INTEGRAL CALCULUS

Chapter 0

AN INTRODUCTION TO INTEGRAL CALCULUS

§ 0.1. Definition

The process inverse to differentiation is defined as integration. Thus if $\frac{d}{dx} F(x) = f(x)$, we say that $f(x)$ is an integral of $f(x)$ *i.e.*,

$$\int F(x)\,dx = f(x)$$

The process of determining an integral of a function is called *integration* and the function to be integrated is called integrand.

§ 0.2. Constant of Integration

Since the differential coefficients of a constant is zero we find that

$$\frac{d}{dx}\{f(x) + c\} = \frac{d}{dx}\{f(x)\} = F(x)$$

$$\therefore \qquad \int F(x)\,dx = f(x) + c, \text{ where } c \text{ is constant.}$$

Here c can be given any number of values and consequently the integral is known as indefinite integral. The common practice is not to write the constant of integration. Thus we have following two classes of integrals.

(a) Indefinite Integrals,

(b) Definite Integrals.

We have already mentioned about indefinite integrals above. Now if $F(x)$ is an integral of $f(x)$ then the integral of $f(x)$ between the limits a and b is denoted by

$$\int_a^b f(x)\,dx$$

and is defined as

$$\int_a^b f(x)\,dx = {}_a^b\big[\mathbf{F}(x)\big] = \mathbf{F}(b) - \mathbf{F}(a)$$

which is the difference of the values of $F(x)$ at the upper limit and lower limit. If we take the integral of $f(x)$ as $F(x) + c$ where c is any constant, even then

$$\int_a^b f(x)\,dx = {}_a^b\big[F(x) + c\big]$$

$$= \{F(b) + c\} - \{F(a) + c\}$$

$$= F(b) - F(a).$$

Since the constant of integration is eliminated in this case, the integral $\int_a^b$ **f (x) dx** is known as definite integral. a is called the lower limit and b the upper limit of the integral.

§ 0.3. Important Theorems

(1) The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function.

Thus if α is a constant, then

$$\int \alpha f(x)\, dx = \alpha \int f(x)\, dx$$

(2) The integral of a sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the functions *i.e.*,

$$\int [f_1(x) \pm f_2(x) \pm \text{........} \pm f_n(x)]\, dx$$
$$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \text{........} \pm \int f_n(x)\, dx$$

§ 0.4. Fundamental Formulae

From differential calculus we know that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)\,x^n}{(n+1)} = x^n$$

Thus $$\int x^n\, dx = \frac{x^{n+1}}{n+1}, \ (n \neq -1)$$

In words the above formula may be stated as :

"To find the integral of x^n w.r.t. 'x', increase the index (power) of x by unity (one) and then divide by the increased index".

However if $n = -1$ we have

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \log x \qquad \left[\because \frac{d}{dx}\log x = \frac{1}{x}\right]$$

§ 0.5 Standard Results

1. $\int e^x\, dx = e^x$
2. $\int a^x\, dx = \dfrac{a^x}{\log_e a}, a \neq 1$
3. $\int \sin x\, dx = -\cos x$
4. $\int \cos x\, dx = \sin x$
5. $\int \tan x\, dx = \log \sec x$
6. $\int \cot x\, dx = \log \sin x$
7. $\int \sec x\, dx = \log(\sec x + \tan x)$ or $\log \tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

8. $\int \operatorname{cosec} x\, dx = \log \tan \dfrac{x}{2} = \log (\operatorname{cosec} x - \cot x)$

9. $\int \sec^2 x\, dx = \tan x$

10. $\int \operatorname{cosec}^2 x\, dx = -\cot x$

11. $\int \sec x \tan x\, dx = \sec x$

12. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$

13. $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right)$$

14. $$\int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left(\frac{x-a}{x+a}\right),\ x > a;$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left(\frac{a+x}{a-x}\right),\ a > x$$

15. $$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \left(\frac{x}{a}\right)$$

16. $$\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \sinh^{-1} \frac{x}{a} = \log \{x + \sqrt{(a^2 + x^2)}\}$$

17. $$\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1} \left(\frac{x}{a}\right) = \log \{x + \sqrt{(x^2 - a^2)}\}$$

18. $$\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a} \sec^{-1} \left(\frac{x}{a}\right)$$

19. $$\int \sqrt{(a^2 - x^2)}\, dx = \frac{x}{2} \sqrt{(a^2 - x^2)} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

20. $$\int \sqrt{(a^2 + x^2)}\, dx = \frac{x}{2} \sqrt{(a^2 + x^2)} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a}\right)$$

21. $$\int \sqrt{(x^2 - a^2)}\, dx = \frac{x}{2} \sqrt{(x^2 - a^2)} - \frac{a^2}{2} \cosh^{-1} \left(\frac{x}{a}\right)$$

22. $\int \sinh x\, dx = \cosh x$

23. $\int \cosh x\, dx = \sinh x$

0.6 Extended forms of Fundamental Formulae

(i) $$\int (ax + b)^n\, dx = \frac{1}{a} \frac{(ax + b)^{n+1}}{(n+1)},$$

(ii) $$\int \frac{1}{(ax + b)^n}\, dx = \frac{1}{a\,(n-1)\,(ax + b)^{n-1}},$$

(iii) $$\int \frac{1}{ax + b}\, dx = \frac{1}{a} \log (ax + b),$$

(iv) $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b}$ and $\int a^{px+q}\, dx = \frac{1}{p} \frac{a^{px+q}}{\log_e a}$,

(v) $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b)$,

(vi) $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b)$,

(vii) $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b)$,

(viii) $\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b)$,

(ix) $\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b)$;

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b)$.

§ 0.7. Methods of Integration

There are four principal methods of integration :

(a) Integration by substitution,
(b) Integration by parts,
(c) Integration by decomposition into sum,
(d) Integration by successive reduction.

(a) Integration by substitution :

Method of substitution is very useful in the sense that it transforms the integral into rather simpler forms.

If we have the integral of the form

$$\int f\{\phi(x)\}\,\phi'(x)\,dx$$

where $\phi'(x)$ denotes the differential coefficient of $\phi(x)$ with respect to x then in order to evaluate it we put

$$\phi(x) = t \quad \text{...(i)}$$

so that $\phi'(x)\,dx = dt$

then the given integral reduces to $\int f(t)\,dt$ which can be evaluated in terms of t and then the value of t will be replaced in terms of x with the help of (i).

Note—In the problems to be solved the function $\int f\{\phi(x)\}$ and $\phi'(x)$ is so mixed up that the proper substitution becomes a guess work. At the initial stages the students may find some difficulty but after a short period they develop the ability of such guess works. Following examples will make the process clear.

Example 1. *Find the value of* $\int \tan x\, dx$.

Sol. $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

Put $\cos x = t$ so that $-\sin x\, dx = dt$, therefore

$$\int \tan x\,dx = -\int \frac{dt}{t} = -\log t$$
$$= -\log\cos x = \log\sec x.$$

Example 2. *Evaluate* $\int x e^{x^2}\,dx$.

Sol. Putting $x^2 = t$ so that $2x\,dx = dt$, we have

$$\int x e^{x^2}\,dx = \frac{1}{2}\int e^t\,dt$$
$$= \frac{1}{2}e^t = \frac{1}{2}e^{x^2}.$$

(b) Integration by parts :

Let $f_1(x)$ and $f_2(x)$ be any two functions of x, then

$$\frac{d}{dx}\{f_1(x)\cdot f_2(x)\} = f_1(x)\frac{d}{dx}\{f_2(x)\} + f_2(x)\frac{d}{dx}\{f_1(x)\}$$

Hence by definition

$$f_1(x)\cdot f_2(x) = \int\left[f_1(x)\frac{d}{dx}\{f_2(x)\} + f_2(x)\frac{d}{dx}\{f_1(x)\}\right]dx$$
$$= \int f_1(x)\frac{d}{dx}\{f_2(x)\}\,dx + \int f_2(x)\frac{d}{dx}\{f_1(x)\}\,dx$$
$$\therefore \int f_1(x)\frac{d}{dx}\{f_2(x)\}\,dx = f_1(x)\cdot f_2(x) - \int f_2(x)\frac{d}{dx}\{f_1(x)\}\,dx \quad \text{...(i)}$$

Now if $\frac{d}{dx}\{f_2(x)\} = F(x)$

$\int F(x)\,dx = f_2(x)$ by definition

Then from (i) we get

$$\int f_1(x)F(x)\,dx = f_1(x)\int F(x)\,dx - \int\left[\left\{\frac{d}{dx}f_1(x)\right\}f_2(x)\right]dx \quad \text{...(ii)}$$

Thus the integral of a product of two functions $f_1(x)$ and $F(x)$ termed respectively as the first and second function may be stated as follows :

Int. of Product of two functions

= First function × Int. of second

– Int. of [Diff. Coeff. of first × Int. of second]

Note 1 – Above formula provides the integral in parts and the process may be repeated if desired.

Note 2 – While making a choice of first and second functions we should take second function as that function which is readily integrable and the first function reduces to a simpler function after differentiaion.

Note 3 – If there is only one function, we can make use of this formula by taking unity as the second function.

Example 3. *Evaluate* $\int x^2 e^x\, dx$.

Sol. Here x^2 should be taken as first function. Integrating by parts taking e^x as second function we have

$$\int x^2 e^x\, dx = x^2 e^x - \int e^x (2x)\, dx$$
$$= x^2 e^x - 2\int xe^x\, dx.$$

Integral on the right is again a product of two functions. Repeating the process again with x as first function, we get

$$\int x^2 e^x\, dx = x^2 e^x - 2\left[xe^x - \int 1 \cdot e^x\, dx\right]$$
$$= x^2 e^x - 2x e^x + 2e^x$$
$$= (x^2 - 2x + 2)\, e^x.$$

Example 4. *Evaluate* $\int \sin^{-1} x\, dx$.

Solution. Integrand in this case cantains only one function. Integrating therefore by parts taking unity as second function, we have

$$\int \sin^{-1} x\, dx = \int \sin^{-1} x\, (1)\, dx$$
$$= x \sin^{-1} x - \int \frac{x\, dx}{\sqrt{(1 - x^2)}}$$

Putting $1 - x^2 = t^2$

so that $-2x\, dx = 2t\, dt$

$$= x \sin^{-1} x + \int \frac{t\, dt}{t}$$
$$= x \sin^{-1} x + t$$

or $\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{(1 - x^2)}$

(c) Integration by decomposition into a sum :

This process is generally used when the integrand is a rational algebraic function or a product of trigonometric functions. In the first case it can be resolved into partial fractions and in the second case it can be decomposed into a sum by the use of proper trigonometric identities.

Example 5. *Evaluate* $\int \frac{1}{(e^x - 1)^2}\, dx$.

Sol. We have $\int \frac{1}{(e^x - 1)^2}\, dx = \int \frac{e^x}{e^x (e^x - 1)^2}\, dx$,

[multiplying the Nr. and Dr. by ...

$$= \int \frac{dt}{t\,(t - 1)^2},$$ putting $e^x = t$ so that $e^x\, dx = dt$.

Now $\dfrac{1}{t(t-1)^2} \equiv \dfrac{A}{t} + \dfrac{B}{t-1} + \dfrac{C}{(t-1)^2}$, [on resolving into partial fractions]

$\therefore\ 1 \equiv A(t-1)^2 + Bt(t-1) + Ct.$ ...(1)

To find A, putting $t = 0$ on both sides of (1), we get $A = 1$.

To find C, put $t = 1$ and we get $C = 1$.

Thus $1 \equiv (t-1)^2 + Bt(t-1) + t.$

Comparing the coefficients of t^2 on both sides, we get

$$0 = 1 + B \quad \text{or} \quad B = -1.$$

$$\therefore \quad \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}.$$

Hence $\displaystyle\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}\,dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2}$

$$= \log t - \log(t-1) - \{1/(t-1)\}$$

$$= \log e^x - \log(e^x - 1) - \{1/(e^x - 1)\}$$

$$= x - \log(e^x - 1) - \{1/(e^x - 1)\}.$$

Example 6. *Evaluate* $\int e^{2x} \cos x\, dx$.

Sol. Integrating by parts taking e^{2x} as first function, we have

$$\int e^{2x} \cos x\, dx = e^{2x} \sin x - \int 2e^{2x} . \sin x\, dx$$

$$= e^{2x} \sin x - 2\left\{- e^{2x} \cos x - \int 2e^{2x} (-\cos x)\, dx\right\}$$

(on repeating the process again)

$$= e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x\, dx$$

Transposing the last term on the right to the left, we get

$$5 \int e^{2x} \cos x\, dx = e^{2x} \sin x + 2e^{2x} \cos x$$

or $$\int e^{2x} \cos x\, dx = \frac{e^{2x}}{5} \{\sin x + 2 \cos x\}.$$

(d) Integration by successive reduction :

Here we first integrate by parts and after making some suitable transformation we connect the given integral with another integral whose integrand is of the same type but a simpler one. The process of reducing a given integral to a simpler integrai of the same kind is known as the method of reduction and the relationship so obtained is known as the reduction formula. Use of the formula may be repeated to evaluate the integral.

EXERCISE 0.1

Evaluate the following :

1. $\displaystyle\int \frac{dx}{4 + 3x - 2x^2}$ 2. $\displaystyle\int x \cos^3 x^2 . \sin x^2\, dx$

3. $\displaystyle\int \frac{\sec x}{\sqrt{3} + \tan x}\, dx$

4. $\displaystyle\int \frac{xe^x}{(x+1)^2}\, dx$

5. $\displaystyle\int e^x \frac{x^2 + 3x + 3}{(x+2)^2}\, dx$

6. $\displaystyle\int e^x \left[\frac{1 + \sqrt{1 - x^2}\, \sin^{-1} x}{\sqrt{1 - x^2}}\right] dx$

7. $\displaystyle\int \frac{x^2\, dx}{(x \sin x + \cos x)^2}$

8. $\displaystyle\int x^2 \tan^{-1} x\, dx$

9. $\displaystyle\int x^3 \tan^{-1} x\, dx$

10. $\displaystyle\int x \sin^{-1} x\, dx$

11. $\displaystyle\int x^2 \sin^{-1} x\, dx$

12. $\displaystyle\int \sin x \log(\sec x + \tan x)\, dx$

13. $\displaystyle\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

14. $\displaystyle\int \frac{(x-1)\, dx}{(x-3)(x-2)}$

15. $\displaystyle\int \frac{x}{x^2 + x - 6}\, dx$

16. $\displaystyle\int \frac{x}{(x^2 - a^2)(x^2 - b^2)}\, dx$

17. $\displaystyle\int_1^2 \frac{(1 + \log x)^4}{x}\, dx$

18. $\displaystyle\int_0^2 x^2 e^{2x}\, dx$

19. $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x\, dx}{(1 - x^2)^{3/2}}$

20. $\displaystyle\int_0^1 x (\tan^{-1} x)^2\, dx$