperbola differs from that of the asymptotes by the same constant.

#### (10) Rectangular Hyperbola.

(i) The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots(1)$$

and the asymptotes are given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \text{ or } y = \pm \frac{b}{a}x.$$

If  $\alpha$  is the angle which either asymptotes makes with transverse axis. Then the angle between the asymptotes is given by

$$2\alpha = 2\tan^{-1}\left(\frac{b}{a}\right).$$

If the asymptotes are at right angle then

$$2\alpha = \frac{\pi}{2} \left( \alpha = \frac{\pi}{4} \right)$$

$$2\tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{2}.$$

$$\frac{b}{a} = 1.$$

$$b = a.$$

The equation of the hyperbola is  $x^2 - y^2 = a^2$ , which is known as rectangular hyperbola (Here sum of the coeffs of  $x^2$  and  $y^2 = 0$ )

The eccentricity of the rectangular hyperbola is given by

$$a^2 = a^2(e^2 - 1) \quad [b^2 = a^2(e^2 - 1)]$$

$$e = \sqrt{2}$$

(ii) If the asymptotes are co-ordinate axes

$$\text{Then } \alpha = \frac{\pi}{4}$$

Rotating the axis by

$-\frac{\pi}{4}$ , we get

$$x = \frac{x' + y'}{\sqrt{2}}$$

$$y = \frac{y' - x'}{\sqrt{2}}$$

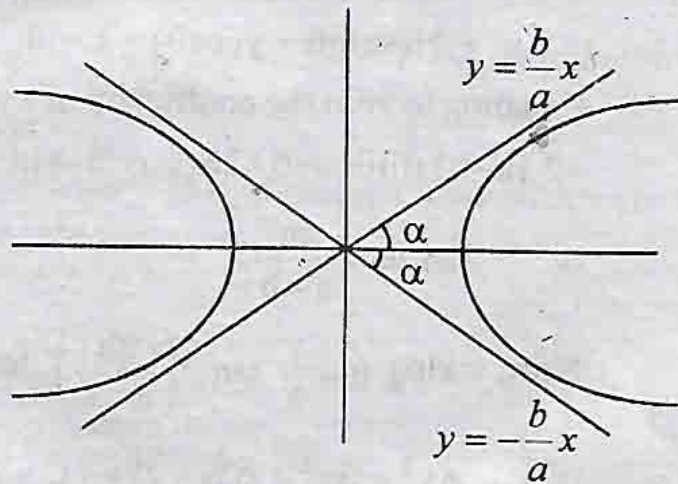


Fig. 10

Putting this value in  $x^2 - y^2 = a^2$  and dropping suffix, we get

$$4xy = 2a^2$$

$$xy = \frac{a^2}{2}$$

$$xy = c^2 \text{ where } c^2 = \frac{a^2}{2}$$

The parametric co-ordinate of any point on the rectangular hyperbola

$xy = c^2$  is  $\left(ct, \frac{c}{t}\right)$  where  $t$  is parameter and the equation of

tangent at  $\left(ct, \frac{c}{t}\right)$  is

$$\frac{x}{t} + ty = 2c.$$

### 1.17. The general equation of second degree.

To prove that the general equation of the second degree always represents a conic.

[Purv., 92]

Let  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  ... (1)

be the general equation of the second degree.

In order to remove the term  $xy$  from (1) let us rotate the rectangular axes through an angle  $\theta$  without changing the position of the origin. Now the new equation becomes

$$\begin{aligned} & a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) \\ & + b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta) \\ & + 2f(x\sin\theta + y\cos\theta) + c = 0. \end{aligned} \quad \dots (2)$$

Equating to zero the coefficient of  $xy$  in (2), we get

$$-2(a-b)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta) = 0$$

$$\text{or} \quad \tan 2\theta = \frac{2h}{a-b}.$$

Now, taking  $\theta = \frac{1}{2} \tan^{-1}\left(\frac{2h}{a-b}\right)$ , let the changed equation is

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0. \quad \dots (3)$$

**Case I.** Let neither  $A$  nor  $B$  be zero. The equation (3) can be written as

$$\begin{aligned} & A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 \\ & = \frac{G^2}{A} + \frac{F^2}{B} - C. \end{aligned} \quad \dots (4)$$

Shifting the origin at  $\left(-\frac{G}{A}, -\frac{F}{B}\right)$  the equation of conic becomes

$$AX^2 + BY^2 = K, \quad \text{where} \quad K = \frac{G^2}{A} + \frac{F^2}{B} - C$$

$$\text{or} \quad \frac{X^2}{K/A} + \frac{Y^2}{K/B} = 1. \quad \dots (5)$$

Equation (5) represents an ellipse if  $\frac{K}{A}$  and  $\frac{K}{B}$  are both positive

and a hyperbola if  $\frac{K}{A}$  and  $\frac{K}{B}$  are of opposite sign. If  $\frac{K}{A}$  and  $\frac{K}{B}$  are both negative then represents an imaginary ellipse.

If  $K = 0$  then (5) becomes

$$AX^2 + BY^2 = 0,$$

which is homogeneous equation of second degree and hence represents a pair of straight lines (real or imaginary).

**Case II.** Let either A or B be zero. For definiteness let A be zero and  $B \neq 0$ .

The equation (3) becomes

$$B\left(y + \frac{F}{B}\right)^2 = \frac{F^2}{B} - 2Gx - C \quad \dots(6)$$

If  $G = 0$  then (6) represents a pair of straight lines which are real, coincident or imaginary according as

$$\frac{1}{B}\left(\frac{F^2}{B} - C\right) > = < 0.$$

If  $G \neq 0$  then (6) can be written as

$$\left(y + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x - \frac{F^2}{2BG} + \frac{C}{2G}\right). \quad \dots(7)$$

Shifting the origin at  $\left(\frac{F^2}{2BG} - \frac{C}{2G}, -\frac{F}{B}\right)$ , (7) becomes

$$Y^2 = -\frac{2G}{B}X,$$

which is the equation of a parabola.

Thus we find that the general equation of the second degree always represents a conic.

### 1.18. Centre of a conic.

**Definition.** The centre of a conic is a point such that all the chords of the conic, which pass through it, are bisected there.

(i) To show that when the conic is of the form

$$ax^2 + 2hxy + by^2 + c = 0 \quad \dots(1)$$

the origin is the centre.

Let  $(x_1, y_1)$  be any point on (1) then

$$ax_1^2 + 2hx_1y_1 + by_1^2 + c = 0$$

$$\text{or } a(-x_1)^2 + 2h(-x_1)(-y_1) + b(-y_1)^2 + c = 0$$

which shows that  $(-x_1, -y_1)$  also lies on (1). The points  $(x_1, y_1)$  and  $(-x_1, -y_1)$  lie on the same straight line through the origin and are at equal distance from the origin.

Therefore, the chord of the conic (1) which passes through the origin and any point  $(x_1, y_1)$  of the curve is bisected at the origin. The origin is therefore the centre of the conic.

(ii) When the conic is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots(1)$$

the origin is centre when both  $f = g = 0$ .

If origin is the centre of conic (1), then corresponding to each point  $(x_1, y_1)$  on (1), there must be a point  $(-x_1, -y_1)$  lying on the curve. Hence

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots(2)$$

$$ax_1^2 + 2hx_1y_1 + by_1^2 - 2gx_1 - 2fy_1 + c = 0 \quad \dots(3)$$

Subtracting (3) from (2) we get

$$gx_1 + fy_1 = 0$$

The relation is true for all points  $(x_1, y_1)$  on (1) only when

$$g = f = 0$$

Hence the proposition.

(iii) To find the co-ordinate of the centre of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots(1)$$

Let the co-ordinate of the centre of the conic (1) be  $(x_1, y_1)$ , shifting the origin at  $(x_1, y_1)$  and keeping the axes parallel to their old directions the new equation of the conic (1) becomes

$$a(x+x_1)^2 + 2h(x+x_1)(y+y_1) + b(y+y_1)^2 + 2g(x+x_1) + 2f(y+y_1) + c = 0$$

$$\text{or } ax^2 + 2hxy + by^2 + 2x(ax_1 + hy_1 + g) + 2y(hx_1 + by_1 + f) + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad \dots(2)$$

The point  $(x_1, y_1)$  the new origin will be centre of (1) if

$$ax_1 + hy_1 + g = 0, \quad \dots(3)$$

$$hx_1 + by_1 + f = 0. \quad \dots(4)$$

Solving (3) and (4) we get the co-ordinate of the centre of the conic (1) as

$$\left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right).$$

[Equations (3) and (4) can also be obtained by differentiating the equation of conic (1) partially with respect to  $x$  and  $y$  respectively.]

If  $ab - h^2 = 0$  and  $hf - bg \neq 0$ ,  $gh - af \neq 0$  then equation (1) represents a parabola which has no centre. On the other hand if

$$ab - h^2 = 0, hf - bg = 0 \text{ and } gh - af = 0,$$

$$\text{Then } \frac{a}{h} = \frac{h}{b} = \frac{g}{f}.$$

In this case equations (3) and (4) reduce to only one equation and equation of the conic (1) can be written as

$$(ax + hy)^2 + 2g(ax + hy) + ac = 0$$

$$\text{or } (ax + hy + g)^2 = g^2 - ac. \quad \dots(5)$$

Equation (5) represents a pair of straight lines which are parallel, coincident or imaginary according as

$$g^2 - ac > = < 0.$$

Now, equation (2) can be written as

$$ax^2 + 2hxy + by^2 + c' = 0,$$

where

$$c' = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

$$= (ax_1 + hy_1 + g)x_1 + (hx_1 + by_1 + f)y_1 + (gx_1 + fy_1 + c)$$

Making use of (3) and (4), we get

$$c' = gx_1 + fy_1 + c.$$

Again, substituting the values of  $x_1$  and  $y_1$  we find that

$$\begin{aligned} c' &= g \left( \frac{hf - bg}{ab - h^2} \right) + f \left( \frac{gh - af}{ab - h^2} \right) + c \\ &= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{ab - h^2}, \end{aligned}$$

where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$

Hence shifting the origin at the centre of the conic (1) its equation becomes  $ax^2 + 2hxy + by^2 + \frac{\Delta}{ab-h^2} = 0$

**Note :** Let the conic be

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Centre is obtained by solving

$$\frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0.$$

### 1.19 To find the equation of asymptotes of the conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots(1)$$

Since equation of a conic and the combined equation of its asymptotes differ only by a constant, let us assume that equation of asymptotes of (1) be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0. \quad \dots(2)$$

Equation (2) should now represent a pair of straight lines, so

$$ab(c+\lambda) + 2fgh - af^2 - bg^2 - (c+\lambda)h^2 = 0$$

$$\text{or } (abc + 2fgh - af^2 - bg^2 - ch^2) + \lambda(ab - h^2) = 0$$

$$\text{or } \lambda = -\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = -\frac{\Delta}{ab - h^2}.$$

Substituting this value of  $\lambda$  in (2) we get the equation of asymptotes of the conic (1) as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0.$$

### 1.20 Nature of the conic.

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots(1)$$

**Case I.** Equation (1) represents a pair of straight lines if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

In this case if  $\theta$  is the angle between the lines then

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

- (i)  $h^2 - ab > 0$  lines are real and different.
- (ii)  $h^2 - ab = 0$  lines are parallel or coincident.
- (iii)  $h^2 - ab < 0$  lines are imaginary.

Again, if

- (iv)  $h^2 - ab > 0$  and  $a + b = 0$  then lines are real and mutually perpendicular.

### Case II.

Let  $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$   
 then equation (1) does not represent a pair of straight lines. In this case equation

$$ax^2 + 2hxy + by^2 = 0 \quad \dots(2)$$

represents two lines which are parallel to the asymptotes of the conic (1).

If  $\theta$  is the angle between the asymptotes then

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}.$$

- (i)  $h^2 - ab > 0$  asymptotes of conic (1) are real so conic (1) is a hyperbola.
- (ii)  $h^2 - ab > 0$  and  $a + b = 0$  then asymptotes are real and mutually perpendicular so conic (1) is a rectangular hyperbola.
- (iii)  $h^2 - ab = 0$  then second degree terms of equation (1) form a perfect square and hence conic (1) is a parabola.
- (iv)  $h^2 - ab < 0$  then asymptotes of the conic (1) are imaginary so conic (1) is an ellipse.

### 1.21 To find the equation of the tangent at any point on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots(1)$$

[GKP, 72; Kan., 86]

Let  $(x_1, y_1)$  be a point on the conic (1) then

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots(2)$$

Differentiating (1), we get

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}.$$

So  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$

Hence equation of tangent at  $(x_1, y_1)$  to the conic (1) is

$$y - y_1 = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1),$$

or  $(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$

or  $(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y = -gx_1 - fy_1 - c$

or  $axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

## 1.22 Polar.

Equation of the polar in the case of every conic is of the same form as the equation of the tangent. So equation of the polar of the point  $(x_1, y_1)$  with respect to the conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

## 1.23 To find the equation of the chord of the conic.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  whose middle point is  $(x_1, y_1)$ .

Let the chord of middle point  $(x_1, y_1)$  cuts the given conic at  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  then its slope is

$$\frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1}.$$

Equation of the chord can be written as

$$y - y_1 = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1}(x - x_1). \quad \dots(1)$$

since  $(x_1, y_1)$  is the middle point,

$$x_1 = \frac{\alpha_1 + \alpha_2}{2} \text{ and } y_1 = \frac{\beta_1 + \beta_2}{2} \quad \dots(2)$$

$(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  are points on the conic so

$$a\alpha_1^2 + 2h\alpha_1\beta_1 + b\beta_1^2 + 2g\alpha_1 + 2f\beta_1 + c = 0 \quad \dots(3)$$

and  $a\alpha_2^2 + 2h\alpha_2\beta_2 + b\beta_2^2 + 2g\alpha_2 + 2f\beta_2 + c = 0 \quad \dots(4)$

Subtracting (3) from (4), we get

$$\begin{aligned} a(\alpha_2^2 - \alpha_1^2) + 2h(\alpha_2\beta_2 - \alpha_1\beta_1) + b(\beta_2^2 - \beta_1^2) \\ + 2g(\alpha_2 - \alpha_1) + 2f(\beta_2 - \beta_1) = 0 \end{aligned}$$

or

$$\frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1} = -\frac{a(\alpha_1 + \alpha_2) + h(\beta_1 + \beta_2) + 2g}{h(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) + 2f} \quad \dots(5)$$

Now putting the value of  $\alpha_1 + \alpha_2$  and  $\beta_1 + \beta_2$  from (2) in (5), we get

$$\frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1} = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} \quad \dots(6)$$

Substituting this value in (1), required equation of the chord becomes

$$y - y_1 = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1)$$

or

$$\begin{aligned} axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c \\ = (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c). \end{aligned}$$

This equation is of the form

$$T = S_1$$

where  $T = axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$

and  $S_1 = (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)$ .

### Conjugate Diameters

**Definition.** Two diameters of the conics are said to be conjugate if each bisects chord parallel to each other.

#### 1.24 To find the condition that $m$ and $m_1$ are slopes of the conjugate diameter of conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots(1)$$

Let  $(x_1, y_1)$  be the co-ordinate of the middle point of one of the system of parallel chords of slope  $m$  of the conic (1)

Then equation of the chord is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Hence,

$$m = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$$

or  $(ax_1 + hy_1 + g) + m(hx_1 + by_1 + f) = 0$

so the locus of the middle point  $(x_1, y_1)$  is

$$ax + hy + g + m(hx + by + f) = 0,$$

$$(a + mh)x + (h + bm)y + (g + mf) = 0 \quad \dots(2)$$

which is a straight line.

The line (2) passes through the middle point of a system of parallel chords of the conic (1) so it is a diameter of the conic.

If  $m_1$  is the slope of (2) then

$$m_1 = -\frac{a + mh}{h + bm}$$

$$\text{or} \quad a + h(m + m_1) + bmm_1 = 0 \quad \dots(3)$$

(3) is the condition that a diameter of slope  $m_1$  of the conic (1) passes through the middle points of a system of parallel chords of slope  $m$  of the conic.

If  $m$  and  $m_1$  are interchanged even then (3) remains unchanged showing that if  $m_1$  is the slope of a diameter of the conic (1) passing through the middle points of a system of parallel chords of slope  $m$  of the conic then there exists a diameter of slope  $m$  which passes through the middle points of a system of parallel chords of slope  $m_1$  of the conic. Hence  $m$  and  $m_1$  are slopes of the conjugate diameters of the conic (1).

Therefore (3) is the required condition,

**Cor. :** Diameters  $y = m_1x$  and  $y = m_2x$  will be conjugate diameters of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad m_1m_2 = -\frac{b^2}{a^2}$$


---

## **Part-I**

---

### **TWO DIMENSIONS**

---