4

POLAR EQUATIONS OF THE CONICS

4.1 Polar Coordinates.

In order to consider the position of a point on a plane in polar coordinate system, a fixed point of the plane is taken as pole or origin and a line passing through it is taken as initial line.

Let O be the pole and OA be the initial line. P is any point of the plane such that $\angle AOP = \theta$ (measured in anti-clockwise) and OP = r.

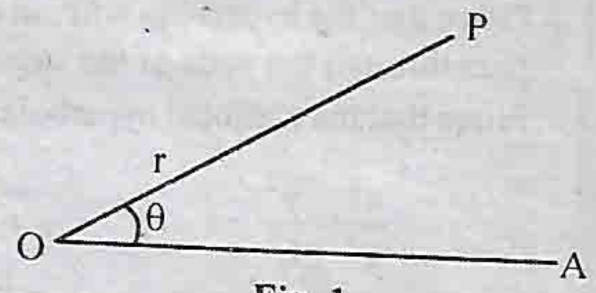

Fig. 1

θ is called the vectorial angle of P and r is called its radius vector. In polar coordinate system the point P (r, θ) can also be represented by following coordinates.

$(r, \theta + 2\pi),\ (r, \theta + 4\pi),\ (r, \theta + 6\pi)$ .....(1)

$(r, \theta - 2\pi),\ (r, \theta - 4\pi),\ (r, \theta - 6\pi)$ .....(2)

$(-r, \theta + \pi),\ (-r, \theta + 3\pi),\ (-r, \theta + 5\pi)$ .....(3)

$(-r, \theta - \pi),\ (-r, \theta - 3\pi),\ (-r, \theta - 5\pi)$ .....(4)

The radius vector is positive if it is measured along the line bounding the vectoral angle, it is negative if it is measured from the pole in opposite direction.

The vectoral angle is positive if it measured from the initial line in anti-clockwise direction, it is negative if it is measured from the initial line in clockwise direction, the polar co-ordinates of a point is not unique, the point P (r, θ) can be represented in infinite number of coordinates.

4.2 To change the polar coordinates of a point into rectangular Cartesian coordinates and vice-versa.

Let O be the pole or origin and x-axis be the initial line.

Let (r, θ) be the polar coordinate and (x, y) be Cartesian coordinate of P. Then

$$x = r\cos\theta \qquad .....(1)$$
$$y = r\sin\theta \qquad .....(2)$$

Also $r = \sqrt{x^2 + y^2}$ .....(3)

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) \qquad .....(4)$$

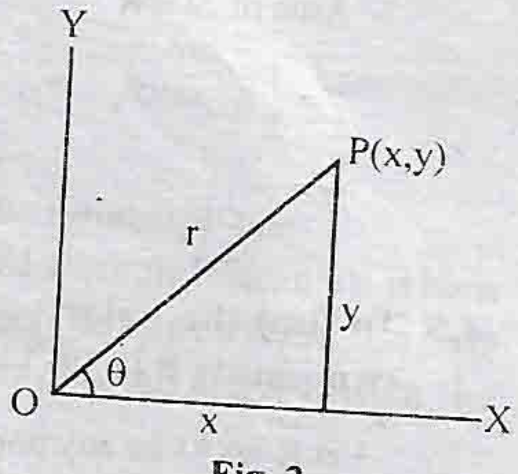

Fig. 2

Any equation in x, y can be transformed into polar coordinates with the help of (1) and (2) whereas equations in polar coordinates are transformed into Cartesian coordinates with the help of (3) and (4).

4.3 Distance between two points

To find the distance between two points.

$P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$

Let $OP = r_1$

$OQ = r_2$

$\angle AOP = \theta_1$

and $\angle AOQ = \theta_2$

In ΔPOQ

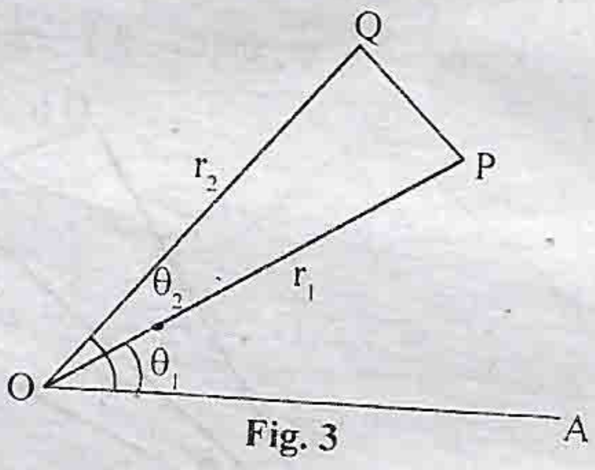

Fig. 3

$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ\cos(POQ)$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1).$$

So $PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

4.4 Area of the triangle.

To find the area of the triangle whose vertices are

$P(r_1, \theta_1)$, $Q(r_2, \theta_2)$ and $R(r_3, \theta_3)$

Area of ΔPQR

= Area of ΔOPQ

+ Area of ΔOQR

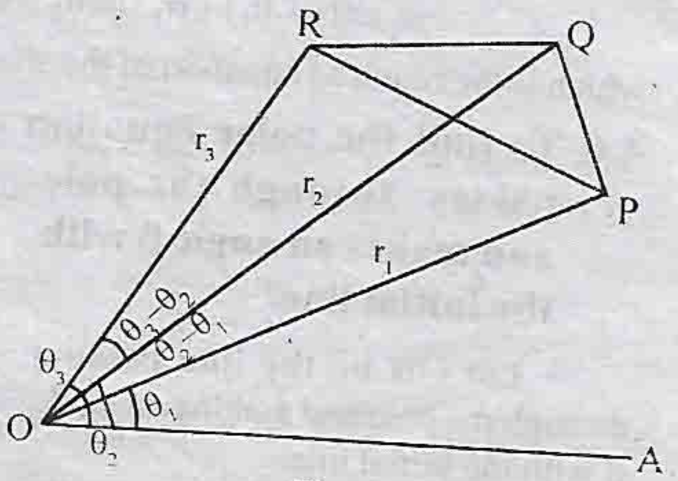

Fig. 4

– Area of ΔOPR

$$= \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) - \frac{1}{2} r_1 r_3 \sin(\theta_3 - \theta_1)$$

$$= \frac{1}{2}\left[r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3)\right].$$

4.5 To find the polar equation of a straight line joining two points P (r_1, θ_1) and Q (r_2, θ_2)

Let R (ρ, ϕ) be any point on the **straight line PQ. Then** Area of ΔPOQ=Area of ΔPOR+Area of ΔROQ.

So $$\frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) = \frac{1}{2} r_1 \rho \sin(\phi - \theta_1) + \frac{1}{2} \rho r_2 \sin(\theta_2 - \phi)$$

or $$r_1 r_2 \sin(\theta_2 - \theta_1) = r_1 \rho \sin(\phi - \theta_1) + \rho r_2 \sin(\theta_2 - \phi).$$

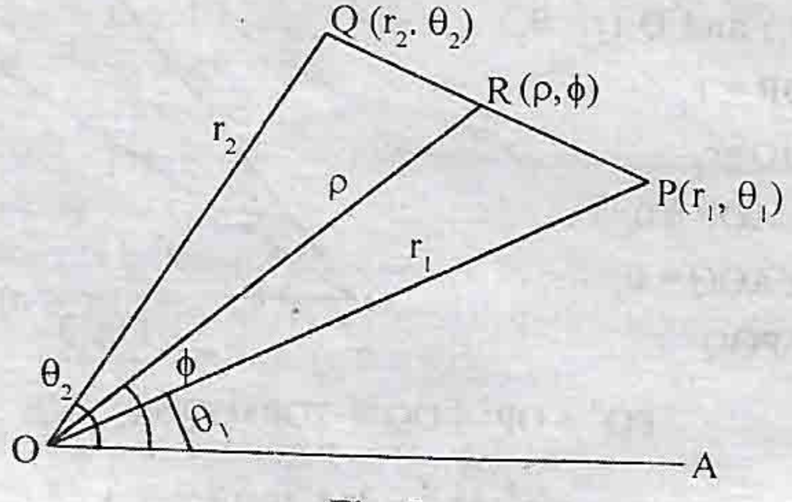

Fig. 5

Hence locus of (ρ, ϕ) is

$$r r_1 \sin(\theta - \theta_1) + r r_2 \sin(\theta_2 - \theta) = r_1 r_2 \sin(\theta_2 - \theta_1),$$

which is the required equation of the straight line.

4.6 To find the polar equation of a straight line which passes through the pole and makes an angle β with the initial line.

Let OB be the line passing through the pole and making an angle β with the initial line.

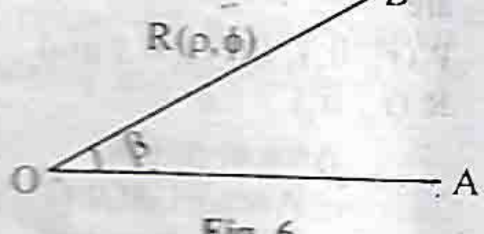

Fig. 6

Let R (ρ, ϕ) be any point on it, then

$$\phi = \beta$$

so the locus of (ρ, ϕ) is $\theta = \beta$

which is the required equation of the straight line.

4.7 To find the polar equation of a straight line on which the length of the perpendicular dropped from the pole is p and this perpendicular makes an angle α with the initial line.

Let MN be a straight line such as the length of the perpendicular dropped from O on it is

$$= OL = p \text{ and } \angle AOL = \alpha.$$

Let R (ρ, ϕ) be any point on the straight line MN.

Now from the ΔLOR

$$\frac{p}{\rho} = \cos(\alpha - \phi).$$

Hence the locus of (ρ, ϕ) is

$$\frac{p}{r} = \cos(\alpha - \theta).$$

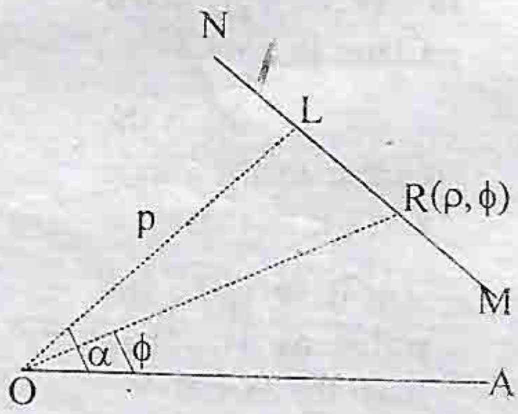

Fig. 7

which is the required equation of the straight line.

4.8 To find the general equation of a straight line in polar form.

Since ax + by = c represents a straight line, so changing it into polar form, we get

$$ar \cos\theta + br \sin\theta = c$$

or $$\frac{c}{r} = a\cos\theta + b\sin\theta,$$

which is the general equation of a straight line in polar form.

Note : The equation of a straight line perpendicular to the line $\frac{c}{r} = a\cos\theta + b\sin\theta$ in polar coordinates is

$$\frac{\lambda}{r} = a\cos\left(\theta + \frac{\pi}{2}\right) + b\sin\left(\theta + \frac{\pi}{2}\right),$$ where λ is an arbitrary constant

because any line perpendicular to ax + by = c is

$$bx - ay = \lambda \qquad \text{.....(i)}$$

Changing (i) into polar coordinates,

$$r\,(b\cos\theta - a\sin\theta) = \lambda$$

$$\frac{\lambda}{r} = a\cos\left(\theta + \frac{\pi}{2}\right) + b\sin\left(\theta + \frac{\pi}{2}\right).$$

The equation of any line perpendicular to a given line in polar coordinates is obtained by replacing θ in given equation by $\theta + \frac{\pi}{2}$ and changing coefficient of $\frac{1}{r}$ to a new constant.

4.9 To find the polar equation of a circle whose centre is (r_1, θ_1) and radius is a

$C(r_1, \theta_1)$ is the centre of the circle and a is its radius.

Let $R(\rho, \phi)$ be any point on it. Then from the ΔOCR

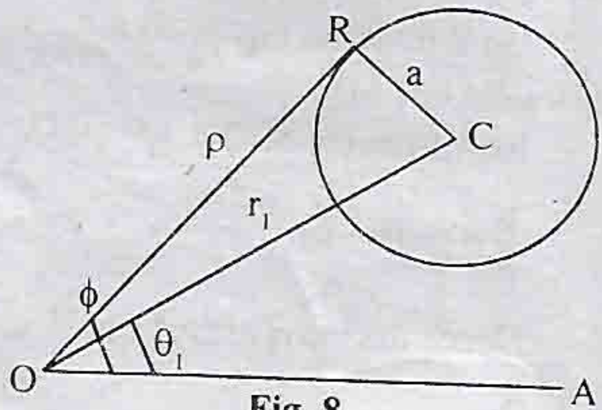

Fig. 8

$$r_1^2 + \rho^2 - 2r_1\rho\cos(\phi - \theta_1) = a^2$$

So the locus of (ρ, ϕ) is

$$r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_1) = a^2 \qquad \text{.....(1)}$$

which is required equation of the circle.

Case I. If pole lies on the circumference of the circle then

$$r_1 = a.$$

so equation of the circle passing through the pole is

$$r = 2a\cos(\theta - \theta_1). \qquad \text{.....(2)}$$

Case II. If pole lies on the circumference of the circle and initial line passes through its centre then

$$r_1 = a \text{ and } \theta_1 = 0.$$

In this case equation of the circle becomes

$$r = 2a\cos\theta \qquad \text{.....(3)}$$

Case III. If pole is at the centre of the circle then for any point $R : (\rho, \phi)$ on it, we have

$$\rho = a.$$

So the locus of (ρ, ϕ) is

$$r = a \qquad .....(4)$$

which is the equation of a circle whose centre is the pole.

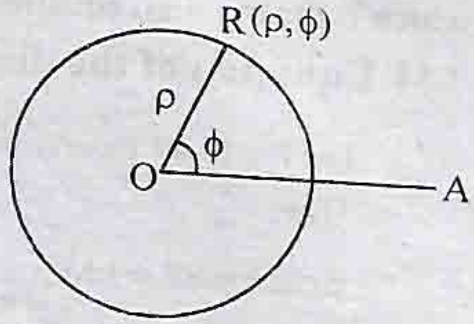

Fig. 9

4.10 Polar equation of a conic.

To find the polar equation of a conic whose focus is the pole.

Let focus S of the conic be the pole and its principal axis SA be the initial line. ZM is the directrix of the conic and $LSL' = 2\ell$ is its latus rectum. Then according to the definition of the conic

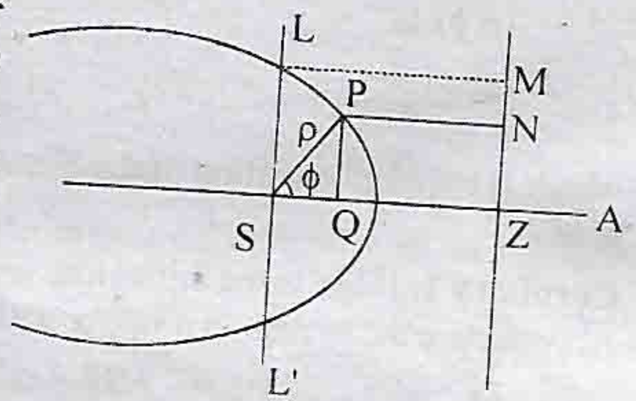

Fig. 10

$$\frac{LS}{LM} = e \qquad (LS = L'S = \ell)$$

or $$\ell = e.LM. \qquad ......(1)$$

Let $P : (\rho, \phi)$ be any point on the conic. Then

$$\frac{PS}{PN} = e$$

or $$\rho = e\,PN \quad \text{or} \quad \rho = e\,QZ$$

$$= e(SZ - SQ) = e(LM - SQ)$$

$$= e\left(\frac{\ell}{e} - \rho\cos\phi\right) = \ell - e\rho\cos\phi$$

or $$\rho(1 + e\cos\phi) = \ell \quad \text{or} \quad \frac{\ell}{\rho} = 1 + e\cos\phi.$$

Hence the locus of (ρ, ϕ) is

$$\frac{\ell}{r} = 1 + e\cos\theta, \qquad ......(2)$$

which is the required equation of the conic.

4.11 Equation of the directrix.

Let P : (ρ, ϕ) be any point on the directrix ZM of the conic

Then

$$\rho\cos\phi = SZ = LM = \frac{\ell}{e}$$

or $$\frac{\ell}{\rho} = e\cos\phi$$

Hence the locus of (ρ, ϕ) is

$$\frac{\ell}{r} = e\cos\theta, \qquad \text{......(2)}$$

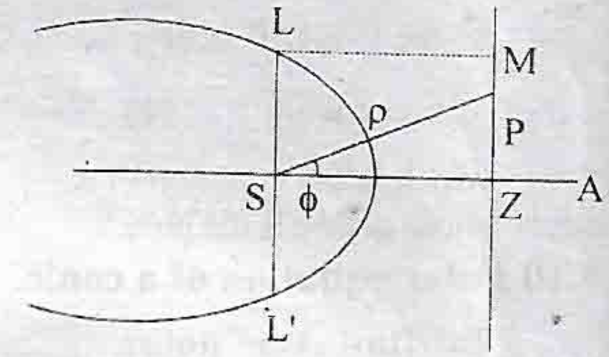

Fig. 11

which is required equation of the directrix which is nearer to focus or pole.

Corollary 1. If SA is the initial line such that the principal axis of the conic SZ makes an angle α with it.

i.e. $\angle ASZ = \alpha$.

Then in order to obtain the equation of the conic in this case

Let P : (ρ, ϕ) be any point on it.

Then $\angle PSA = \phi$, $PS = \rho$,

and $\angle PSZ = \phi - \alpha$.

According to the definition of the conic

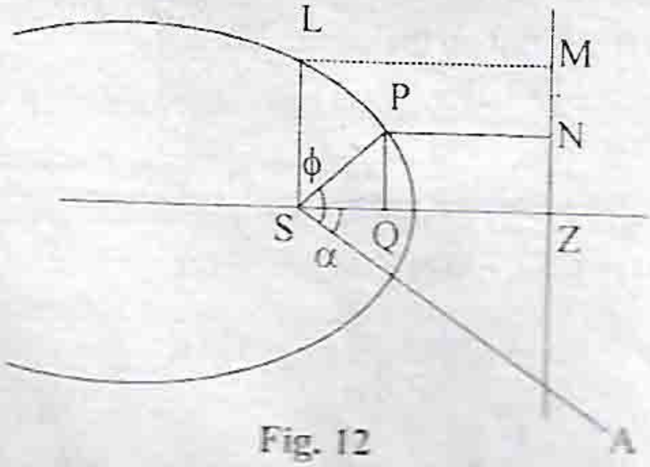

Fig. 12

$$\frac{PS}{PN} = e$$

or $$PS = e\,PN$$

or $\quad \rho = e\,(SZ - SQ)$

$$= e(LM - SQ) = e\left\{\frac{\ell}{e} - \rho\cos(\phi - \alpha)\right\}$$

$$= \ell - e\,\rho\,\cos(\phi - \alpha)$$

or $\quad \rho\{1 + e\cos(\phi - \alpha)\} = \ell$

or $\quad \dfrac{\ell}{\rho} = 1 + e\cos(\phi - \alpha)$

so the locus of (ρ, ϕ) is

$$\frac{\ell}{r} = 1 + e\cos(\theta - \alpha), \qquad .....(1)$$

which is the required equation of the conic.

Corollary 2. In this case, in order to obtain the equation of the directrix let us take a point $P(\rho, \phi)$ on it.

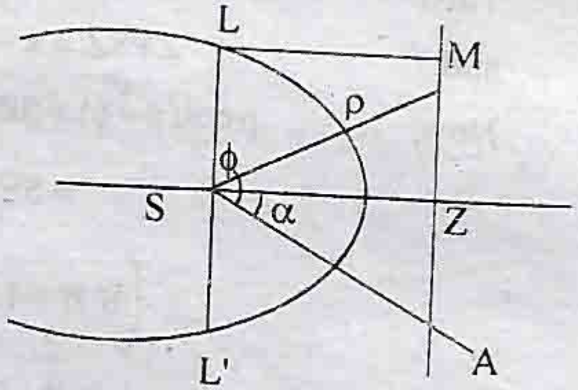

Fig. 13

Then $\quad \rho\cos(\phi - \alpha) = SZ$

$$= LM$$

$$= \frac{\ell}{e}$$

or $\quad \dfrac{\ell}{\rho} = e\cos(\phi - \alpha)$

So the locus of (ρ, ϕ) is

$$\frac{\ell}{r} = e\cos(\theta - \alpha) \qquad .....(2)$$

This is the required equation of the directrix.

Corollary 3. If $\quad \alpha = \pi$

Then equation of the conic becomes

$$\frac{\ell}{r} = 1 - e\cos\theta \qquad .....(3)$$

and equation of the directrix becomes

$$\frac{\ell}{r} = -e\cos\theta. \qquad .....(4)$$

4.12 To find the equation of that directrix of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ which is corresponding to the focus other than the pole.

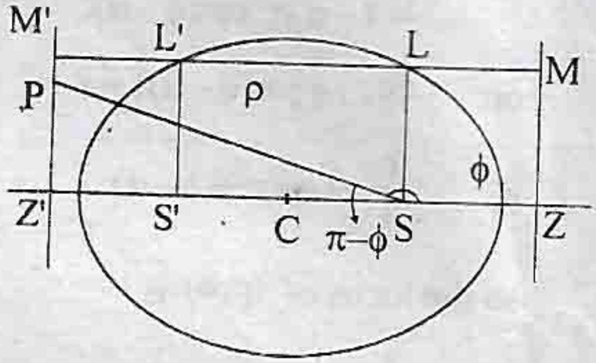

Fig. 14

Let focus S of the conic be pole, corresponding to which ZM is the directrix. S' is the other focus of the conic corresponding to which Z' M' is the directrix.

Let P (ρ, φ) be any point on Z' M'.

Then $\angle ZSP = \phi, SP = \rho,$

and $\angle PSZ' = \pi - \phi$

Now, $\rho\cos(\pi - \phi) = SZ' = SS' + S'Z'$

$$= SS' + \frac{\ell}{e} \qquad .....(1)$$

$$\left(S'Z' = L'M' = \frac{\ell}{e} \right)$$

Since S S' = 2ae where a is the length of semi-principal axis and

$$\ell = a(1 - e^2)$$

So $SS' = \frac{2e\ell}{1 - e^2}$.

Substituting this value in (1), we get :

$$-\rho\cos\phi = \frac{\ell}{e} + \frac{2e\ell}{1 - e^2}$$

$$= \ell\left[\frac{1 - e^2 + 2e^2}{e(1 - e^2)}\right] = \ell\left[\frac{1 + e^2}{e(1 - e^2)}\right]$$

or $\frac{\ell}{\rho} = -\frac{e(1 - e^2)}{1 + e^2}\cos\phi$

Hence the locus of (ρ, ϕ) is

$$\frac{\ell}{r} = -\frac{e(1-e^2)}{(1+e^2)}\cos\theta$$

Which is the required equation of the directrix Z' M'.

Note : Let P be any point on the conic $\frac{\ell}{r} = 1 + e\cos\theta$ whose vectorical angle is α.

Then radius vector of P is $\frac{\ell}{1+e\cos\alpha}$ and polar co-ordinate of point P is

$$\left(\frac{\ell}{1+e\cos\alpha}, \alpha\right).$$

The point P is generally called the point α on the conic.

EXAMPLES

Example 1. Show that in a conic the semi-latus rectum is the harmonic mean between the segments of a focal chord. (Purv., 2003)

(GKP, 92, 94 98, 2007)

Solution. Let $\frac{\ell}{r} = 1 + e\cos\theta$ be the equation of the conic and PS P' be its focal chord such that $\angle ASP = \alpha$.

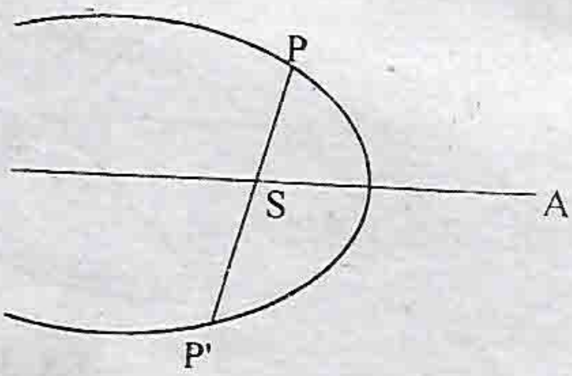

Fig. 15

Then co-ordinate of P' is (SP, α) and that of P' is (S P', $\pi + \alpha$).

Since P and P' lie on the conic,

$$\frac{\ell}{SP} = 1 + e\cos\alpha \qquad .....(1)$$

and $$\frac{\ell}{SP'} = 1 + e\cos(\pi+\alpha) = 1 - e\cos\alpha \qquad .....(2)$$

Adding (1) and (2), we get

$$\frac{\ell}{SP} + \frac{\ell}{SP'} = 2$$

or $$\frac{\frac{1}{SP}+\frac{1}{SP'}}{2}=\frac{1}{\ell}. \qquad .....(3)$$

So $\frac{1}{\ell}$ is the arithmatic mean of $\frac{1}{SP}$ and $\frac{1}{SP}$

or ℓ is the harmonic mean of SP and S P'.
Hence Proved.

Example 2. If PS P' and QS Q' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'}+\frac{1}{QS.SQ'} \text{ is constant.}$$

[Purv., 93, 95, 2001; GKP, 1996, 98, 2000]

Solution. Let PS P' and QS Q' be two mutually perpendicular focal chords of the conic

$$\frac{\ell}{r}=1+e\cos\theta \text{ and}$$

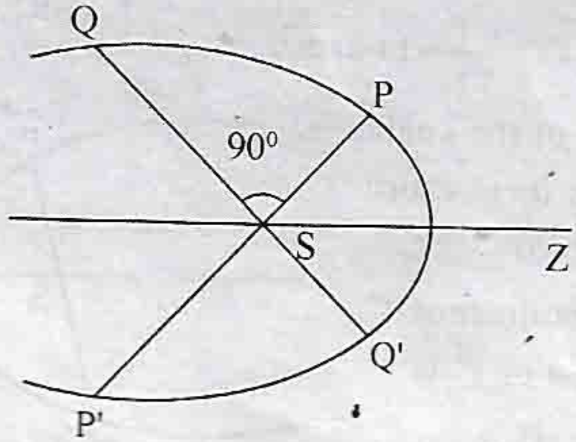

Fig. 16

$\angle ZSP=\alpha$, Then $P(SP,\alpha)$, $P'(SP',\pi+\alpha)$

$Q\left(SQ,\frac{\pi}{2}+\alpha\right)$ and $Q'\left(SQ',\frac{3\pi}{2}+\alpha\right)$

P, P', Q, Q' are points on the conic so

$$\frac{\ell}{SP}=1+e\cos\alpha, \qquad .....(1)$$

$$\frac{\ell}{SP'} = 1 - e\cos\alpha, \qquad .....(2)$$

$$\frac{\ell}{SQ} = 1 - e\sin\alpha \qquad .....(3)$$

and $$\frac{\ell}{SQ'} = 1 + e\sin\alpha. \qquad .....(4)$$

Multiplying (1) and (2), we get

$$\frac{\ell^2}{SP \cdot SP'} = 1 - e^2\cos^2\alpha. \qquad .....(5)$$

Similarly, multiplying (3) and (4), we get

$$\frac{\ell^2}{SQ \cdot SQ'} = 1 - e^2\sin^2\alpha. \qquad .....(6)$$

Adding (5) and (6), we get

$$\frac{\ell^2}{SP \cdot SP'} + \frac{\ell^2}{SQ \cdot SQ'} = 2 - e^2$$

or $$\frac{1}{SP \cdot SP'} + \frac{1}{SQ \cdot SQ'} = \frac{2 - e^2}{\ell^2} = \text{constant}.$$

Hence proved.

Example 3. Show that the equations

$$\frac{\ell}{r} = 1 - e\cos\theta \text{ and } \frac{\ell}{r} = -1 - e\cos\theta$$

represent the same conic. [Purv., 1997, 2000, 2004]

Solution. Let P(ρ, ϕ) be any point on the conic

$$\frac{\ell}{r} = 1 - e\cos\theta.$$

Then $$\frac{\ell}{r} = -1 - e\cos\phi. \qquad ...... (1)$$

P can be also represented by the coordinate $(-\rho, \pi + \phi)$ and it will lie on the conic

$$\frac{\ell}{r} = -1 - e\cos\theta,$$

if $$\frac{\ell}{-\rho} = -1 - e\cos(\pi + \phi)$$

or $$\frac{\ell}{\rho} = 1 - e\cos\phi \qquad \text{...... (2)}$$

This is true from (1) so P also lies on the conic

$$\frac{\ell}{r} = -1 - e\cos\theta.$$

Thus we find that any point lying on the conic

$$\frac{\ell}{r} = 1 - e\cos\theta \text{ also lies on the conic}$$

$$\frac{\ell}{r} = -1 - e\cos\theta.$$

Similarly, we can prove that any point lying on the conic.

$\frac{\ell}{r} = -1 - e\cos\theta$ also lies on the conic $\frac{\ell}{r} = 1 - e\cos\theta$.

Hence these two conics are identical.

Example 4. A circle passing through the focus of a conic whose latus rectum is 2ℓ meets the conic in four points whose distances from the focus are r_1, r_2, r_3 and r_4 respectively. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{\ell}. \qquad \text{[Avadh, 88]}$$

Solution. Taking focus of the conic as pole let its equation be

$$\frac{\ell}{r} = 1 + e\cos\theta \qquad \text{..... (1)}$$

and

Equation of the circle passing through the focus (pole) of the conic be

$$r = 2a\cos(\theta - \alpha). \qquad \text{..... (2)}$$

Eliminating θ from (1) and (2) we get

$$e^2 r^4 + 4ae\cos\alpha\, r^3 + 4a(a - ae^2\sin^2\alpha - e\ell\cos\alpha)\, r^2$$
$$- 8a^2\ell r + 4a^2\ell^2 = 0 \qquad \text{..... (3)}$$

Equation (3) is of fourth degree in r, gives four values of r, corresponding to which we get radius vectors of four points of intersection of (1) and (2). These are given as r_1, r_2, r_3 and r_4.

So r_1, r_2, r_3 and r_4 are roots of the equation (3).

$$\Sigma r_i r_j r_k = r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2$$

$$= \frac{8a^2\ell}{e^2} \qquad \text{...... (4)}$$

$$r_1r_2r_3r_4 = \frac{4a^2\ell^2}{e^2} \qquad \text{...... (5)}$$

Dividing (4) by (5), we get

$$\frac{\Sigma r_i r_j r_k}{r_1r_2r_3r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \left(\frac{2}{\ell}\right).$$

Hence proved.

Example 5. Prove that the locus of the middle points of focal chords of a conic is a conic section of same kind.

[Luck., 85, Purv., 98, 2000]

Solution. Let the equation of the conic be

$$\frac{\ell}{r} = 1 + e\cos\theta \qquad \text{...... (1)}$$

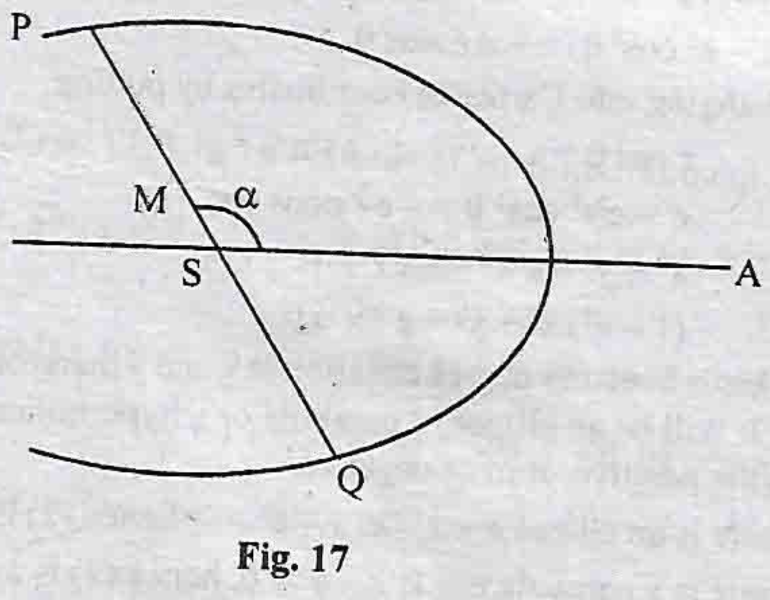

Fig. 17

Let PSQ be a chord of the conic inclined at an angle α to the initial line SA, S being pole (focus).

Let M be the middle point of chord PQ whose polar coordinate be (ρ, ϕ). Then

$$SM = \rho, \qquad \angle MSA = \phi.$$

Then $\alpha = \phi$. The point $P(SP, \alpha)$ and $Q(SQ, \pi + \alpha)$ be on the conic (1) Therefore,

$$\frac{\ell}{SP} = 1 + e\cos\alpha \quad \text{and} \quad \frac{\ell}{SQ} = 1 + e\cos(\pi + \alpha)$$

$$PQ = SP + SQ = \frac{\ell}{(1+e\cos\alpha)} + \frac{\ell}{(1-e\cos\alpha)}$$

$$= \frac{2\ell}{(1-e^2\cos^2\alpha)}$$

$$PM = MQ = \frac{1}{2}\ PQ = \frac{\ell}{(1-e^2\cos^2\alpha)}.$$

$$SM = SP - PM = \frac{\ell}{(1+e\cos\alpha)} - \frac{\ell}{(1-e^2\cos^2\alpha)}$$

$$SM = -\frac{e\ell\cos\alpha}{1-e^2\cos^2\alpha} = -\frac{e\ell\cos\phi}{1-e^2\cos^2\phi} \qquad \text{since } \alpha = \phi$$

$$\rho(1 - e^2\cos^2\phi) = -e\ell\cos\phi \qquad \text{since } SM = \rho.$$

Therefore locus of (ρ, ϕ) is

$$r(1 - e^2\cos^2\theta) = -e\ell\cos\theta.$$

Changing into Cartesian coordinates by putting

$$r\cos\theta = x, \qquad r\sin\theta = y$$

$$r^2 - e^2r^2\cos^2\theta = -e\ell r\cos\theta$$

$$x^2 + y^2 - e^2x^2 = -e\ell x$$

$$(1 - e^2)x^2 + y^2 + e\ell x = 0.$$

This is a second degree equation in x and y therefore represents a conic. It will be an ellipse, a parabola or a hyperbola according as $1 - e^2$ is the positive, zero or negative.

If the conic is an ellipse $e < 1$, i.e. $1 - e^2 > 0$, hence (2) is an ellipse.

If the conic is a parabola $e = 1$, $1 - e^2 = 0$, hence (2) is a parabola.

If the conic is a hyperbola $e > 1$, $1 - e^2 < 0$, hence (2) is a hyperbola.

Therefore locus (2) represents a conic of same kind as the given conic.

EXAMPLES

1. Show that equations $\frac{\ell}{r} = 1 + e\cos\theta$ and $\frac{\ell}{r} = -1 + e\cos\theta$ represent the same conic. (GKP, 2008) Purv. '2004]

2. In any conic prove that
   (i) The sum of the reciprocals of the segments of any focal chord is constant.
   (ii) The sum of the reciprocals of two perpendicular focal chords is constant. [Purv., 91; GKP, 96; IAS, 87; Avadh, 97]

3. A circle of given radius passing through the focus S of a given conic intersects it in A, B, C and D. Show that SA·SB·SC·SD is constant. [Purva, 90]

4. A point moves so that the sum of its distance from two fixed points S, S′ is constant and equal to 2a. Show that P lies on the conic

$$\frac{a(1-e^2)}{r} = 1 - e\cos\theta$$

referred to S as pole and SS′ as initial line, SS′ being equal to 2 ae.

5. QR is a chord of the conic $\frac{\ell}{r} = 1 - e\cos\theta$ subtanding an angle 2α at its focus S and SP, the bisector of the angle QSR, meets QR in P. Show that the locus of P is the conic.

$$\frac{\ell\cos\alpha}{r} = 1 - e\cos\alpha\cos\theta.$$

6. If PSQ and PHR be two chords of an ellipse through the foci S and H. Show that $\frac{PS}{SQ} + \frac{PH}{HR}$ is independent of the position of P. [GKP, 99]

4.13 To find the equation of tangent at a point on the conic

$$\frac{\ell}{r} = 1 + e\cos\theta. \quad \text{[GKP, 88, 98; Purv., 96, 97]}$$

Proof. Given the equation of the conic

$$\frac{\ell}{r} = 1 + e\cos\theta. \quad .....(1)$$

Let P and Q be two points on the conic whose vectorial angles are $\alpha - \beta$ and $\alpha + \beta$ respectively then

$$\frac{\ell}{SP} = 1 + e\cos(\alpha - \beta) \qquad .....(2)$$

$$\frac{\ell}{SQ} = 1 + e\cos(\alpha + \beta)\,. \qquad .....(3)$$

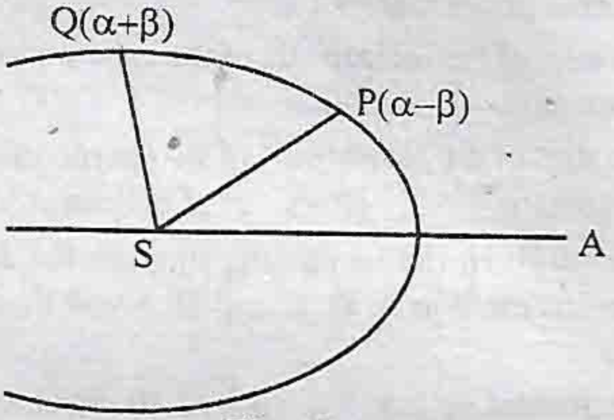

Fig. 18

Let the equation of the chord be

$$\frac{\ell}{r} = A\cos(\theta - \alpha) + B\cos\theta\,.$$

Since this chord passes through P and Q, then

$$\frac{\ell}{SP} = A\cos\beta + B\cos(\alpha - \beta) \qquad .....(4)$$

and
$$\frac{\ell}{SQ} = A\cos\beta + B\cos(\alpha + \beta)\,. \qquad .....(5)$$

Then by (2) and (4) we get

$$A\cos\beta + B\cos(\alpha - \beta) = 1 + e\cos(\alpha - \beta) \qquad .....(6)$$

Similarly by (3) and (5) we get

$$A\cos\beta + B\cos(\alpha + \beta) = 1 + e\cos(\alpha + \beta) \qquad .....(7)$$

Solving (6) and (7)

$$A\cos\beta = 1 \quad \text{or} \quad A = \sec\beta$$

$$B = e.$$

The equation of the chord PQ becomes

$$\frac{\ell}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta \qquad .....(8)$$

If the points P and Q coincides i.e.

$$\alpha - \beta = \alpha + \beta \qquad \text{or} \qquad \beta = 0.$$

When $\beta = 0$ the chord PQ becomes tangents to the conic at point θ

$= \alpha$ Therefore, the equation of tangent at point $\theta = \alpha$ is

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta .$$

Note: Equation of chord joining points on the conic $\ell/r = 1 + e\cos\theta$, whose vectorial angles are $\alpha-\beta$ and $\alpha+\beta$

$$\ell/r = \sec\beta\cos(\theta-\alpha) + e\cos\theta .$$

4.14 To find the equation of the normal at a point on the conic $\frac{\ell}{r} = 1 + e\cos\theta$.

[Purv., 99, 2000, 2003; GKP, 2004]

Let P be a point on the conic $\frac{\ell}{r} = 1 + e\cos\theta$ whose vectorial angle is α i.e.

$$P\left(\frac{\ell}{1+e\cos\alpha}, \alpha\right)$$

Let Cartesian coordinate of P be (x_1, y_1). Then

$$x_1 = \frac{\ell\cos\alpha}{1+e\cos\alpha} \quad .....(1)$$

$$y_1 = \frac{\ell\sin\alpha}{1+e\cos\alpha} \quad .....(2)$$

Equation of the tangent at P is

$$\frac{\ell}{r} = \cos(\theta-\alpha) + e\cos\theta \quad .....(3)$$

Changing (3) into Cartesian form, we get

$$\ell = x(\cos\alpha + e) + y\sin\alpha .$$

Slope of the tangent at P is

$$= -\frac{\cos\alpha + e}{\sin\alpha},$$

so slope of the normal at P is

$$= \frac{\sin\alpha}{\cos\alpha + e}.$$

Equation of the normal at P is

$$y - y_1 = \frac{\sin\alpha}{\cos\alpha + e}(x - x_1) \qquad .....(4)$$

Changing (4) into polar form, we get

$$r\sin\theta - \frac{\ell\sin\alpha}{1 + e\cos\alpha} = \frac{\sin\alpha}{\cos\alpha + e}\left(r\cos\theta - \frac{\ell\cos\alpha}{1 + e\cos\alpha}\right) \qquad .....(5)$$

Now, simplifying (5) we get the equation of normal at α is

$$r\left[\sin\theta - \frac{\sin\alpha\cos\theta}{\cos\alpha + e}\right] = \frac{\ell\sin\alpha}{1 + e\cos\alpha}\left[1 - \frac{\cos\alpha}{\cos\alpha + e}\right]$$

or $$r[\sin(\theta - \alpha) + e\sin\theta] = \frac{\ell e\sin\alpha}{1 + e\cos\alpha}$$

or $$\frac{\ell}{r}\left[\frac{e\sin\alpha}{1 + e\cos\alpha}\right] = \sin(\theta - \alpha) + e\sin\theta.$$

4.15. To find the equation of asymptotes of the conic

$$\frac{\ell}{r} = 1 + e\cos\theta.$$ Alld. 74, GKP 72, 75, 2003)

Proof. The equation of the conic is

$$\frac{\ell}{r} = 1 + e\cos\theta \qquad .....(1)$$

Let (r', α) be a point on (1) then

$$\frac{\ell}{r'} = 1 + e\cos\alpha \qquad .....(2)$$

The equation of tangent at the point α is

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta \qquad .....(3)$$

As $r' \to \infty$ the tangent (3) tends to become an asymptote of the conic. Therefore from (2)

$$1 + e\cos\alpha = 0 \ (r' \to \infty)$$

$$\cos\alpha = -\frac{1}{e}$$

$$\sin\alpha = \pm\sqrt{1 - \frac{1}{e^2}}.$$

The tangent (3) can be written as

$$\frac{\ell}{r} = \cos\theta(e + \cos\alpha) + \sin\theta\,\sin\alpha$$

$$= \left(e - \frac{1}{e}\right)\cos\theta \pm \sqrt{1 - \frac{1}{e^2}}\sin\theta$$

or $$\frac{\ell}{r} = \frac{\sqrt{(e^2-1)}}{e}\left[\sqrt{(e^2-1)}\cos\theta \pm \sin\theta\right].$$

These asymptotes are real if $e^2 - 1 > 0$ or $e > 1$.

4.16. Chord of contact.

To find the equation of the chord of contact of the point $P(r_1, \theta_1)$ with respect to the conic $\frac{\ell}{r} = 1 + e\cos\theta$.

(GKP 2003.

Let α and β be the vectorial angles of the points of contact A, B of the tangents drawn from $P(r_1, \theta_1)$ to the given conic.

AB is the chord of contact of the point P with respect to the conic. So from 4.13 equation of the chord AB is

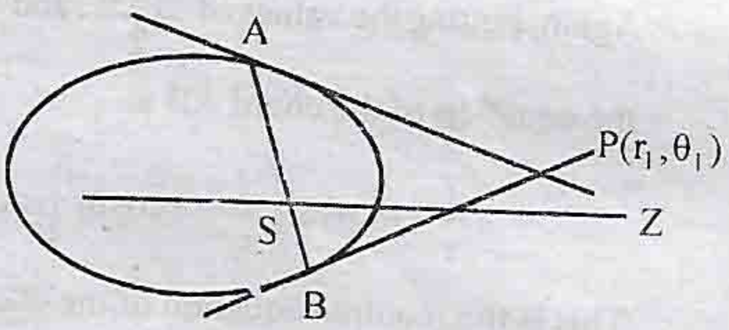

Fig. 19

$$\frac{\ell}{r}\cos\frac{\beta-\alpha}{2} = \cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta\cos\frac{\beta-\alpha}{2}$$

or $$\left(\frac{\ell}{r} - e\cos\theta\right)\cos\frac{\beta-\alpha}{2} = \cos\left(\theta - \frac{\beta+\alpha}{2}\right) \quad .....(1)$$

Equations of tangents at A and B are

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta \quad .....(2)$$

and $$\frac{\ell}{r} = \cos(\theta - \beta) + e\cos\theta \quad .....(3)$$

These tangents pass through (r_1, θ_1) so

$$\frac{\ell}{r_1} = \cos(\theta_1 - \alpha) + e\cos\theta_1 \qquad .....(4)$$

$$\frac{\ell}{r_1} = \cos(\theta_1 - \beta) + e\cos\theta_1 \qquad .....(5)$$

Comparing (4) and (5), we get

$$\cos(\theta_1 - \alpha) = \cos(\theta_1 - \beta)$$

or $$\theta_1 - \alpha = \pm(\theta_1 - \beta) \text{ or } \theta_1 = \frac{\alpha+\beta}{2} \qquad .....(6)$$

Putting this value of θ_1 in (5), we get

$$\left(\frac{\ell}{r_1} - e\cos\theta_1\right) = \cos\left(\frac{\alpha+\beta}{2} - \beta\right)$$

$$= \cos\left(\frac{\alpha-\beta}{2}\right). \qquad ......(7)$$

Again, putting the values of $\frac{\alpha+\beta}{2}$. and $\cos\left(\frac{\alpha-\beta}{2}\right)$ in (1), we get the equation of the chord AB as

$$\left(\frac{\ell}{r} - e\cos\theta\right)\left(\frac{\ell}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1) \qquad .....(8)$$

This is the required equation of the chord of contact.

4.17. Polar.

To find the equation of the polar of the point $P(r_1, \theta_1)$ with respect to the conic $\frac{\ell}{r} = 1 + e\cos\theta$. [GKP, 2000; Avadh, 97]

The polar of the point $P(r_1, \theta_1)$ is the chord of contact of tangents drawn from it to the conic $\frac{\ell}{r} = 1 + e\cos\theta$.

Proceeding in the same ways as 4.16, we get the equation of the polar of the point (r_1, θ_1) as

$$\left(\frac{\ell}{r} - e\cos\theta\right)\left(\frac{\ell}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1).$$

4.18. Director-circle.

To find the equation of the director - circle of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ [Purv.,95, 98, 2002; IAS. 81; Avadh, 87; GKP, 75]

Definition. The director-circle is the locus of the point from which tangents drawn to the conic are mutually perpendicular.

Proof. Let P be the point (ρ, θ) from which tangents drawn to the conic $\frac{\ell}{r} = 1 + e\cos\theta$ are mutually perpendicular and these tangents touch the conic at A, B whose vectorial angles are α and β respectively.

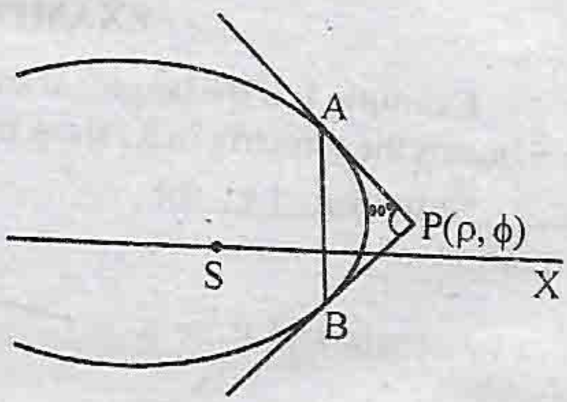

Fig. 20

Now, from 4.16 (6) and 4.16 (7), we get

$$\phi = \frac{\alpha + \beta}{2} \qquad .....(1)$$

and

$$\left(\frac{\ell}{\rho} - e\cos\phi\right) = \cos\frac{\alpha - \beta}{2} \qquad .....(2)$$

Slopes of tangents at α and β are

$$-\frac{\cos\alpha + e}{\sin\alpha} \quad \text{and} \quad -\frac{\cos\beta + e}{\sin\beta}.$$

Since these tangents are mutually perpendicular,

$$\frac{(\cos\alpha + e)(\cos\beta + e)}{\sin\alpha\sin\beta} = -1$$

or

$$\cos(\alpha - \beta) + e(\cos\alpha + \cos\beta) + e^2 = 0$$

or

$$2\cos^2\frac{\alpha - \beta}{2} + 2e\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} + (e^2 - 1) = 0 \qquad .....(3)$$

Substituting the values of $\frac{\alpha + \beta}{2}$ and $\cos\frac{\alpha - \beta}{2}$ in (3), we get

$$2\left(\frac{\ell}{\rho} - e\cos\phi\right)^2 + 2e\cos\phi\left(\frac{\ell}{\rho} - e\cos\phi\right) + (e^2 - 1) = 0$$

or $$\rho^2(1-e^2)+2e\ell\rho\cos\phi-2\ell^2=0.$$

Hence the locus of (ρ,ϕ) is

$$r^2(1-e^2)+2e\ell r\cos\theta-2\ell^2=0.$$

This is the required equation of the director-circle.

EXAMPLES

Example 1. If the tangent at any point P on a conic whose focus is S, meets the directrix in K, show the angle PSK is a right angle.

Solution. Let the equation of the conic be $\frac{\ell}{r}=1+e\cos\theta$ and P be a point on it whose vectorial angle is α.

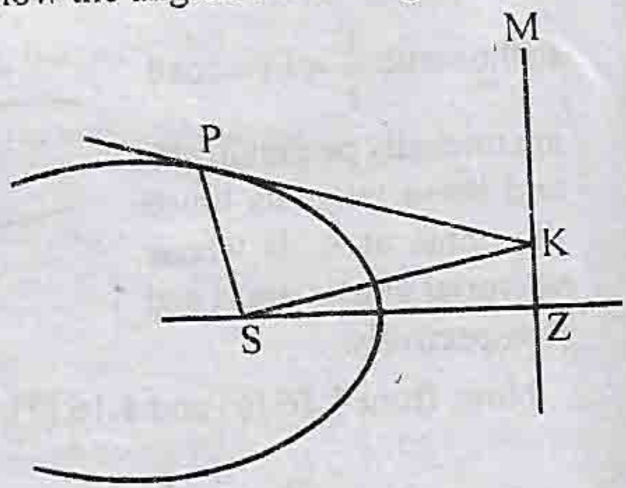

Fig. 21

Equation of the tangent at P is

$$\frac{\ell}{r}=\cos(\theta-\alpha)+e\cos\theta \qquad .....(1)$$

Equation of the directrix of the conic is

$$\frac{\ell}{r}=e\cos\theta \qquad .....(2)$$

Let (r_1, θ_1) be the coordinate of the point of intersection K of (1) and (2), then

$$\frac{\ell}{r_1}=\cos(\theta_1-\alpha)+e\cos\theta_1 \qquad .....(3)$$

and $$\frac{\ell}{r_1}=e\cos\theta_1 \qquad .....(4)$$

From (3) and (4), we get

$$\cos(\theta_1-\alpha)=0$$

i.e., $$\theta_1-\alpha=\pm\frac{\pi}{2}.$$

Hence $\angle$ PSK is a right angle.

Example 2. Show that the locus of the point of intersection of two tangents to the parabola $\frac{\ell}{r} = 1 + \cos\theta$ which cut one another at a constant angle α, is the hyperbola $\frac{\ell}{r} = \cos\alpha + \cos\theta$.

Solution. Let θ_1 and θ_2 be the vectorial angles of the points of contact A, B of the tangents to the parabola

$$\frac{\ell}{r} = 1 + \cos\theta$$

which cut one another at P such that $\angle APB = \alpha$.

Let us assume that (ρ, ϕ) be the coordinate of P.

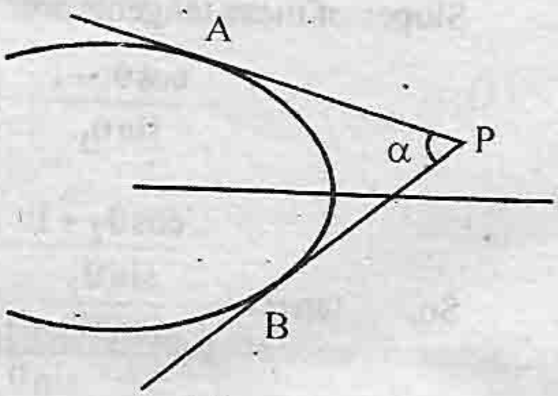

Fig. 22

Equations of tangents at A and B are

$$\frac{\ell}{r} = \cos(\theta - \theta_1) + \cos\theta \qquad .....(1)$$

$$\frac{\ell}{r} = \cos(\theta - \theta_2) + \cos\theta. \qquad .....(2)$$

These tangents pass through P so

$$\frac{\ell}{\rho} = \cos(\phi - \theta_1) + \cos\phi \qquad .....(3)$$

and

$$\frac{\ell}{\rho} = \cos(\phi - \theta_2) + \cos\phi \qquad .....(4)$$

Comparing (3) and (4), we get

$$\cos(\phi - \theta_1) = \cos(\phi - \theta_2) \quad \text{or} \quad \phi - \theta_1 = \pm(\phi - \theta_2)$$

or

$$\phi = \frac{\theta_1 + \theta_2}{2} \qquad .....(5)$$

Putting this value of ϕ in (4), we get

$$\frac{\ell}{\rho} - \cos\phi = \cos\left(\frac{\theta_1 + \theta_2}{2} - \theta_2\right)$$

$$= \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \qquad .....(6)$$

Changing (1) and (2) into Cartesian form, we get

$$\ell = x(\cos\theta_1 + 1) + y\sin\theta_1 \qquad .....(7)$$

$$\ell = x(\cos\theta_2 + 1) + y\sin\theta_2 . \qquad .....(8)$$

Slopes of these tangents are

$$-\frac{\cos\theta_1 + 1}{\sin\theta_1} \quad \text{and} \quad -\frac{\cos\theta_2 + 1}{\sin\theta_2}.$$

So, $$\tan\alpha = \frac{\dfrac{\cos\theta_2 + 1}{\sin\theta_2} - \dfrac{\cos\theta_1 + 1}{\sin\theta_1}}{1 + \dfrac{(\cos\theta_1 + 1)(\cos\theta_2 + 1)}{\sin\theta_1 \sin\theta_2}}.$$

$$= \frac{\sin(\theta_1 - \theta_2) + (\sin\theta_1 - \sin\theta_2)}{\cos(\theta_1 - \theta_2) + (\cos\theta_1 + \cos\theta_2) + 1}$$

$$= \frac{2\sin\dfrac{\theta_1 - \theta_2}{2}\left[\cos\dfrac{\theta_1 - \theta_2}{2} + \cos\dfrac{\theta_1 + \theta_2}{2}\right]}{2\cos\dfrac{\theta_1 - \theta_2}{2}\left[\cos\dfrac{\theta_1 - \theta_2}{2} + \cos\dfrac{\theta_1 + \theta_2}{2}\right]}$$

$$= \tan\frac{\theta_1 - \theta_2}{2}.$$

Hence $$\alpha = \frac{\theta_1 - \theta_2}{2}.$$

Now, putting this value in (6), we get

$$\frac{\ell}{\rho} - \cos\phi = \cos\alpha \quad \text{or} \quad \frac{\ell}{\rho} = \cos\alpha + \cos\phi.$$

So the locus of (ρ, ϕ) in

$$\frac{\ell}{r} = \cos\alpha + \cos\theta.$$

Hence proved.

Example 3. Show that the condition that the line $\frac{\ell}{r} = A\cos\theta + B\sin\theta$ may touch the conic $\frac{\ell}{r} = 1 + e\cos\theta$ is $(A-e)^2 + B^2 = 1$. [Purv., 99, 2000, 01; GKP, 84]

Solution. Let the given line touches the given conic at a point whose vectorial angle is α. So the tangent at α

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta \qquad .....(1)$$

and $$\frac{\ell}{r} = A\cos\theta + B\sin\theta \qquad .....(2)$$

must be identical.

Changing (1) and (2) into Cartesian form the equations become

$$\ell = x(\cos\alpha + e) + y\sin\alpha \qquad .....(3)$$

$$\ell = Ax + By \qquad .....(4)$$

Now, comparing (3) and (4), we get

$$1 = \frac{\cos\alpha + e}{A} = \frac{\sin\alpha}{B}.$$

Eliminating α from these equations we get the required condition as

$$(A-e)^2 + B^2 = 1 \cdot$$

Example 4. Prove that the two conics $\frac{\ell_1}{r} = 1 + e_1\cos\theta$ and $\frac{\ell_2}{r} = 1 + e_2\cos(\theta - \alpha)$ will touch each other if $\ell_1^{\,2}(1 - e_2^{\,2}) + \ell_2^2(1 - e_1^{\,2}) = 2\ell_1\ell_2(1 - e_1 e_2\cos\alpha)$.

[Purv. 96, 2001; GKP, 84; Avadh, 86; Alld., 87]

Solution. Given conics are

$$\frac{\ell_1}{r} = 1 + e_1\cos\theta \qquad .....(1)$$

and $$\frac{\ell_2}{r} = 1 + e_2\cos(\theta - \alpha) \qquad .....(2)$$

Let (1) and (2) touch each other at a point whose vectorial angle is β.

Equation of the tangent at β to the conic (1) is

$$\frac{\ell_1}{r} = \cos(\theta - \beta) + e_1 \cos\theta \qquad .....(3)$$

Equation of the tangent at β to the conic (2) is

$$\frac{\ell_2}{r} = \cos(\theta - \beta) + e_2 \cos(\theta - \alpha) \qquad .....(4)$$

Straight lines (3) and (4) must be identical so changing these equations into Cartesian form and comparing, we get

$$\frac{\ell_1}{\ell_2} = \frac{\cos\beta + e_1}{\cos\beta + e_2\cos\alpha} = \frac{\sin\beta}{\sin\beta + e_2 \sin\alpha}$$

or $\qquad (\ell_2 - \ell_1)\cos\beta = (\ell_1 e_2 \cos\alpha - \ell_2 e_1) \qquad .....(5)$

and $\qquad (\ell_2 - \ell_1)\sin\beta = \ell_1 e_2 \sin\alpha \qquad .....(6)$

Squaring and adding (5) and (6), we get the required condition as

$$(\ell_2 - \ell_1)^2 = \ell_1^2 e_2^2 + \ell_2^2 e_1^2 - 2\ell_1\ell_2 e_1 e_2 \cos\alpha$$

or $\qquad \ell_1^2(1 - e_2^2) + \ell_2^2(1 - e_1^2) = 2\ell_1\ell_2(1 - e_1e_2\cos\alpha)$

Hence proved.

Example 5. Prove that the locus of the foot of the perpendicular from the focus of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ on a tangent to it, is $r^2(e^2 - 1) - 2\ell e r\cos\theta + \ell^2 = 0.$

[Avadh, 86; GKP, 83; Purv., 91, 92, 93, 94]

Solution. Let A be a point on the conic $\frac{\ell}{r} = 1 + e\cos\theta$ whose vectorial angle is α and P be the foot of the perpendicular dropped from focus S on the tangent at A.

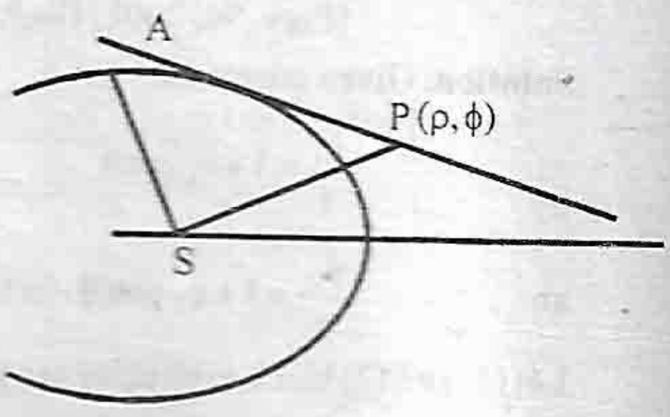

Fig. 23

Let us assume that the coordinate of P be (ρ, ϕ). Equation of the tangent at A is

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta \qquad .....(1)$$

$P(\rho, \phi)$ is a point on (1) so

$$\frac{\ell}{\rho} = \cos(\phi - \alpha) + e\cos\phi$$

or
$$\frac{\ell}{\rho} - e\cos\phi = \cos(\phi - \alpha) \qquad .....(2)$$

Slope of the tangent (1) is $= -\dfrac{\cos\alpha + e}{\sin\alpha}$

Slope of the perpendicular SP is = tan ϕ.

Hence,
$$\tan\phi\left(-\frac{\cos\alpha + e}{\sin\alpha}\right) = -1$$

or
$$-\sin\phi\cos\alpha - e\sin\phi = -\cos\phi\sin\alpha$$

or
$$-e\sin\phi = \sin(\phi - \alpha) \qquad .....(3)$$

Squarring and adding (2) and (3), we get

$$\frac{\ell^2}{\rho^2} + e^2 - \frac{2\ell e\cos\phi}{\rho} = 1$$

or
$$(e^2 - 1)\rho^2 - 2\ell e\rho\cos\phi + \ell^2 = 0 \qquad .....(4)$$

So the locus of (ρ, ϕ) is

$$(e^2 - 1)r^2 - 2\ell er\cos\theta + \ell^2 = 0$$

Hence Proved.

Example 6. Two equal ellipses of eccentricity e are placed with their axes at right angles and have a common focus S. If PQ be a common tangent to the two ellipses, show that the angle PSQ is

$$2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right)$$

[IAS, 82; Agra, 82; Luck., 81]

Solution. Let the equation of the given two equal ellipses be

$$\frac{\ell}{r} = 1 + e\cos\theta \qquad .....(1)$$

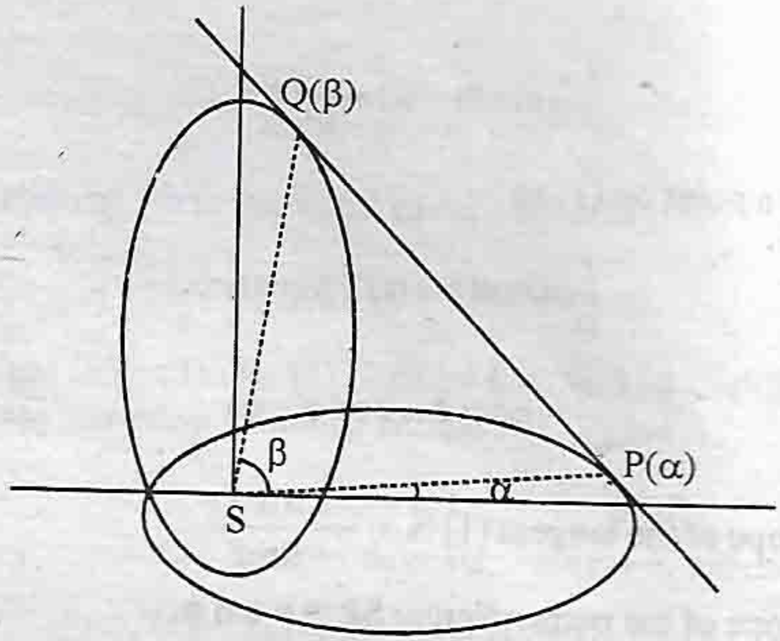

Fig. 24

$$\frac{\ell}{r} = 1 + e\cos\left(\theta - \frac{\pi}{2}\right) \qquad .....(2)$$

Let P (α) be any point on ellipse (1) and Q (β) be any point on the ellipse (2) and PQ be the common tangent, then the equations

$$\frac{\ell}{r} = \cos(\theta - \alpha) + e\cos\theta$$

$$\frac{\ell}{r} = \cos(\theta - \beta) + e\cos\left(\theta - \frac{\pi}{2}\right)$$

are identical. These equations can be written as

$$\frac{\ell}{r} = (\cos\alpha + e)\cos\theta + \sin\theta\sin\alpha$$

$$\frac{\ell}{r} = \cos\beta\cos\theta + (\sin\beta + e)\sin\theta\,.$$

Comparing the coefficients, we get

$$1 = \frac{\cos\alpha + e}{\cos\beta} = \frac{\sin\alpha}{\sin\beta + e}$$

or $\cos\alpha + e = \cos\beta$ i.e. $\cos\beta - \cos\alpha = e$ .....(4)

$\sin\beta + e = \sin\alpha$ $\qquad \sin\alpha - \sin\beta = e$ .....(5)

From (4),

$$e = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\,.$$

From (5),

$$e = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

Squarring and adding

$$2e^2 = 4\sin^2\frac{\alpha-\beta}{2}$$

$$\sin\frac{\alpha-\beta}{2} = \frac{e}{\sqrt{2}}$$

$$\alpha-\beta = 2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right)$$

$$\angle PSQ = 2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right).$$

Example 7. P, Q, R are three points on the conic $\frac{\ell}{r} = 1 + e\cos\theta$ the focus S being the pole; SP and SR meet the tangent at Q in M and N so that SM = SN = ℓ. Prove that PR touches the curve

$$\frac{\ell}{r} = 1 + 2e\cos\theta.$$

Solution. Let the vectorial angles of points P,Q,R be α, β, γ respectively.

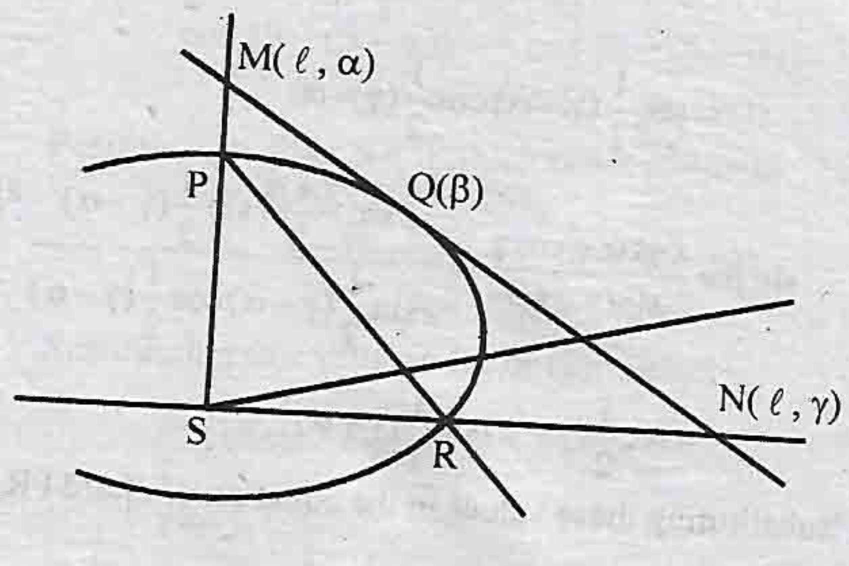

Fig. 25

The equation of chord PR is

$$\frac{\ell}{r} = \sec\left(\frac{\gamma-\alpha}{2}\right)\cos\left(\theta - \frac{\gamma+\alpha}{2}\right) + e\cos\theta$$

$$= \sec\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\gamma+\alpha)\cos\theta + \sec\left(\frac{\gamma-\alpha}{2}\right).$$

$$\sin\frac{1}{2}(\gamma+\alpha)\sin\theta + e\cos\theta \quad \text{......(1)}$$

Equation of **tangent at** Q whose vectorial angle is β, is

$$\frac{\ell}{r} = \cos(\theta-\beta) + e\cos\theta.$$

This tangent passes through M (ℓ , α) and N (ℓ , γ), therefore

$$\cos(\alpha-\beta) + e\cos\alpha = 1$$

$$\cos(\gamma-\beta) + e\cos\gamma = 1$$

or $$\cos\alpha(\cos\beta + e) + \sin\alpha\sin\beta = 1 \quad \text{......(2)}$$

$$\cos\gamma(\cos\beta + e) + \sin\gamma\sin\beta = 1 \quad \text{.....(3)}$$

Solving (2) and (3) we get

$$\cos\beta + e = \frac{\sin\gamma - \sin\alpha}{\sin(\gamma-\alpha)} = \frac{2\cos\frac{\gamma+\alpha}{2}\sin\frac{1}{2}(\gamma-\alpha)}{2\sin\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\gamma-\alpha)}$$

$$= \sec\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\gamma+\alpha)$$

$$\sin\beta = \frac{\cos\alpha - \cos\gamma}{\sin(\gamma-\alpha)} = \frac{2\sin\frac{\gamma+\alpha}{2}\sin\frac{1}{2}(\gamma-\alpha)}{2\sin\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\gamma-\alpha)}$$

$$= \sec\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\gamma+\alpha).$$

Substituting these values in the equation of chord PR, we get

$$\frac{\ell}{r} = (\cos\beta + e)\cos\theta + \sin\beta\sin\theta + e\cos\theta$$

$$= \cos(\theta-\beta) + 2e\cos\theta.$$

It is the equation of tangent at point $\theta = \beta$ to the conic

$$\frac{\ell}{r} = 1 + 2e\cos\theta .$$

Example 8. Find the locus of the poles of the chords of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ which subtends a constant angle 2α at the focus.

[GKP, 98, 99; Purv., 97; Kan., 77]

Solution. Let the chord P Q subtends an angle 2α at the focus S of the conic. The vectorial angles of P and Q are $\beta - \alpha$ and $\beta + \alpha$ respectively.

$$\angle QSP = (\beta + \alpha) - (\beta - \alpha) = 2\alpha.$$

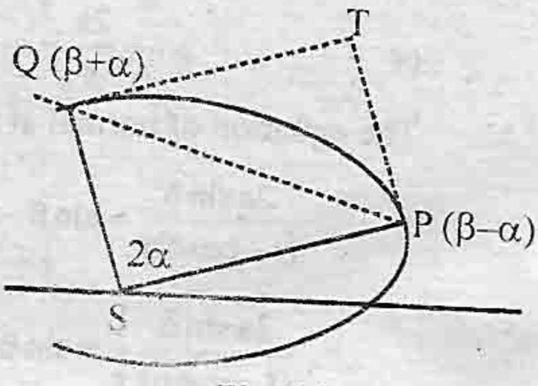

Fig. 26

The equation of tangents P and Q to the conic are

$$\frac{\ell}{r} = \cos\{\theta - (\beta - \alpha)\} + e\cos\theta \qquad .....(1)$$

$$\frac{\ell}{r} = \cos\{\theta - (\beta + \alpha)\} + e\cos\theta \qquad .....(2)$$

The point of intersection of the tangents at P and Q will be obtained by eliminating β from (1) and (2). From (1) and (2) we get

$$\cos\{\theta - (\beta - \alpha)\} = \cos\{\theta - (\beta + \alpha)\}$$

$$\theta - (\beta - \alpha) = \pm\{\theta - (\beta + \alpha)\}$$

Positive sign gives $\alpha = 0$ which is not admissble, Negative sign gives

$$\theta - (\beta - \alpha) = -\{\theta - (\beta + \alpha)\}$$

$$\theta = \beta.$$

Substituting this value in (1) or (2), we get

$$\frac{\ell}{r} = \cos\alpha + e\cos\theta$$

$$\frac{\ell\sec\alpha}{r} = 1 + (e\sec\alpha)\cos\theta$$

which is a conic having its focus S and its semi-latus rectum $= \ell\sec\alpha$ and eccentricity $= e\sec\alpha$.

Example 9. If the normals at the three points of the parabola $r = a\,\mathrm{cosec}^2 \dfrac{\theta}{2}$ whose vectorial angles are α, β, γ meet in a point whose vectorial angle is ϕ, prove that

$$\alpha + \beta + \gamma - \pi = 2\phi .$$

Solution. The equation of the parabola is

$$r = a\,\mathrm{cosec}^2 \frac{\theta}{2}$$

or $$\frac{2a}{r} = 1 - \cos\theta . \qquad .....(1)$$

The equation of normal at a point δ to the parabola is

$$\frac{2a\sin\delta}{(1-\cos\delta)r} = \sin\theta - \sin(\theta - \delta)$$

$$\frac{2a\sin\delta}{(1-\cos\delta)}\frac{1}{r} = \sin\theta(1-\cos\delta) + \cos\theta\sin\delta .$$

It passes through (ρ, ϕ), then

$$\frac{2a\sin\delta}{1-\cos\delta}\frac{1}{\rho} = \sin\phi(1-\cos\delta) + \cos\phi\sin\delta . \qquad .....(2)$$

Putting $\sin\delta = \dfrac{2\tan\dfrac{\delta}{2}}{1-\tan^2\dfrac{\delta}{2}}$ $\quad \cos\delta = \dfrac{1-\tan^2\dfrac{\delta}{2}}{1+\tan^2\dfrac{\delta}{2}}$

in (2) and solving, we get

$$\rho\sin\phi\tan^3\frac{\delta}{2} + (\rho\cos\phi - a)\tan^2\frac{\delta}{2} - a = 0 .$$

This is a cubic equation in $\tan\dfrac{\delta}{2}$ giving three values of $\tan\dfrac{\delta}{2}$.

Let three values are $\tan\dfrac{\alpha}{2}, \tan\dfrac{\beta}{2}, \tan\dfrac{\gamma}{2}$, then by the theory of equation

$$S_1 = \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{a - \rho\cos\phi}{\rho\sin\phi}$$

$$S_2 \equiv \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=0$$

$$S_3 \equiv \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=\frac{a}{\rho\sin\phi}$$

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\frac{S_1-S_3'}{1-S_2}$$

$$=\frac{\frac{a-\rho\cos\phi}{\rho\sin\phi}-\frac{a}{\rho\sin\phi}}{1-0}$$

$$=-\cot\phi$$

$$=\tan\left(\frac{\pi}{2}+\phi\right)$$

$$\alpha+\beta+\gamma=\pi+2\phi$$

$$2\phi=\alpha+\beta+\gamma-\pi.$$

EXERCISE 4.

1. PSP′ is a focal chord of a conic. Prove that the angle between the tangents at P and P′ is $\tan^{-1}\dfrac{2e\sin\alpha}{1-e^2}$ where α is the angle between the chord and the major axis. (GKP 2007)
[GKP, 84, 2004; Avadh., 85; Alld., 86; Agra, 88; Purv., 99]
2. If PQ is the chord of contact of the tangents drawn from a point T to a conic whose focus is S, prove that
(i) $ST^2 = SP \cdot SQ$ if the conic is a parabola ; and
(ii) if the conic is central and b is its semi-minor axis, then
$$\frac{1}{SP\cdot SQ}-\frac{1}{ST^2}=\frac{1}{\ell^2}\sin^2\left(\frac{1}{2}PSQ\right).$$
3. If S is the focus and P and Q two points on a conic such that the angle PSQ is constant and equal to 2δ, prove that :
(i) The locus of the intersection of tangents at P and Q is a conic section whose foucs is S.

(ii) The line PQ always touches a conic whose focus is S.

[UPSC, 88; Luck., 85]

4. A conic is described having the same focus and eccentricity as the conic $\frac{\ell}{r} = 1 + e\cos\theta$ and the two conics touch at a point $\theta = \alpha$, prove that the length of its latus rectum is

$$\frac{2\ell(1-e^2)}{e^2 + 2\ell\cos\alpha + 1}.$$

[Agra, 86; IAS, 83]

5. If the normal at L, one of the extremities of the latus rectum of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ meets the curve again at Q, show that

$$SQ = \frac{\ell(1 + 3e^2 + e^4)}{1 + e^2 - e^4}.$$

6. If the normals at α, β, γ on $\frac{\ell}{r} = 1 + e\cos\theta$ meet in the point (ρ, ϕ). show that $2\phi = \alpha + \beta + \gamma$.

[Purv., 96, 2002]

7. If the normals at the points $\theta_1, \theta_2, \theta_3, \theta_4$ on the conic

$$\frac{\ell}{r} = 1 + e\cos\theta$$

meet in the point (ρ, ϕ), show that

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\pi + 2\phi$$

[Kan., 80; Purv., 2001]

8. If SM and SN are perpendiculars from the focus S on the **tangent** and normal at any point on the conic and SL is **perpendicular to** MN, obtain in polar coordinates the locus of L.

9. Two conics have a common focus and directrix. If any tangent to one intersects the other in P and Q, show that

$\sec\left(\frac{1}{2}PSQ\right) = \frac{e_1}{e}$; e, e_1 being their eccentricities. [IFS, 87]

10. Prove that the equation of a pair of tangents from a point (r_1, θ_1) to the conic $\frac{\ell}{r} = 1 + e\cos\theta$ is

$$\left[\left(\frac{\ell}{r}-1+e\cos\theta\right)^2-1\right]\left[\left(\frac{\ell}{r_1}-e\cos\theta_1\right)^2-1\right]$$

$$=\left[\left(\frac{\ell}{r}-e\cos\theta\right)\left(\frac{\ell}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)\right]^2 \qquad \text{[GKP, 75]}$$

11. OPQ is a conic one of whose foci is S and PQ passes through a fixed point O, show that the product

$$\tan\frac{PSO}{2}\tan\frac{QSO}{2} \text{ is constant.} \qquad \text{[Luck., 75]}$$

12. Show that the two conics

$$2\ell\sqrt{3}=r\left(\sqrt{3}+\cos\theta\right)$$

and $$\ell\sqrt{3}=r\left[\sqrt{3}+\cos\left(\theta+\frac{\pi}{3}\right)\right]$$

touch where $\theta=\frac{\pi}{2}$ [Luck., 88]

13. If a chord PQ of a conic whose eccentricity is e and semi-latus rectum is ℓ, sutends a right angle at the focus S, prove that

$$\left(\frac{1}{SP}-\frac{1}{\ell}\right)^2+\left(\frac{1}{SQ}-\frac{1}{\ell}\right)^2=\frac{e^2}{\ell^2} \qquad \text{[GKP, 2004]}$$