3

CONFOCAL CONICS

3.1 Definition.

All those conics which have common foci are called confocal conics. Since one axis of a central conic passes through its foci and other is the perpendicular bisector of the line joining them so confocal conics have common axes and common centre.

Let us consider the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad .....(1)$$

Co-ordinates of its foci are

$$\left\{\pm\sqrt{(a^2+\lambda)-(b^2+\lambda)},0\right\}$$

i.e. $\left\{\pm\sqrt{a^2-b^2},0\right\}$.

These co-ordinates are free from λ, so equation (1), in which λ is a parameter represents a family of conics having same foci and hence confocal conics.

Cor. : Parabolas having common focus and common axis are called confocal parabolas. If the common focus is taken as origin and the common axis as axis of x, then the equation of confocal parabola is

$$y^2 = 4\lambda(x+\lambda).$$

3.2 Confocal conics through a given point.

Theorem . Two confocal conics pass through a given point, out of which one is a hyperbola and other is an ellipse.

[Purv., 97, 99; Avadh '77, 83: **(GKP, 98,2003, 2008; Purv., 97,99; Avadh, 77, 83,**

Let the confocal conics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

passes through the given point (α, β). Then

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}=1$$

or $$(\lambda+a^2)(\lambda+b^2)-\alpha^2(\lambda+b^2)-\beta^2(\lambda+a^2)=0 \qquad .....(1)$$

This being an equation of second degree in λ, gives two values of λ, corresponding to which we get two confocal conics passing through (α, β).

Let $f(\lambda)=(\lambda+a^2)(\lambda+b^2)-\alpha^2(\lambda+b^2)-\beta^2(\lambda+a^2)$ .....(2)

and $a > b$.

Now $f(-a^2)=-\alpha^2(b^2-a^2)$ is positive,

$f(-b^2)=-\beta^2(a^2-b^2)$ is negative,

$f(+\infty)=+\infty$ is positive.

So one root of the equation (1) lies between $-a^2$ and $-b^2$ and other between $-b^2$ and $+\infty$. Let the roots be λ_1 and λ_2 such that

$$-a^2<\lambda_1<-b^2 \text{ and } -a^2<-b^2<\lambda_2<+\infty$$

i.e., $\lambda_1+a^2>0$, and $\lambda_2+a^2>0$,

$\lambda_1+b^2<0$ and $\lambda_2+b^2>0$.

Corresponding to these two roots confocal conics though (α, β) are

$$\frac{x^2}{a^2+\lambda_1}-\frac{y^2}{b^2+\lambda_1}=1 \qquad .....(3)$$

and $$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1 \qquad .....(4)$$

Equation (3) represents a hyperbola, whereas (4) represents an ellipse.

3.3 Confocal conics cut orthogonally.

Theorem. Two confocal conics passing through a given point intersect orthogonally.

[GKP, 97, 2000, 2004; 2007) Avadh, 1980, 82, 96; Purv, 98, 2000]

Let two confocal conics passing through (α, β) be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1. \qquad .....(1)$$

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1. \qquad .....(2)$$

Then $\dfrac{\alpha^2}{a^2+\lambda_1}+\dfrac{\beta^2}{b^2+\lambda_1}=1.$ .....(3)

and $\dfrac{\alpha^2}{a^2+\lambda_2}+\dfrac{\beta^2}{b^2+\lambda_2}=1.$ .....(4)

Subtracting (4) from (3) and then dividing by $\lambda_2-\lambda_1$, we get

$$\frac{\alpha^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{\beta^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0. \quad .....(5)$$

The equation of tangent at (α, β) to conic (1) is

$$\frac{\alpha x}{a^2+\lambda_1}+\frac{\beta y}{b^2+\lambda_1}=1. \quad .....(6)$$

The equation of tangent at (α, β) to conic (2) is

$$\frac{\alpha x}{a^2+\lambda_2}+\frac{\beta y}{b^2+\lambda_2}=1. \quad .....(7)$$

Tangents (6) and (7) will be mutually perpendicular if

$$\frac{\alpha^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{\beta^2}{(b^2+\lambda_1)(b^2+\lambda_2)}$$

which is true from (5).

Hence conic (1) and (2) intersect o..hogona.

3.4 Propositions on confocals.

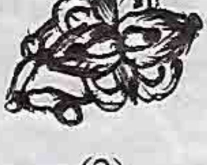
(i) Only one confocal conic touc.es a given straight line.

[GKP, 1985, 96, 2000]

Let the given straight line be

$$y = mx + c \quad .....(1)$$

The conic $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$



touches the line (1)

if $c^2=(a^2+\lambda)m^2+(b^2+\lambda).$ .....(2)

Equation (2) being of first degree in λ, gives one and only one value of λ, corresponding to which we get only one confocal conic touching the given line (1).

(ii) The point of intersection of two perpendicular tangents one to each of two given confocals lies on a circle.

[Avadh, 1979; GKP, 83, 96, 99; Purv.; 89, 92, 95]

Let the two given confocals be.

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \qquad .....(1)$$

and $$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1 \qquad .....(2)$$

Now, $$y-mx=\sqrt{(a^2+\lambda_1)m^2+(b^2+\lambda_1)} \qquad .....(3)$$

and $$y+\frac{1}{m}x=\sqrt{\left(a^2+\lambda_2\right)\left(-\frac{1}{m}\right)^2+(b^2+\lambda_2)}$$

i.e. $$my+x=\sqrt{(a^2+\lambda_2)+(b^2+\lambda_2)m^2} \qquad .....(4)$$

are two perpendicular straight lines touching conics (1) and (2) respectively for all values of m.

Let (x_1, y_1) be the point of intersection of (3) and (4)

Then $$y_1-mx_1=\sqrt{(a^2+\lambda_1)m^2+(b^2+\lambda_1)} \qquad .....(5)$$

and $$my_1+x_1=\sqrt{(a^2+\lambda_2)+(b^2+\lambda_2)m^2} \qquad .....(6)$$

Squaring and adding (5) and (6) we get :

$$x_1^2+y_1^2=a^2+b^2+\lambda_1+\lambda_2$$

Hence the locus of (x_1, y_1) is

$$x^2+y^2=a^2+b^2+\lambda_1+\lambda_2,$$

which is a circle.

(iii) The difference of the squares of the perpendiculars drawn from the centre, on any two parallel tangents one to each of two given confocals is constant. (GKP, 1983; Purv., 2002)

Let $$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1. \qquad .....(1)$$

and $$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1. \qquad .....(2)$$

be the two given confocal conics having a common centre (0, 0).

Now $$y-mx=\sqrt{(a^2+\lambda_1)m^2+(b^2+\lambda_1)} \qquad .....(3)$$

and $\quad y - mx = \sqrt{(a^2+\lambda_2)m^2+(b^2+\lambda_2)}$ .....(4)

are two parallel straight lines touching the conics (1) and (2) respectively for all values of m.

So the difference of the squares of the perpendiculars drawn from (0, 0) on (3) and (4) is

$$= \frac{(a^2+\lambda_1)m^2+(b^2+\lambda_1)}{1+m^2} - \frac{(a^2+\lambda_2)m^2+(b^2+\lambda_2)}{1+m^2}$$

$= (\lambda_1 - \lambda_2)$, which is a constant.

(iv) The locus of the pole of a given straight line with respect to a system of confocal conics is a straight line.

[GKP, 1984; Purv., 89, 91, 98]

Let $\quad y = mx + c$ .....(1)

be a given line and

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad .....(2)$$

be the equation of a conic of the given system of confocals.

Let (α, β) be the pole of the line (1) with respect to the confocal conic (2).

Now, the polar of (α, β) with respect to (2) is

$$\frac{\alpha x}{a^2+\lambda} + \frac{\beta y}{b^2+\lambda} = 1 \quad .....(3)$$

So (1) and (3) must be identical, hence comparing these equations we get :

$$\frac{-\alpha}{(a^2+\lambda)m} = \frac{\beta}{b^2+\lambda} = \frac{1}{c}. \quad .....(4)$$

Eliminating λ from these relations we get

$$\alpha c + m\beta c + m(a^2 - b^2) = 0. \quad .....(5)$$

Hence the locus of (α, β) is

$$cx + mcy + m(a^2 - b^2) = 0, \quad .....(6)$$

which is a straight line perpendicular to the given line (1)

EXAMPLES

Example 1. Prove that the equation of the confocal hyperbola through the point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

whose eccentric angle is α, is

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2$$

(Purv., 92, 94, 99, 2000; Avadh, 1975, 78; GKP, 1982, 2004)

Solution. The point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

whose eccentric angle is α, be $(a\cos\alpha, b\sin\alpha)$

Let $$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

be a conic confocal to the given ellipse and passing through $(a\cos\alpha, b\sin\alpha)$.

Then $$\frac{a^2\cos^2\alpha}{a^2+\lambda}+\frac{b^2\sin^2\alpha}{b^2+\lambda}=1$$

or $$a^2\cos^2\alpha(b^2+\lambda)+b^2\sin^2\alpha(a^2+\lambda)=(a^2+\lambda)(b^2+\lambda)$$

or $$\lambda^2+\lambda(a^2\sin^2\alpha+b^2\cos^2\alpha)=0 \quad .....(1)$$

Equation (1) gives

$$\lambda=0 \quad \text{or} \quad \lambda=-(a^2\sin^2\alpha+b^2\cos^2\alpha)$$

Since $\lambda \neq 0$ so taking $\lambda=-(a^2\sin^2\alpha+b^2\cos^2\alpha)$ the equation of the required confocal hyperbola is obtained as

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2.$$

Example 2. Show that the locus of the point of contact of parallel tangents to a system of confocal conics is rectangular hyperbola.

Solution. Let $$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad .....(1)$$

be a member of the given system of confocals and (x_1, y_1) be a point on it at which a straight line of slope m touching the conic (1).

Therefore (x_1, y_1) lies on (1),

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1 \quad .....(2)$$

Tangent to (1) at (x_1, y_1) is

$$\frac{xx_1}{a^2+\lambda}+\frac{yy_1}{b^2+\lambda}=1 \quad .....(3)$$

If m be its slope then

$$m=-\frac{x_1}{y_1}\frac{b^2+\lambda}{a^2+\lambda}$$

or $\quad my_1(a^2+\lambda)+x_1(b^2+\lambda)=0,$

or $\quad \lambda=-\dfrac{my_1a^2+x_1b^2}{my_1+x_1} \quad .....(4)$

Substituting the value of λ from (4) in (2), we get

$$\frac{x_1^2}{a^2-\dfrac{my_1a^2+x_1b^2}{my_1+x_1}}+\frac{y_1^2}{b^2-\dfrac{my_1a^2+x_1b^2}{my_1+x_1}}=1$$

or $\quad x_1(my_1+x_1)-\dfrac{y_1}{m}(my_1+x_1)=a^2-b^2$

or $\quad x_1^2-y_1^2+\left(m-\dfrac{1}{m}\right)x_1y_1=a^2-b^2.$

Hence the locus of (x_1, y_1) is

$$x^2-y^2+xy\left(m-\frac{1}{m}\right)=a^2-b^2,$$

which is the equation of a rectanglar hyperbola.

Example 3. Prove that the locus of the points lying on a system of confocal ellipses and having the same eccentric angle α, is a confocal hyperbola whose asymptotes are inclined at an angle 2α.

Solution. Let $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$

be a member of the system of confocal ellipses and (x_1, y_1) be a point on it whose eccentric angle is α.

Then $$x_1 = \sqrt{a^2+\lambda}\cos\alpha \qquad .....(1)$$

and $$y_1 = \sqrt{b^2+\lambda}\sin\alpha. \qquad .....(2)$$

Eliminating λ from (1) and (2), we get

$$\frac{x_1^2}{\cos^2\alpha} - \frac{y_1^2}{\sin^2\alpha} = (a^2-b^2)$$

So the locus of (x_1, y_1) is

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = (a^2-b^2) \qquad .....(3)$$

This is a hyperbola confocal with the given system of ellipses and its asymptotes are

$$\frac{x}{\cos\alpha} - \frac{y}{\sin\alpha} = 0 \quad \text{and} \quad \frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} = 0.$$

Slopes of these asymptotes are

$m_1 = \tan\alpha$ and $m_2 = -\tan\alpha$.

If θ is the angle between these asymptotes then

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$$

So $\theta = 2\alpha$.

Hence asymptotes of the hyperbola (3) are inclined at an angle 2α.

Example 4. If λ, μ be the parameters of the confocals which pass through points P and Q on a given ellipse respectively. Show that if P and Q are extremities of conjugate diameters then $\lambda+\mu$ is constant. Show also that if the tangents at P and Q are at right angle then $\frac{1}{\lambda}+\frac{1}{\mu}$ is constant.

[GKP, 73]

Solution. Let confocal conics be

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad .....(1)$$

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} = 1 \qquad .....(2)$$

and $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad .....(3)$$

is the given ellipse

Case I.

Let P and Q be extremities of conjugate diameters of the ellipse (3) So if ϕ is the eccentric angle of P then eccentric angle of Q is $\phi+\frac{\pi}{2}$

Conic (1) passes through P, so

$$\frac{a^2\cos^2\phi}{a^2+\lambda}+\frac{b^2\sin^2\phi}{b^2+\lambda}=1$$

or $\quad \lambda^2+\lambda(a^2\sin^2\phi+b^2\cos^2\phi)=0$

i.e., either $\quad \lambda=0,$

or $\quad \lambda=-(a^2\sin^2\phi+b^2\cos^2\phi).$

Since $\quad \lambda\neq 0,\ \lambda=-(a^2\sin^2\phi+b^2\cos^2\phi).$

Conic (2) passes through Q, so

Either $\quad \mu=0$

or $$\mu=-\left\{a^2\sin^2\left(\phi+\frac{\pi}{2}\right)+b^2\cos^2\left(\phi+\frac{\pi}{2}\right)\right\}.$$

Since $\quad \mu\neq 0,\ \mu=-(a^2\cos^2\phi+b^2\sin^2\phi).$

Adding λ and μ we get

$$\lambda+\mu=-(a^2\sin^2\phi+b^2\cos^2\phi)-(a^2\cos^2\phi+b^2\sin^2\phi).$$
$$=-(a^2+b^2),$$

which is a constant.

Case II.

Let P and Q be points on the ellipse (3) whose eccentric angles are θ and ϕ. Tangents at P and Q are mutually perpendicular.

Equation of tangent at P on the ellipse (3) is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad .....(4)$$

Equation of tangent at Q on the ellipse (3) is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \qquad .....(5)$$

since (4) and (5) are mutually perpendicular so

$$\frac{\cos\theta\cos\phi}{a^2}+\frac{\sin\theta\sin\phi}{b^2}=0$$

or $$\tan\theta\tan\phi=-\frac{b^2}{a^2} \qquad .....(6)$$

Conic (1) passes through P, so

$$\lambda=-(a^2\sin^2\theta+b^2\cos^2\theta).$$

Conic (2) passes through Q,

so $$\mu=-(a^2\sin^2\phi+b^2\cos^2\phi).$$

Now $$\frac{1}{\lambda}+\frac{1}{\mu}=-\left[\frac{1}{a^2\sin^2\theta+b^2\cos^2\theta}+\frac{1}{a^2\sin^2\phi+b^2\cos^2\phi}\right]$$

$$=-\left[\frac{1+\tan^2\theta}{b^2+a^2\tan^2\theta}+\frac{1+\tan^2\phi}{b^2+a^2\tan^2\phi}\right]$$

$$=-\left[\frac{2b^2+(a^2+b^2)(\tan^2\theta+\tan^2\phi)+2a^2\tan^2\theta\tan^2\phi}{b^4+a^2b^2(\tan^2\theta+\tan^2\phi)+a^4\tan^2\theta\tan^2\phi}\right]$$

Putting the value of tan θ tan ϕ from (6) we get

$$\frac{1}{\lambda}+\frac{1}{\mu}=-\left[\frac{2b^2+(a^2+b^2)(\tan^2\theta+\tan^2\phi)+\frac{2b^4}{a^2}}{b^4+a^2b^2(\tan^2\theta+\tan^2\phi)+b^4}\right]$$

$$=-\frac{a^2+b^2}{a^2b^2}$$

which is a constant.

Example 5. Prove that the two conics

$ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$

can be placed so as to be confocal, if

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}$$

[Luck., 82, 86; Avadh, 78, GKP, 97, Purv., 97, 2000]

Solution. Let us rotate the axes (keeping the axes rectangular) through such an angle that equation of conics

$ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$

became

$$\alpha x^2+\beta y^2=1 \quad\text{and}\quad \alpha' x^2+\beta' y^2=1$$

respectively. If these conics are placed so as to be confocal then

$$\frac{1}{\alpha}-\frac{1}{\beta}=\frac{1}{\alpha'}-\frac{1}{\beta'} \quad\text{or}\quad \frac{\alpha-\beta}{\alpha\beta}=\frac{\alpha'-\beta'}{\alpha'\beta'} \qquad .....(1)$$

Now by the principle of invariants we have

$$\alpha+\beta=a+b \qquad \alpha\beta=ab-h^2$$

$$\alpha'+\beta'=a'+b' \qquad \alpha'\beta'=a'b'-h'^2$$

So, $\alpha-\beta=\sqrt{(a+b)^2-4(ab-h^2)}=\sqrt{(a-b)^2+4h^2}$

and $\alpha'-\beta'=\sqrt{(a'+b')^2-4(a'b'-h'^2)}=\sqrt{(a'-b')^2+4h'^2}$

Substituting these values in (1), we get

$$\frac{\sqrt{(a-b)^2+4h^2}}{ab-h^2}=\frac{\sqrt{(a'-b')^2+4h'^2}}{a'b'-h'^2}$$

Now squaring both sides we get the required condition :

$$\frac{(a-b)^2+4h^2}{(ab-h^{\hat{}})^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Example 6. Show that the ends of equal conjugate diameters of a series of confocal ellipses are on a confocal rectangular hyperbola. [Purv., 89; GKP, 99]

Solution. Let one member of the series of confocal ellipses be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

Let (x_1, y_1) be the end of one *of its equal* conjugate diameters

Then

$$x=\pm\sqrt{a^2+\lambda}\cos\frac{\pi}{4} \qquad .....(1)$$

$$y_1=\pm\sqrt{b^2+\lambda}\sin\frac{\pi}{4} \qquad .....(2)$$

Eliminating λ from (1) and (2), we get

$$2x_1^2-2y_1^2=a^2-b^2 \quad \text{or} \quad x_1^2-y_1^2=\left(\frac{a^2-b^2}{2}\right)$$

So the locus of (x_1, y_1) is

$$x^2-y^2=\left(\frac{a^2-b^2}{2}\right)$$

ich is a rectangular hyperbola confocal with the given system of ellipses.

EXERCISE 3

1. Find the conics confocal with $x^2 + 2y^2 = 2$ which pass through the point (1, 1) (Purv., 90, 95, 2004; GKP, 97)

   (Ans. $3x^2 - y^2 \pm \sqrt{5}(x^2 - y^2) = 2$)

2. Prove that the locus of the pcints of contact of tangents drawn from a given point on the major axis to a system of confocals is a circle.

3. If two conics confocal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and passing through (x_1, y_1) are

$$\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1 \text{ and } \frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1$$

then show that

(i) $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = -\frac{\lambda_1\lambda_2}{a^2b^2}$

(ii) $x_1^2 + y_1^2 - a^2 - b^2 = \lambda_1 + \lambda_2$

4. Prove that the locus of the points of contact of the tangents drawn from a given point to a system of conics confocal with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a cubic curve

$$\frac{x}{y-\beta} + \frac{y}{x-\alpha} = -\frac{a^2 - b^2}{\alpha y - \beta x},$$

which passes through (α, β) and through the foci of the confocals. (Avadh, 1980; Purv., 99)

5. Show that points of contact of the tangents of the confocals

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1,$$

which are also tangents to the parabola

$$y^2 = 4x\sqrt{a^2 - b^2}$$

lie on a straight line. (GKP, 1996)

6. *Show that only one* member of a given system of confocals can have a given straight line as a normal.

7. If the product of the perpendiculars let fall on a straight line from its pole with respect to the e''ipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and from the centre of the ellipse is a constant quantity λ then prove that the straight line is a tangent to the confocal

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

8. An ellipse and a hyperbola are confocal and the asymptotes of the hyperbola lie along the equiconjugate diameters of the ellipse. Prove that the hyperbola will cut at right angles all conics which pass through the ends of the axes of the ellipse.

9. Prove that the confocal hyperbola through the point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

whose ecentric angle is 45° is $2(x^2-y^2)=a^2-b^2$ (GKP. 86)

(GKP, 2009)