

Chapter 1

INTEGRATION OF IRRATIONAL ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS

1.1. Integration of $1/\sqrt{ax^2 + bx + c}$.

(a) When a is positive

$$\begin{aligned}
 \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt[\sqrt]{[x^2 + (b/a)x + (c/a)]}}} \\
 &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt[\sqrt]{[x + (b/2a)]^2 + [(c/a) - (b^2/4a^2)]}}} \\
 &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt[\sqrt]{[x + (b/2a)]^2 - [(b^2 - 4ac)/4a^2]}}} \quad \dots(1)
 \end{aligned}$$

Now two cases arise viz., $b^2 - 4ac > 0$ and $b^2 - 4ac < 0$.

Case I. $b^2 > 4ac$.

Then from (1) we get

$$\begin{aligned}
 \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{\sqrt{a}} \int dt / \sqrt{\left[t^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]}, \\
 &\quad \text{where } t = x + (b/2a) \\
 &= \frac{1}{\sqrt{a}} \cosh^{-1} \left[t / \frac{\sqrt{b^2 - 4ac}}{2a} \right] \\
 &= \frac{1}{\sqrt{a}} \cosh^{-1} \left[\left(x + \frac{b}{2a}\right) / \frac{\sqrt{b^2 - 4ac}}{2a} \right] \\
 &= \frac{1}{\sqrt{a}} \cosh^{-1} \left[\frac{2ax + b}{\sqrt{b^2 - 4ac}} \right]
 \end{aligned}$$

Case II. $b^2 < 4ac$.

If $b^2 < 4ac$, then $b^2 - 4ac$ is negative.

∴ from (1) we get

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{[x + (b/2a)]^2 + [(4ac - b^2)/4a^2]}},$$

which is of the form $\int \frac{dx}{\sqrt{x^2 + a^2}}$

$$= \frac{1}{\sqrt{a}} \sinh^{-1} \left[\left(x + \frac{b}{2a} \right) / \sqrt{\frac{4ac - b^2}{4a^2}} \right]$$

$$= \frac{1}{\sqrt{a}} \sinh^{-1} \left[\frac{2ax + b}{\sqrt{4ac - b^2}} \right].$$

In case of $b^2 - 4ac = 0$, $\sqrt{ax^2 + bx + c}$ is a rational function.

(b) When a is negative. Let $a = -A$.

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \int \frac{dx}{\sqrt{(c + ax - Ax^2)}}$$

$$= \frac{1}{\sqrt{-A}} \int \frac{dx}{\sqrt{[(c/A) + (b/A)x] - x^2}}$$

$$= \frac{1}{\sqrt{-A}} \int \frac{dx}{\sqrt{[(b^2/4A^2) + (c/A)] - [x - (b/2A)]^2}}$$

$$= \frac{1}{\sqrt{-A}} \int \frac{dx}{\sqrt{[(b^2 + 4cA)/4A^2] - [x - (b/2A)]^2}}$$

which is of the form $\int \frac{dx}{a^2 - x^2}$

$$= \frac{1}{\sqrt{-A}} \sin^{-1} \left[\frac{x - (b/2A)}{\sqrt{[(b^2 + 4cA)/4A^2]}} \right]$$

$$= \frac{1}{\sqrt{-A}} \sin^{-1} \left[\frac{2Ax - b}{\sqrt{(b^2 + 4cA)}} \right]$$

$$= \frac{1}{\sqrt{(-a)}} \sin^{-1} \left[\frac{-2ax - b}{\sqrt{(b^2 - 4ac)}} \right],$$

$\sqrt{(-a)}$ is real, $\therefore a < 0$

Example 1. Integrate $1/\sqrt{1 - x - x^2}$.

Sol. $\int \frac{dx}{\sqrt{1 - x - x^2}} = \int \frac{dx}{\sqrt{[1 - (x^2 + x)]}}$

$$\begin{aligned}
 &= \int \frac{dx}{\sqrt{\left[\frac{5}{4} - (x + \frac{1}{2})^2\right]}} = \sin^{-1} \left\{ \frac{(x + \frac{1}{2})}{\sqrt{(5/4)}} \right\} \\
 &= \sin^{-1} \left(\frac{2x + 1}{\sqrt{5}} \right).
 \end{aligned}$$

Example 2. Evaluate $\int \frac{dx}{\sqrt{x(1-x)}}$.

$$\begin{aligned}
 \text{Sol. } \int \frac{dx}{\sqrt{x-x^2}} &= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - (x-\frac{1}{2})^2}} \\
 &= \sin^{-1} \left\{ \frac{(x-\frac{1}{2})}{\frac{1}{2}} \right\} = \sin^{-1} (2x-1).
 \end{aligned}$$

Example 3. Integrate $1/\sqrt{x^2+x+1}$.

$$\begin{aligned}
 \text{Sol. } \frac{dx}{\sqrt{x^2+x+1}} &= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + (\sqrt{3}/2)^2}} \\
 &= \sinh^{-1} \left[\frac{x+\frac{1}{2}}{\sqrt{3}/2} \right] \\
 &= \sinh^{-1} [(2x+1)/\sqrt{3}]
 \end{aligned}$$

EXERCISE 1·1

Integrate the following with respect to x :

- | | |
|-----------------------------------|--|
| 1. $1/\sqrt{2x-x^2}$ | 2. $1/[2\sqrt{x(1+x)}]$ |
| 3. $1/\sqrt{3x-x^2-2}$ | 4. $1/\sqrt{x^2+x-2}$ |
| 5. $1/\sqrt{2+x-3x^2}$ (GKP 2005) | 6. $1/\sqrt{x^2+2x+5}$ |
| 7. $1/\sqrt{4+3x-2x^2}$ | 8. $\int_{\alpha}^{\beta} \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx$ |

§ 1·2. Integration of $\sqrt{ax^2+bx+c}$

(a) When a is positive

$$\begin{aligned}
 \int \sqrt{ax^2+bx+c} dx &= \sqrt{a} \int \sqrt{\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)} dx \\
 &= \sqrt{a} \int \sqrt{\left(x + \frac{b}{2}a\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)} dx
 \end{aligned}$$

$$= \sqrt{a} \int \sqrt{\left\{ \left(x + \frac{b}{2} \right)^2 + \left(\frac{4ac - b^2}{4a^2} \right) \right\}} dx$$

which as in § 1.1 is one of the two standard forms

$$\int \sqrt{x^2 + a^2} dx, \int \sqrt{x^2 - a^2} dx$$

and hence can be integrated.

(b) When a is negative

Let $a = -A$. Then

$$\begin{aligned} \int \sqrt{(ax^2 + bx + c)} dx &= \int \sqrt{(c + bx - Ax^2)} dx \\ &= \sqrt{-A} \int \sqrt{\left(\frac{c}{A} + \frac{b}{A}x - x^2 \right)} dx \\ &= \sqrt{-A} \int \sqrt{\left(\frac{b^2}{4A^2} + \frac{c}{A} - \left(\frac{b}{2A} \right)^2 \right)} dx \\ &= \sqrt{-A} \int \sqrt{\left[\frac{b^2 + 4Ac}{4A^2} - \left(x - \frac{b}{2A} \right) \right]} dx \end{aligned}$$

which is of the form $\int \sqrt{a^2 - x^2} dx$ and hence can be integrated.

Example 1. Integrate $\sqrt{(2x^2 + 3x + 4)}$.

$$\begin{aligned} \text{Sol. } \int \sqrt{(2x^2 + 3x + 4)} dx &= \sqrt{2} \int \sqrt{\left(x^2 + \frac{3}{2}x + 2 \right)} dx \\ &= \sqrt{2} \int \sqrt{\left\{ \left(x + \frac{3}{4} \right)^2 + \frac{23}{16} \right\}} dx \\ &= \sqrt{2} \int \sqrt{\left\{ \left(x + \frac{3}{4} \right)^2 + \left(\frac{\sqrt{23}}{4} \right)^2 \right\}} dx \\ &= \sqrt{2} \left[\frac{1}{2} \left(x + \frac{3}{4} \right) \sqrt{\left\{ \left(x + \frac{3}{4} \right)^2 + \frac{23}{16} \right\}} + \frac{1}{2} \cdot \frac{23}{16} \sinh^{-1} \left\{ \frac{x + \frac{3}{4}}{\sqrt{23/4}} \right\} \right] \\ &= \sqrt{2} \left[\frac{1}{8} (4x + 3) \sqrt{\left(x^2 + \frac{3}{2}x + 2 \right)} + \frac{23}{32} \sinh^{-1} \left(\frac{4x + 3}{\sqrt{23}} \right) \right] \\ &= \frac{1}{8} (4x + 3) \sqrt{(2x^2 + 3x + 4)} + \frac{23\sqrt{2}}{32} \sinh^{-1} \left(\frac{4x + 3}{\sqrt{23}} \right). \end{aligned}$$

Example 2. Evaluate $\int_3^4 \sqrt{\{(4-x)(x-3)\}} dx$.

(Gorakhpur 2005, 10)

$$\begin{aligned}
 \text{Sol. } & \int_3^4 [(4-x)(x-3)] dx = \int_3^4 \sqrt{(7x-x^2-12)} dx \\
 & = \int_3^4 \sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{7}{2}\right)^2} dx \\
 & = \left[\frac{1}{2} \left(x-\frac{7}{2}\right) \sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{7}{2}\right)^2} + \frac{1}{2} \left(\frac{1}{2}\right)^2 \sin^{-1} \left\{ \frac{x-\frac{7}{2}}{\frac{1}{2}} \right\} \right]_3^4 \\
 & = (1/8) [\frac{1}{2}\pi + \frac{1}{2}\pi] = \pi/8.
 \end{aligned}$$

Example 3. Integrate $(x^2 + 1)/\sqrt{x^2 + 3}$.

$$\begin{aligned}
 \text{Sol. } & \int \frac{(x^2 + 1) dx}{\sqrt{x^2 + 3}} = \int \frac{(x^2 + 3) - 2}{\sqrt{x^2 + 3}} dx \\
 & = \int \sqrt{x^2 + 3} dx - 2 \int \frac{dx}{\sqrt{x^2 + 3}} \\
 & = \frac{1}{2}x \sqrt{x^2 + 3} + \frac{1}{2} \cdot 3 \cdot \sinh^{-1}(x/\sqrt{3}) - 2 \sinh^{-1}(x/\sqrt{3}) \\
 & = \frac{1}{2}[x \sqrt{x^2 + 3} - \sinh^{-1}(x/\sqrt{3})].
 \end{aligned}$$

EXERCISE 1·2

1. Evaluate $\int \sqrt{x^2 + 3} dx$.
2. Integrate $\sqrt{x^2 - x + 1}$.
3. Evaluate $\int_1^2 \sqrt{\{(x-1)(2-x)\}} dx$.
4. $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)} dx$.

§ 1·3. Integration of $(px+q)/\sqrt{ax^2+bx+c}$.

In this case we should break up the numerator $(px+q)$ into two parts, one of which is some multiple of the exact differential coefficient of (ax^2+bx+c) and the other is free from x .

Hence we write

$$\begin{aligned}
 px+q &= A \frac{d}{dx} (ax^2+bx+c) + B \\
 &= A(2ax+b) + B
 \end{aligned}$$

Comparing the coefficients of x and constant terms, we get

$$A = \frac{p}{2a}, B = \left(q - \frac{pb}{2a}\right)$$

So we have $(px + q) = \frac{p}{2a}(2ax + b) + \left(q - \frac{bp}{2a}\right)$

$$\begin{aligned} \therefore \int \frac{(px + q) dx}{\sqrt{ax^2 + bx + c}} &= \frac{p}{2a} \int \frac{(2ax + b) dx}{\sqrt{ax^2 + bx + c}} \\ &\quad + \left(q - \frac{bp}{2a}\right) \int \frac{dx}{\sqrt{ax^2 + bx + c}} \\ &= \frac{p}{2a} \cdot 2 \sqrt{ax^2 + bx + c} + \left(q - \frac{bp}{2a}\right) \int \frac{dx}{\sqrt{ax^2 + bx + c}} \end{aligned}$$

Integral on the right can be evaluated by the method of § 1.1.

§ 1.4. Integration of $\frac{c_0x^n + c_1x^{n-1} + \dots + c_{n-1}x + c_n}{\sqrt{ax^2 + bx + c}}$.

To integrate this function, we assume

$$\begin{aligned} \int \frac{c_0x^n + c_1x^{n-1} + \dots + c_{n-1}x + c_n}{\sqrt{ax^2 + bx + c}} \\ = (A_0x^{n-1} + A_1x^{n-2} + \dots + A_{n-2}x + A_{n-1}) \sqrt{ax^2 + bx + c} \\ + A_n \int \frac{dx}{\sqrt{ax^2 + bx + c}} \end{aligned}$$

where $A_0, A_1, \dots, A_{n-1}, A_n$ are arbitrary constants.

Differentiating both sides and multiplying by $\sqrt{ax^2 + bx + c}$, we get

$$\begin{aligned} c_0x^n + c_1x^{n-1} + \dots + c_{n-1}x + c_n \\ = [(n-1)A_0x^{n-2} + (n-2)A_1x^{n-3} + \dots + A_{n-2}] (ax^2 + bx + c) \\ + (A_0x^{n-1} + A_1x^{n-2} + \dots \\ + (A_{n-2}x + A_{n-1}) \frac{2ax + b}{2} + A_n \end{aligned}$$

Comparing the coefficients of like powers of x from both sides we obtain the constants A_0, A_1, \dots, A_n .

Also $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be evaluated by the method as in § 1.1.

Example 1. Integrate $x/\sqrt{x^2 + x + 1}$.

$$\begin{aligned} \text{Sol. } \int \frac{x dx}{\sqrt{x^2 + x + 1}} &= \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{\sqrt{x^2 + x + 1}} dx \\ &= \frac{1}{2} \int \frac{(2x+1) dx}{\sqrt{x^2 + x + 1}} - \frac{1}{2} \int \frac{dx}{\sqrt{x^2 + x + 1}} \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{x^2 + x + 1} - \frac{1}{2} \int \frac{dx}{\sqrt{\{(x + \frac{1}{2})^2 + (3/4)\}}} \\
 &= \sqrt{x^2 + x + 1} - \frac{1}{2} [\sinh^{-1} \{(x + \frac{1}{2})/\sqrt{(3/4)}\}] \\
 &= \sqrt{x^2 + x + 1} - \frac{1}{2} \sinh^{-1} \{(2x + 1)/\sqrt{3}\}.
 \end{aligned}$$

Example 2. Integrate $(2x + 3)/\sqrt{x^2 + x + 1}$.

$$\begin{aligned}
 \text{Sol. } \int \frac{(2x + 3) dx}{\sqrt{x^2 + x + 1}} &= \int \frac{(2x + 1) + 2}{\sqrt{x^2 + x + 1}} dx, \\
 \therefore \frac{d}{dx}(x^2 + x + 1) &= 2x + 1 \\
 &= \int \frac{(2x + 1) dx}{\sqrt{x^2 + x + 1}} + 2 \int \frac{dx}{\sqrt{x^2 + x + 1}} \\
 &= 2\sqrt{x^2 + x + 1} + 2 \int \frac{dx}{\sqrt{[(x + \frac{1}{2})^2 + (\sqrt{3}/2)^2]}} \\
 &= 2\sqrt{x^2 + x + 1} + 2 \sinh^{-1} \{(x + \frac{1}{2})/(\sqrt{3}/2)\} \\
 &= 2\sqrt{x^2 + x + 1} + 2 \sinh^{-1} \{(2x + 1)/\sqrt{3}\}.
 \end{aligned}$$

Example 3. Integrate $\sqrt{[(a + x)/x]}$.

$$\begin{aligned}
 \text{Sol. } \int \sqrt{\left(\frac{a+x}{x}\right)} dx &= \int \frac{(a+x) dx}{\sqrt{x(a+x)}} \\
 &\quad \text{multiplying num. and denom. by } \sqrt{a+x} \\
 &= \int \frac{(a+x) dx}{\sqrt{ax+x^2}} = a \int \frac{dx}{\sqrt{ax+x^2}} + \int \frac{x dx}{\sqrt{ax+x^2}} \quad \dots(i) \\
 \text{Now } \int \frac{dx}{\sqrt{ax+x^2}} &= \int \frac{dx}{\sqrt{[(x+\frac{1}{2}a)^2 - (\frac{1}{2}a)^2]}} = \cosh^{-1} \left[\frac{(x+\frac{1}{2}a)}{(\frac{1}{2}a)} \right] \\
 &= \cosh^{-1} [(2x+a)/a] \quad \dots(ii)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \int \frac{x dx}{\sqrt{ax+x^2}} &= \frac{1}{2} \int \frac{(2x+a)-a}{\sqrt{ax+x^2}} dx, \because \frac{d}{dx}(ax+x^2) = a+2x \\
 &= \frac{1}{2} \int \frac{(2x+a) dx}{\sqrt{ax+x^2}} - \frac{1}{2} a \int \frac{dx}{\sqrt{ax+x^2}} \\
 &= \frac{1}{2} \int \frac{2u du}{u} - \frac{1}{2} a \cosh^{-1} \left(\frac{2x+a}{a} \right),
 \end{aligned}$$

where $u^2 = ax + x^2$ and from (ii)

$$= u - \frac{1}{2} a \cosh^{-1} \{(2x+a)/a\}$$

$$= \sqrt{ax+x^2} - \frac{1}{2} a \cosh^{-1} \{(2x+a)/a\}$$

Substituting values from (ii) and (iii) in (i) we get

$$\int \sqrt{\left(\frac{a+x}{a}\right)} dx = a \cosh^{-1} \left(\frac{2x+a}{a}\right)$$

$$+ \sqrt{ax+x^2} - \frac{1}{2} a \cosh^{-1} \{(2x+a)/a\}$$

$$= \frac{1}{2} a \cosh^{-1} \{(2x+a)/a\} + \sqrt{ax+x^2}$$

Example 4. Integrate $\frac{x^3}{\sqrt{x^2 - 2x + 2}}$.

Sol. Let $\int \frac{x^3}{\sqrt{x^2 - 2x + 2}} dx = (A_0x^2 + A_1x + A_2)\sqrt{x^2 - 2x + 2} + A_3 \int \frac{dx}{x^2 - 2x + 2}$

Differentiating both sides with respect to x and multiplying by $\sqrt{x^2 - 2x + 2}$ we get

$$x^3 = (2A_0x + A_1)(x^2 - 2x + 2) + (A_0x^2 + A_1x + A_2) \frac{1}{2}(2x - 2) + A_3$$

$$= 3A_0x^3 + (2A_1 - 5A_0)x^2 + (4A_0 - 3A_1 + A_2)x + 2A_1 - A_2 + A_3$$

Comparing the coefficients of like powers of x , we get

$$3A_0 = 1, 2A_1 - 5A_0 = 0, 4A_0 - 3A_1 + A_2 = 0, 2A_1 - A_2 + A_3 = 0$$

which gives $A_0 = \frac{1}{3}, A_1 = \frac{5}{6}, A_2 = \frac{7}{6}, A_3 = -\frac{1}{2}$

$$\begin{aligned} \therefore \int \frac{x^3}{\sqrt{x^2 - 2x + 2}} &= \left(\frac{1}{3}x^2 + \frac{5}{6}x + \frac{7}{6} \right) \sqrt{x^2 - 2x + 2} - \frac{1}{2} \int \frac{dx}{\sqrt{x^2 - 2x + 2}} \\ &= \frac{1}{6}(2x^2 + 5x + 7) \sqrt{x^2 - 2x + 2} - \frac{1}{2} \int \frac{dx}{\sqrt{[(x-1)^2 + 1]}} \\ &= \frac{1}{6}(2x^2 + 5x + 7) \sqrt{x^2 - 2x + 2} - \frac{1}{2} \sinh^{-1}(x-1) \end{aligned}$$

Example 5. Evaluate $\int \frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{(2x^2 - 2x + 1)}} dx$.

(Purvanchal 2005, 2008)

Sol. Let $\int \frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{(2x^2 - 2x + 1)}} dx$

$$= (A_0x^2 + A_1x + A_2)\sqrt{(2x^2 - 2x + 1)} + A_3 \int \frac{dx}{\sqrt{(2x^2 - 2x + 1)}}$$

Differentiating both sides with respect to x and multiplying by $(2x^2 - 2x + 1)$, we get

$$\begin{aligned} 6x^3 + 15x^2 - 7x + 6 &= (2A_0x + A_1)(2x^2 - 2x + 1) \\ &\quad + (A_0x^2 + A_1x + A_2) \frac{1}{2}(4x - 2) + A_3 \\ &= 6A_0x^3 + (-5A_0 + 4A_1)x^2 + (2A_0 - 3A_1 + 2A_2)x + A_1 - A_2 + A_3 \end{aligned}$$

Comparing the coefficients of x^3, x^2, x and constant terms respectively from two we get

$$6A_0 = 6, -5A_0 + 4A_1 = 15, 2A_0 - 3A_1 + 2A_2 = -7,$$

$$A_1 - A_2 + A_3 = 6 \text{ which gives}$$

$$A_0 = 1, A_1 = 5, A_2 = 3, A_3 = 4$$

$$\begin{aligned} \int \frac{6x^2 + 15x^2 - 7x + 6}{\sqrt{(2x^2 - 2x + 1)}} dx &= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} \\ &\quad + \frac{4}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x^2 - x + \frac{1}{2}\right)}} \\ &= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \int \frac{dx}{\sqrt{\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right]}} \\ &= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \sinh^{-1} \frac{x - \frac{1}{2}}{1/2} \\ &= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \sinh^{-1}(2x - 1) \end{aligned}$$

EXERCISE 1·3

Integrate the following with respect to x :

$$(3 - 2x)/\sqrt{4 + 2x - x^2} \quad 2. \quad (x - 1)/\sqrt{9 - 4x^2}$$

$$(2x + 3)/\sqrt{x^2 + 1} \quad (GKP 2013) \quad 4. \quad \sqrt{[(a - x)/x]}$$

$$(6x + 5)/\sqrt{6 + x - 2x^2} \quad 6. \quad (2x + 5)/\sqrt{x^2 + 3x + 2}$$

$$(x + 1)/\sqrt{x^2 - x + 1}$$

$$(x^2 - x + 1)/\sqrt{2x^2 - x + 2}$$

$$\text{Evaluate } \int_0^1 \frac{(1 + 4x + 2x^2) dx}{\sqrt{2x - x^2}}$$

$$\text{Evaluate } \sqrt{\frac{x^3 + 3}{\sqrt{x^2 + 1}}}$$

(Purvanchal 2007; Gorakhpur 1991, 95, 97)

§ 1·5. Integration of $(px + q) \sqrt{ax^2 + bx + c}$.

We write $px + q = A \left[\frac{d}{dx} (ax^2 + bx + c) \right] + B$ and compare the coefficient of x and constant term to find A and B .

$$\text{Thus } px + q = \frac{p}{2a} (2ax + b) + \left(q - \frac{pb}{2a} \right)$$

$$\begin{aligned} \therefore \int (px + q) \sqrt{(ax^2 + bx + c)} dx &= \frac{p}{2a} \int (2ax + b) \sqrt{(ax^2 + bx + c)} dx \\ &\quad + \left(q - \frac{pb}{2a} \right) \int \sqrt{(ax^2 + bx + c)} dx \\ &= \frac{p}{2a} \int t^{1/2} dt + \left(q - \frac{pb}{2a} \right) \int \sqrt{(ax^2 + bx + c)} dx, \end{aligned}$$

(putting $t = ax^2 + bx + c$ in the 1st integral)

$$= \frac{p}{2a} \cdot \frac{2}{3} (ax^2 + bx + c)^{3/2} + \left(q - \frac{pb}{2a} \right) \int \sqrt{(ax^2 + bx + c)} dx.$$

The integral on the right can be evaluated by the method of § 1.2.

Example 1. Integrate $(2x - 5) \sqrt{2 + 3x - x^2}$.

(Purvanchal 2)

Sol. $\because (d/dx)(2 + 3x - x^2) = 3 - 2x$

writing $(2x - 5) = -(3 - 2x) - 2$

$$\begin{aligned} \therefore \int (2x - 5) \sqrt{2 + 3x - x^2} dx &= \int (3 - 2x) \sqrt{2 + 3x - x^2} dx - 2 \int \sqrt{2 + 3x - x^2} dx \\ &= \int (3 - 2x) \sqrt{2 + 3x - x^2} dx - 2 \int \sqrt{[(17/4) - \{x - (3/2)\}^2]} dx \\ &= (2/3) (2 + 3x - x^2)^{3/2} - 2 \left[\frac{1}{2} \{x - (3/2)\} \right] \sqrt{[(17/4) - \{x - (3/2)\}^2]} \\ &\quad - 2 \frac{1}{2} (17/4) \sin^{-1} [\{x - (3/2)\}/\sqrt{(17/4)}] \\ &= - (2/3) (2 + 3x - x^2)^{3/2} - \frac{1}{2} (2x - 3) \sqrt{2 + 3x - x^2} \\ &\quad - (17/4) \sin^{-1} \{(2x - 3) \sqrt{(17/4)}\}. \end{aligned}$$

Example 2. Integrate $(x + 2) \sqrt{2x^2 - 6x + 5}$.

Sol. $\int (x + 2) \sqrt{2x^2 - 6x + 5} dx$

$$= \frac{1}{4} \int [(4x - 6) + 14] \sqrt{2x^2 - 6x + 5} dx,$$

$$\begin{aligned}
 & \because \frac{d}{dx}(2x^2 - 6x + 5) = 4x - 6 \\
 & = \frac{1}{4} \int (4x - 6) \sqrt{(2x^2 - 6x + 5)} dx + \frac{7\sqrt{2}}{2} \int \sqrt{(x^2 - 3x + \frac{5}{2})} dx \\
 & = \frac{1}{4} \int 2t^2 dt + \left(\frac{7}{2}\right) \sqrt{2} \int \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx, \text{ putting} \\
 & \quad 2x^2 - 6x + 5 = t^2 \\
 & \therefore (4x - 6) dx = 2t dt \\
 & = \frac{1}{6} t^3 + \left(\frac{7}{2}\right) \sqrt{2} \left[\frac{1}{2} \left(x - \frac{3}{2}\right) \sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{1}{2} \left(\frac{1}{2}\right)^2} \sinh^{-1} \{(2x - 3)\} \right] \\
 & = \frac{1}{6} (2x^2 - 6x + 5)^{3/2} + \frac{7\sqrt{2}}{4} (2x - 3) \sqrt{(x^2 - 3x + \frac{5}{2})} + \frac{7\sqrt{2}}{16} \sinh^{-1} (2x - 3) \\
 & = \frac{1}{6} (2x^2 - 6x + 5)^{3/2} + \frac{7}{4} (2x - 3) \sqrt{(2x^2 - 6x + 5)} + \frac{7\sqrt{2}}{16} \sinh^{-1} (2x - 3).
 \end{aligned}$$

1.6. Integration of $1/(Ax + B) \sqrt{(Cx + D)}$

In such cases put $Cx + D = t^2$, so that $C dx = 2t dt$.

Also $x = (t^2 - D)/C$

$$\begin{aligned}
 \therefore \int \frac{dx}{(Ax + B) \sqrt{(Cx + D)}} &= \int \frac{(2t/C) dt}{\left[A \left(\frac{t^2 - D}{C}\right) + B\right] t} \\
 &= \frac{2}{C} \int \frac{C dt}{At^2 - DA + BC} \\
 &= \frac{2}{A} \int \frac{dt}{t^2 + k}, \text{ where } k \text{ is some positive} \\
 &\quad \text{or negative constant.}
 \end{aligned}$$

This reduces to one of the forms $\int \frac{dt}{t^2 + a^2}$ or $\int \frac{dt}{t^2 - a^2}$ hence can be integrated.

Example 1. Evaluate $\int_8^{15} \frac{dx}{(x - 3) \sqrt{(x + 1)}}$.

$$\begin{aligned}
 \text{Sol. } \int_8^{15} \frac{dx}{(x - 3) \sqrt{(x + 1)}} &= \int_3^4 \frac{2t dt}{(t^2 - 4) \sqrt{(t^2)}} \text{, where } x + 1 = t^2 \\
 &= 2 \int_3^4 \frac{dt}{(t^2 - 2^2)} = 2 \left[\frac{1}{2(2)} \log \left(\frac{t - 2}{t + 2} \right) \right]_3^4 \\
 &= \frac{1}{2} \log (2/6) - \frac{1}{2} \log (1/5) \\
 &= \frac{1}{2} \log [(2/6) \times (5/1)] = \frac{1}{2} \log (5/3).
 \end{aligned}$$

Example 2. Integrate $(x+1)/(x-1)\sqrt{x+2}$.

$$\begin{aligned}\text{Sol. } \int \frac{(x+1)dx}{(x-1)\sqrt{x+2}} &= \int \frac{(x-1)+2}{(x-1)\sqrt{x+2}} dx \\ &= \int \frac{dx}{\sqrt{x+2}} + 2 \int \frac{dx}{(x-1)\sqrt{x+2}} \\ &= 2\sqrt{x+2} + 2 \int \frac{2t dt}{(t^2-3)t}, \text{ putting } x+2=t^2 \\ &= 2\sqrt{x+2} + 4 \int \frac{dt}{t^2 - (\sqrt{3})^2} \\ &= 2\sqrt{x+2} + 4 \cdot \frac{1}{2\sqrt{3}} \log \left\{ \frac{t-\sqrt{3}}{t+\sqrt{3}} \right\} \\ &= 2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \left\{ \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right\}\end{aligned}$$

Example 3. Integrate $1/(x-3)\sqrt{x+2}$.

Sol. Put $x+2=t^2$ so that $dx = 2t dt$.

$$\begin{aligned}\therefore \int \frac{dx}{(x-3)\sqrt{x+2}} &= \int \frac{2t dt}{[(t^2-2)-3]t} \\ &= 2 \int \frac{dt}{t^2 - 5} = 2 \int \frac{dt}{t^2 - (\sqrt{5})^2} \\ &= \frac{2}{2\sqrt{5}} \log \left(\frac{t-\sqrt{5}}{t+\sqrt{5}} \right) \\ &= \frac{1}{\sqrt{5}} \log \left\{ \frac{\sqrt{x+2}-\sqrt{5}}{\sqrt{x+2}+\sqrt{5}} \right\}.\end{aligned}$$

EXERCISE 1·4

Integrate the following functions :

1. $x[(x-1)\sqrt{x+2}]$.
2. $x[(x+3)\sqrt{x+1}]$.
3. $x[(x-1)\sqrt{x+2}]$.
4. $x[(x+2)\sqrt{x-1}]$.
5. $x[(x+2)\sqrt{x+3}]$.

§ 1·7. Integrate $1/(ax^2+bx+c)\sqrt{px+q}$.

In some cases put $(px+q)=t^2$

so that $p dx = 2t dt$ and $x = (t^2-q)/p$.

$$\therefore \int \frac{dx}{(ax^2 + bx + c) \sqrt{px + q}} = \int \frac{(2t/p) dt}{\left[\frac{(t^2 - q)^2}{p^2} + b \frac{(t^2 - q)}{p} + c \right] \cdot t}$$

$$= 2p \int \frac{dt}{(At^4 + Bt^2 + C)}.$$

This integrand can be splitted into partial fractions and thus is integrable.

Example. Integrate $(x+2)/(x^2+3x+3) \sqrt{x+1}$.

Sol. Put $x+1=t^2$, so that $dx=2t dt$.

$$\begin{aligned}\therefore \int \frac{(x+2) dx}{(x^2+3x+3) \sqrt{x+1}} &= \int \frac{(t^2+1) 2t dt}{\{(t^2-1)^2 + 3(t^2-1) + 3\} t} \\&= 2 \int \frac{(t^2+1) dt}{t^4 + t^2 + 1} = 2 \int \frac{(t^2+1) dt}{(t^2+t+1)(t^2-t+1)} \\&= \int \left[\frac{1}{(t^2-t+1)} + \frac{1}{(t^2+t+1)} \right] dt \\&= \int \frac{dt}{(t-\frac{1}{2})^2 + (\frac{3}{4})} + \int \frac{dt}{(t+\frac{1}{2})^2 + (\frac{3}{4})} \\&= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t-1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t+1}{\sqrt{3}} \\&= \frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{2\sqrt{x+1}-1}{\sqrt{3}} \right] + \frac{2}{\sqrt{3}} \tan^{-1} \left\{ \frac{2\sqrt{x+1}+1}{\sqrt{3}} \right\}.\end{aligned}$$

8. Integration of $1/(px+q) \sqrt{ax^2+bx+c}$.

In such a case put $px+q=1/t$.

so that $p dx = -\frac{1}{t^2} dt$ and $x = \frac{1}{p} \left(\frac{1}{t} - q \right)$

$$\begin{aligned}\therefore \int \frac{dx}{(px+q) \sqrt{ax^2+bx+c}} &= \int \frac{\left(\frac{1}{p} \left(\frac{1}{t} - q \right) \right) \left(-\frac{1}{t^2} \right) dt}{\frac{1}{t} \sqrt{\left[\frac{a}{p^2} \left(\frac{1}{t} - q \right)^2 + \frac{b}{p} \left(\frac{1}{t} - q \right) + c \right]}}$$

reduces to the form $\int \frac{dt}{\sqrt{At^2+Bt+C}}$ which can be integrated by the method

§ 1.9. Integration of $\frac{1}{(x - k)^r \sqrt{ax^2 + bx + c}}$.

When $x > k$, put $x - k = \frac{1}{t}$ so that

$$\begin{aligned} dx &= \left(-\frac{1}{t^2}\right) dt \text{ and } x = k + \frac{1}{t} = \frac{1+kt}{t} \\ \therefore \int \frac{dx}{(x-k)^r \sqrt{ax^2+bx+c}} &= \int \frac{\left(-\frac{1}{t^2}\right) dt}{\left(\frac{1}{t}\right)^r \sqrt{\left[a\left(\frac{1+kt}{t}\right)^2 + b\left(\frac{1+kt}{t}\right) + c\right]}} \\ &= \int \frac{-t^{-1}}{\sqrt{[a(1+kt)^2 + bt(1+kt) + ct^2]}} \\ &= \int \frac{-t^{-1}}{\sqrt{[(ak^2 + bk + c)t^2 + (2ak + b)t + a]}} \\ &= - \int \frac{t^{r-1}}{\sqrt{At^2 + Bt + C}} \end{aligned}$$

where $A = ak^2 + bk + c$, $B = 2ak + b$, $C = a$

If $r = 1$, this can be integrated by § 1.1 and if $r > 1$ and integer, we can integrate it by the method of § 1.3 or § 1.4. When $x < k$, we put $k - x = \frac{1}{t}$ and proceed as above.

Example 1. Integrate $1/(x+1)\sqrt{x^2+1}$.

Sol. Put $x+1 = 1/t$, so that $dx = -(1/t^2) dt$

$$\begin{aligned} \therefore \int \frac{dx}{(x+1)\sqrt{x^2+1}} &= - \int \frac{(1/t^2) dt}{\frac{1}{t} \sqrt{\left(\frac{1}{t}-1\right)^2 + 1}} \\ &= - \int \frac{dt}{\sqrt{\{(1-t)^2 + t^2\}}} = - \int \frac{dt}{\sqrt{1-2t+2t^2}} \\ &= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2 - t + \frac{1}{2}}} = \frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{[(t-\frac{1}{2})^2 + (\frac{1}{2})^2]}} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{\sqrt{2}} \sinh^{-1} \left\{ \frac{(t - \frac{1}{2})}{\frac{1}{2}} \right\} = -\frac{1}{\sqrt{2}} \sinh^{-1} (2t - 1) \\
 &= -\frac{1}{\sqrt{2}} \sinh^{-1} \left(\frac{2}{1+x} - 1 \right) = -\frac{1}{\sqrt{2}} \sinh^{-1} \left(\frac{1-x}{1+x} \right).
 \end{aligned}$$

Example 2. Integrate $1/[(x+2)\sqrt{x^2+6x+7}]$.

Sol. Put $x+2 = 1/t$, so that $dx = (-1/t^2) dt$

$$\begin{aligned}
 &\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}} \\
 &= \int \frac{(-1/t^2) dt}{\frac{1}{t}\sqrt{\left(\frac{1-2t}{t}\right)^2 + 6\left(\frac{1-2t}{t}\right) + 7}}, \quad \because x = \frac{1}{t} - 2 = \frac{1-2t}{t} \\
 &= -\int \frac{dt}{\sqrt{(1-2t)^2 + 6t(1-2t) + 7t^2}} \\
 &= -\int \frac{dt}{\sqrt{1+2t-t^2}} \\
 &= -\int \frac{dt}{\sqrt{[(\sqrt{2})^2 - (t-1)^2]}} \\
 &= -\sin^{-1} \left[\frac{t-1}{\sqrt{2}} \right] \\
 &= -\sin^{-1} \left[\frac{1/(x+2) - 1}{\sqrt{2}} \right] = -\sin^{-1} \left[\frac{1-(x+2)}{(x+2)\sqrt{2}} \right] \\
 &= \sin^{-1} [(x+1)/\{(x+2)\sqrt{2}\}].
 \end{aligned}$$

Example 3. Integrate $\sqrt{x+1}/[(x+2)\sqrt{x+3}]$.

$$\text{Sol. } \int \frac{\sqrt{x+1} dx}{(x+2)\sqrt{x+3}} = \int \frac{(x+1) dx}{(x+2)\sqrt{[(x+3)(x+1)]}},$$

multiplying num. and denom. by $\sqrt{x+1}$

$$\begin{aligned}
 &= \int \frac{(x+2-1) dx}{(x+2)\sqrt{x^2+4x+3}} \\
 &= \int \frac{dx}{\sqrt{x^2+4x+3}} - \int \frac{dx}{(x+2)\sqrt{[(x^2+4x+3)]}} \\
 &= \int \frac{dx}{\sqrt{[(x+2)^2-1]}} - \int \frac{(-1/t^2) dt}{(1/t)\sqrt{[\{(1-2t)/t\}^2 + 4\{(1-2t)/t\} + 3]}}
 \end{aligned}$$

By putting $x+2 = 1/t$ in the second integral)

$$\begin{aligned}
 &= \cosh^{-1}(x+2) + \int \frac{dt}{\sqrt{[(1-2t)^2 + 4t(1-2t) + 3t^2]}} \\
 &= \cosh^{-1}(x+2) + \int \frac{dt}{\sqrt{(1-t^2)}} = \cosh^{-1}(x+2) + \sin^{-1} t \\
 &= \cosh^{-1}(x+2) + \sin^{-1}\{1/(x+2)\}.
 \end{aligned}$$

Example 4. Integrate $\frac{1}{(x-1)^2 \sqrt{1-x^2}}$.

Sol. Here $x < 1$, so we put $1-x = \frac{1}{t}$ and so $dx = \frac{1}{t^2} dt$ and $x = \frac{t-1}{t}$

$$\begin{aligned}
 \therefore \int \frac{dx}{(x-1)^2 \sqrt{1-x^2}} &= \int \frac{\frac{1}{t^2} dt}{\frac{1}{t^2} \sqrt{\left[1 - \left(\frac{t-1}{t}\right)^2\right]}} \\
 &= \int \frac{t dt}{\sqrt{t^2 - (t-1)^2}} = \int \frac{t dt}{\sqrt{(2t-1)}}
 \end{aligned}$$

Again put $2t-1 = u^2$ so that $dt = u du$

$$\begin{aligned}
 \therefore \int \frac{dx}{(x-1)^2 \sqrt{1-x^2}} &= \int \frac{\frac{1}{2} (u^2 + 1)}{u} u du \\
 &= \frac{1}{2} \int (u^2 + 1) du \\
 &= \frac{1}{2} \left[\frac{u^3}{3} + u \right] \\
 &= \frac{1}{6} u (u^2 + 3) \\
 &= \frac{1}{6} \sqrt{(2t-1)(2t-1+3)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{6} \sqrt{\left[\frac{2}{1-x}-1\right]\left[\frac{2}{1-x}+2\right]} \\
 &= \frac{1}{3} \sqrt{\frac{1+x}{1-x} \left[\frac{2-x}{1-x}\right]} \\
 &= \frac{1}{3} \frac{\sqrt{(1+x)(2-x)}}{(1-x)^{3/2}}
 \end{aligned}$$

EXERCISE 1.5

1. Integrate $2/\{(2x+3)\sqrt{x^2-4}\}$

2. Integrate $1/[(1+x)\sqrt{1+x-x^2}]$

3. Show that $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} = \frac{1}{\sqrt{3}}$
4. Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}, x > 1$
5. Integrate $\frac{1}{(x-2)^2\sqrt{x^2+2x-3}}$
6. Evaluate $\int \frac{(2x-3)dx}{(x-1)\sqrt{x+2}}$.
- (Purvanchal 2004)

§ 1.10. Integration of $1/(Ax^2 + B)\sqrt{(Cx^2 + D)}$.

In such cases put $x = 1/t$, so that $dx = -(1/t^2)dt$.

$$\begin{aligned} \therefore \int \frac{dx}{(Ax^2 + B)\sqrt{(Cx^2 + D)}} &= \int \frac{-(1/t^2)dt}{\left(\frac{A}{t^2} + B\right)\sqrt{\left(\frac{C}{t^2} + D\right)}} \\ &= - \int \frac{t dt}{(A + Bt^2)\sqrt{(C + Dt^2)}} \end{aligned} \quad \dots(1)$$

Now put $C + Dt^2 = u^2$

Then $2Dt dt = 2u du$ or $t dt = (1/D)u du$

\therefore from (1), we get

$$\begin{aligned} \int \frac{dx}{(Ax^2 + B)\sqrt{(Cx^2 + D)}} &= -\frac{1}{D} \int \frac{u du}{[A + B\{(u^2 - C)/D\}]u} \\ &= \int \frac{du}{[(AD - BC) + Bu^2]} = -\frac{1}{B} \int \frac{du}{[u^2 + \{(AD - BC)/B\}]} \end{aligned}$$

which can be easily integrated as the integrant is of the form

$$\int \frac{dx}{x^2 + a^2} \text{ or } \int \frac{dx}{x^2 - a^2}.$$

Example 1. Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)dt$

$$\begin{aligned} \therefore \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} &= \int \frac{-(1/t^2)dt}{\{1+(1/t^2)\}\sqrt{\{1-(1/t^2)\}}} \\ &= - \int \frac{t dt}{(t^2+1)\sqrt{(t^2-1)}} = - \int \frac{z dz}{(z^2+2)z}, \text{ putting } t^2 - 1 = z^2 \end{aligned}$$

$$\begin{aligned}
 &= - \int \frac{dz}{z^2 + 2} = - \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{z}{\sqrt{2}} \right] \\
 &= - \frac{1}{\sqrt{2}} \tan^{-1} \left\{ \frac{\sqrt{(t^2 - 1)}}{\sqrt{2}} \right\} = - \frac{1}{\sqrt{2}} \tan^{-1} \left\{ \frac{\sqrt{(1 - x^2)}}{x \sqrt{2}} \right\}.
 \end{aligned}$$

Example 2. Integrate $1/[(x^2 + 1)\sqrt{(x^2 - 1)}]$.

Sol. Put $x = 1/t$, so that $dx = -(1/t^2) dt$

$$\begin{aligned}
 \therefore \int \frac{dx}{(x^2 + 1)\sqrt{(x^2 - 1)}} &= \int \frac{-(1/t^2) dt}{\{(1/t^2) + 1\}\sqrt{\{(1/t^2) - 1\}}} \\
 &= - \int \frac{t dt}{(1 + t^2)\sqrt{(1 - t^2)}} = - \int \frac{-z dz}{(2 - z^2).z}, \text{ where } 1 - t^2 = z^2 \\
 &= \int \frac{dz}{(\sqrt{2})^2 - z^2} = \frac{1}{2\sqrt{2}} \log \frac{z - \sqrt{2}}{z + \sqrt{2}} \\
 &= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{(1+t)^2} - \sqrt{2}}{\sqrt{(1-t)^2} + \sqrt{2}} \\
 &= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{(x^2 - 1)} - x\sqrt{2}}{\sqrt{(x^2 - 1)} + x\sqrt{2}}
 \end{aligned}$$

Example 3. Integrate $1/[(x^2 + 1)\sqrt{(1 + x^2)}]$.

Sol. Put $x = 1/t$, so that $dx = -(1/t^2) dt$

$$\begin{aligned}
 \therefore \int \frac{dx}{(x^2 + 1)\sqrt{(1 + x^2)}} &= \int \frac{-(1/t^2) dt}{\{(1/t^2) + 1\}\sqrt{1 + (1/t)^2}} \\
 &= - \int \frac{t dt}{(1 + t^2)\sqrt{(1 + t^2)}} = - \int \frac{z dz}{z^2.z}, \text{ putting } 1 + t^2 = z^2 \\
 &= - \int \frac{1}{z^2} dz = \frac{1}{z} = \frac{1}{\sqrt{1 + t^2}} \\
 &= \frac{1}{\sqrt{1 + (1/x)^2}} = \frac{x}{\sqrt{1 + x^2}}.
 \end{aligned}$$

Example 4. Integrate $x\sqrt{(a^2 - x^2)/(a^2 + x^2)}$.

$$\begin{aligned}
 \text{Sol. } \int \frac{x\sqrt{(a^2 - x^2)} dx}{(a^2 + x^2)} &= - \int \frac{t \cdot t dt}{a^2 + (a^2 - t^2)}, \text{ putting } a^2 - x^2 = t^2 \\
 &= - \int \frac{t^2 dt}{(2a^2 - t^2)} = - \int \left(-1 + \frac{2a^2}{2a^2 - t^2} \right) dt \\
 &= \int dt - 2a^2 \int \frac{dt}{(2a^2 - t^2)}
 \end{aligned}$$

$$\begin{aligned}
 &= t - 2a^2 \frac{1}{2\sqrt{(2a^2)}} \log \left(\frac{a\sqrt{2} + t}{a\sqrt{2} - t} \right) \\
 &= \sqrt{(a^2 - x^2)} - \frac{a}{\sqrt{2}} \log \left\{ \frac{a\sqrt{2} + \sqrt{(a^2 - x^2)}}{a\sqrt{2} - \sqrt{(a^2 - x^2)}} \right\}.
 \end{aligned}$$

EXERCISE 1·6

Show that $\int \frac{dx}{(x^2 + a^2)\sqrt{x^2 + b^2}}$

$$= \frac{1}{2a\sqrt{a^2 - b^2}} \log \frac{x\sqrt{(a^2 - b^2)} + a\sqrt{(x^2 + b^2)}}{x\sqrt{(a^2 - b^2)} - a\sqrt{(x^2 + b^2)}}, b^2 < a^2$$

Show that

$$\int \frac{dx}{(2x^2 + 3)\sqrt{x^2 - 4}} = \frac{1}{2\sqrt{33}} \log \left\{ \frac{x\sqrt{11} + \sqrt{3x^2 - 12}}{x\sqrt{11} - \sqrt{3x^2 - 12}} \right\}$$

Integrate $1/[(3 + 4x^2)\sqrt{4 - 3x^2}]$.

Integrate $1/[(x^2 + 4)\sqrt{x^2 + 9}]$.

1·11. Integration of $x^m (a + bx^n)^p$, where m, n and p are not necessarily integers.

The integration of the above function will be considered in the following three cases.

Case 1. When p is a positive integer, expand $(a + bx^n)^p$ by the binomial theorem and each term may then be integrated.

Case 2. When $\frac{m+1}{n}$ is an integer and $p = \frac{r}{s}$ when r and s are integers, put $a + bx^n = u^s$. With this substitution the given integral is reduced to the form

$$\frac{s}{bn} \int \left(\frac{u^s - a}{b} \right)^k u^{ps+s-1} du$$

where $k = \frac{m+1}{n} - 1$ is an integer. It can be evaluated by expanding $(u^s - a)^k$ by binomial theorem, if k is positive. If k is negative, the method of partial fraction may be applied.

Case 3. When $p + \frac{m+1}{n}$ is an integer, but p is not an integer. In this case we

put $x = \frac{1}{t}$. With this substitution the given integral is reduced to the form

$$- \int t^{-(m+np+2)} (b + at^n)^p dt$$

Now if $\frac{m+np+1}{n}$ is an integer i.e. if $\frac{m+1}{n} + p$ is an integer, the above integral will become under the case 2, which has been discussed above.

Remark. If these cases fails, the method of successive reduction can be applied which will be discussed in the chapter 2, § 2.10.

Example 1. Integrate $x^{1/2} (1+x^{3/4})^3$.

Sol. Expanding $(1+x^{3/4})^3$ by binomial theorem, we get

$$\begin{aligned}\int x^{1/2} (1+x^{3/4})^3 dx &= \int x^{1/2} [1 + 3x^{3/4} + 3x^{3/2} + x^{9/4}] dx \\ &= \int x^{1/2} + 3x^{5/4} + 3x^{3/2} + x^{11/4} dx \\ &= \frac{2}{3}x^{3/2} + \frac{4}{5}x^{9/4} + x^3 + \frac{4}{15}x^{15/4}\end{aligned}$$

Example 2. Integrate $x^3 (2+x^4)^{2/3}$.

Sol. Here $m = 3$, $n = 4$, $p = \frac{2}{3}$ and $\frac{m+1}{n} = \frac{3+1}{4} = 1$ which is an integer.

So we put $2+x^4 = u^3$ so that $4x^3 dx = 3u^2 du$.

$$\begin{aligned}\therefore \int x^3 (2+x^4)^{2/3} dx &= \int \frac{3u^4}{4} du = \frac{3u^5}{20} \\ &= \frac{3}{20} (2+x^4)^{5/3}\end{aligned}$$

Example 3. Integrate $\frac{(x-x^3)^{1/3}}{x^4}$.

$$\begin{aligned}\text{Sol. } I &= \int \frac{(x-x^3)^{1/3}}{x^4} = \int x^{-4} x^{1/3} (1-x^2)^{1/3} dx \\ &= \int x^{-11/3} (1-x^2)^{1/3} dx\end{aligned}$$

Here $m = -\frac{11}{3}$, $n = 2$, $p = \frac{1}{3}$

and $\frac{m+1}{n} + p = -1$ which is an integer.

So, we put $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2} dt$

$$\begin{aligned}\text{Then } I &= \int t^{11/3} \left(1 - \frac{1}{t^2}\right)^{1/3} \left(-\frac{1}{t^2}\right) dt \\ &= - \int t(t^2-1)^{1/3} dt\end{aligned}$$

Put $t^2 - 1 = u^3$ so that $t dt = \frac{3}{2} u^2 du$

$$\begin{aligned}\therefore I &= - \int u \cdot \frac{3}{2} u^2 du = - \frac{3}{2} \int u^3 du \\ &= - \frac{3u^4}{8} \\ &= - \frac{3}{8} [t^2 - 1]^{4/3} \\ &= - \frac{3}{8} \left(\frac{1}{x^2} - 1 \right)^{4/3} = - \frac{3}{8} \left(\frac{1-x^2}{x^2} \right)^{4/3}.\end{aligned}$$

Solved Examples on Integration by Rationalising

Example 4. Integrate $\frac{1}{[\sqrt{x} + \sqrt{(1-x)}]}$ (GKP 2016)

$$\begin{aligned}\text{Sol. } \int \frac{dx}{[\sqrt{x} + \sqrt{(1-x)}]} &= \int \frac{[\sqrt{x} - \sqrt{(1-x)}] dx}{[\sqrt{x} + \sqrt{(1-x)}][\sqrt{x} - \sqrt{(1-x)}]} \\ &= \int \frac{[\sqrt{x} - \sqrt{(1-x)}] dx}{x - (1-x)} = \int \frac{[\sqrt{x} - \sqrt{(1-x)}] dx}{(2x-1)} \\ &= \int \frac{\sqrt{x} dx}{(2x-1)} - \int \frac{\sqrt{(1-x)} dx}{(2x-1)} \\ &= \int \frac{x dx}{(2x-1)\sqrt{x}} - \int \frac{(1-x) dx}{(2x-1)\sqrt{(1-x)}} \\ &= \frac{1}{2} \int \frac{(2x-1+1) dx}{(2x-1)\sqrt{x}} + \frac{1}{2} \int \frac{(2x-1-1) dx}{(2x-1)\sqrt{(1-x)}} \\ &= \frac{1}{2} \left[\int \frac{dx}{\sqrt{x}} + \int \frac{dx}{(2x-1)\sqrt{x}} \right] \\ &\quad + \frac{1}{2} \left[\int \frac{dx}{\sqrt{(1-x)}} - \int \frac{dx}{(2x-1)\sqrt{(1-x)}} \right] \quad \dots(i)\end{aligned}$$

$$\text{Now } \int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx = 2x^{1/2}$$

$$\int \frac{dx}{\sqrt{(1-x)}} = \int (1-x)^{-1/2} dx = -2(1-x)^{1/2};$$

$$\begin{aligned}\int \frac{dx}{(2x-1)\sqrt{x}} &= \int \frac{2t dt}{(2t^2-1)t}, \text{ putting } x = t^2 \\ &= 2 \int \frac{dt}{(2t^2-1)} = \int \frac{dt}{(t^2-\frac{1}{2})} = \frac{1}{2 \cdot (1/\sqrt{2})} \log \left\{ \frac{t - (1/\sqrt{2})}{t + (1/\sqrt{2})} \right\} \\ &= \frac{1}{\sqrt{2}} \log \left(\frac{t\sqrt{2}-1}{t\sqrt{2}+1} \right) = \frac{1}{\sqrt{2}} \log \left\{ \frac{\sqrt{(2x)-1}}{\sqrt{(2x)+1}} \right\}\end{aligned}$$

$$\text{Also } \int \frac{dx}{(2x-1)\sqrt{1-x}} = \int \frac{-2t dt}{\{2(1-t^2)-1\} \cdot t}, \text{ putting } 1-x=t^2$$

$$= -2 \int \frac{dt}{(1-2t^2)} = 2 \int \frac{dt}{(2t^2-1)} = \int \frac{dt}{t^2 - (\frac{1}{2})}$$

$$= \frac{1}{\sqrt{2}} \log \left(\frac{t\sqrt{2}-1}{t\sqrt{2}+1} \right) = \frac{1}{\sqrt{2}} \log \left[\frac{\sqrt{2(1-x)}-1}{\sqrt{2(1-x)}+1} \right]$$

\therefore From (i), the given integral

$$= \frac{1}{2} \left[2x^{1/2} + \frac{1}{\sqrt{2}} \log \left\{ \frac{\sqrt{2x}-1}{\sqrt{2x}+1} \right\} \right]$$

$$+ \frac{1}{2} \left[-2(1-x^2)^{1/2} - \frac{1}{\sqrt{2}} \log \left\{ \frac{\sqrt{2(1-x)}-1}{\sqrt{2(1-x)}+1} \right\} \right].$$

$$= \sqrt{x} + \frac{1}{2\sqrt{2}} \log \left\{ \frac{\sqrt{2x}-1}{\sqrt{2x}+1} \right\} - \sqrt{(1-x)}$$

$$- \frac{1}{2\sqrt{2}} \log \left\{ \frac{\sqrt{2(1-x)}-1}{\sqrt{2(1-x)}+1} \right\}.$$

Example 5. Integrate $1/\{\sqrt{x+a} + \sqrt{x+b}\}$.

$$\text{Sol. } \int \frac{dx}{\{\sqrt{x+a} + \sqrt{x+b}\}} = \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(x+a) - (x+b)} dx.$$

(by rationalising the denominator)

$$= \frac{1}{(a-b)} \int [\sqrt{x+a} - \sqrt{x+b}] dx$$

$$= \frac{1}{(a-b)} [\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2}]$$

$$= \frac{2}{3(a-b)} [(x+a)^{3/2} - (x+b)^{3/2}].$$

Example 6. Integrate $x/[\sqrt{x+a} + \sqrt{x+b}]$.

$$\text{Sol. } \int \frac{x dx}{\{\sqrt{x+a} + \sqrt{x+b}\}} = \int \frac{x \{\sqrt{x+a} - \sqrt{x+b}\} dx}{\{(x+a) - (x+b)\}}$$

(by rationalising the denominator)

$$= \frac{1}{(a-b)} \left[\int x \sqrt{x+a} dx - \int x \sqrt{x+b} dx \right]$$

$$= \frac{1}{(a-b)} \left[\int (t^2-a) \cdot t \cdot 2t dt - \int (u^2-b) \cdot u \cdot 2u du \right],$$

(by putting $x+a=t^2$ and $(x+b)=u^2$ in first and second integrals respectively)

$$= \frac{2}{a-b} \left[(\frac{1}{5}t^5 - \frac{1}{3}at^3) - (\frac{1}{5}u^5 - \frac{1}{3}bu^3) \right]$$

$$= \frac{2}{a-b} \left[\frac{1}{15}t^3(3t^2-5a) - \frac{1}{15}u^3(3u^2-5b) \right]$$

$$\begin{aligned}
 &= \frac{2}{15(a-b)} \left[(x+a)^{3/2} \{3(x+a) - 5a\} \right. \\
 &\quad \left. - (x+b)^{3/2} \{3(x+b) - 5b\} \right] \\
 &= \frac{2}{15(a-b)} \left[(x+a)^{3/2} (3x-2a) - (x+b)^{3/2} (3x-2b) \right].
 \end{aligned}$$

Example 7. Integrate $1/[x + \sqrt{x^2 - 1}]$.

$$\begin{aligned}
 \text{Sol. } \int \frac{dx}{[x + \sqrt{x^2 - 1}]} &= \int \frac{[x - \sqrt{x^2 - 1}] dx}{x^2 - (x^2 - 1)}, \\
 &\quad \text{(by rationalising the denominator)} \\
 &= \int [x - \sqrt{x^2 - 1}] dx = \int x dx - \int \sqrt{x^2 - 1} dx \\
 &= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\log[x + \sqrt{x^2 - 1}].
 \end{aligned}$$

Example 8. Integrate $\frac{1}{\sqrt{1+x} - \sqrt{x}}$. (GKP 2014)

$$\begin{aligned}
 \text{Sol. } I &= \int \frac{1}{\sqrt{1+x} - \sqrt{x}} dx \\
 &= \int \frac{\sqrt{1+x} + \sqrt{x}}{\sqrt{1+x} - \sqrt{x}} \times \frac{1}{\sqrt{1+x} + \sqrt{x}} dx \\
 &= \int \frac{\sqrt{1+x} + \sqrt{x}}{1+x-x} dx \\
 &= \int \sqrt{1+x} dx + \int \sqrt{x} dx \\
 &= \frac{(1+x)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} \\
 I &= \frac{2}{3} [(1+x)^{3/2} + x^{3/2}]
 \end{aligned}$$

Solved Examples on Integration by Algebraic Substitutions

Example 9. Integrate $x^{1/2}/(1-x^{1/3})$.

Sol. The L.C.M. of 2 and 3 i.e., of the denominator of the two fractional numbers namely $\frac{1}{2}$ and $\frac{1}{3}$ is 6. \therefore Put $x = t^6$ so that $dx = 6t^5 dt$.

$$\begin{aligned}
 \therefore \int \frac{x^{1/2} dx}{(1+x^{1/3})} &= \int \frac{t^3 \cdot 6t^5 dt}{(1+t^2)} = 6 \int \frac{t^8 dt}{(t^2+1)} \\
 &= 6 \int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{t^2+1} \right) dt, \text{ (by actual division)}
 \end{aligned}$$

$$= 6 \left[\left(\frac{1}{7} t^7 - \frac{1}{5} t^5 + \frac{1}{3} t^3 - t + \tan^{-1} t \right) \right], \text{ where } t = x^{1/6}$$

$$= 6 \left[\frac{1}{7} x^{7/6} - \frac{1}{5} x^{5/6} - \frac{1}{3} x^{1/2} - x^{1/6} + \tan^{-1} (x^{1/6}) \right].$$

Example 10. Integrate $1/(x^{1/2} + x^{1/3})$.

Sol. As in Example 1 above, put $x = t^6$ or $t = x^{1/6}$

$$\therefore \int \frac{dx}{(x^{1/2} + x^{1/3})} = \int \frac{6t^5 dt}{(t^3 + t^2)} = 6 \int \frac{t^3}{(t+1)} dt$$

$$= 6 \int \left[t^2 - t + 1 - \frac{1}{(t+1)} \right] dt, \text{ (by actual division)}$$

$$= 6 \left[\frac{1}{3} t^3 - \frac{1}{2} t^2 + t - \log(t+1) \right]$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6 \log(x^{1/6} + 1).$$

Example 11. Integrate $1/x \sqrt{x^2 + 1}$.

Sol. Putting $x^2 + 1 = t^2$, so that $x dx = t dt$.

$$\begin{aligned} \therefore \int \frac{dx}{x \sqrt{x^2 + 1}} &= \int \frac{1}{x \cdot t} \frac{t dt}{x} = \int \frac{1}{x^2} dt = \int \frac{1}{(t^2 - 1)} dt \\ &= \frac{1}{2} \log \left(\frac{t-1}{t+1} \right) = \frac{1}{2} \log \left\{ \frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 1} + 1} \right\}. \end{aligned}$$

Example 12. Integrate $1/x \sqrt{(a^n + x^n)}$.

Sol. Put $a^n + x^n = t^2$, so that $nx^{n-1} dx = 2t dt$

$$\therefore \int \frac{dx}{x \sqrt{(a^n + x^n)}} = \frac{1}{n} \int \frac{nx^{n-1} dx}{x^n \sqrt{(a^n + x^n)}},$$

$$\begin{aligned} &\quad \text{(by multiplying num. and denom. by } nx^{n-1}) \\ &= \frac{1}{n} \int \frac{2t dt}{(t^2 - a^n) t} = \frac{2}{n} \int \frac{dt}{t^2 - (a^{n/2})^2} \\ &= \frac{2}{n} \cdot \frac{1}{2a^{n/2}} \log \left(\frac{t - a^{n/2}}{t + a^{n/2}} \right) = \frac{1}{na^{n/2}} \log \left\{ \frac{\sqrt{(a^n + x^n)} - a^{n/2}}{\sqrt{(a^n + x^n)} + a^{n/2}} \right\} \\ &= \frac{1}{na^{n/2}} \log \left\{ \frac{\sqrt{(a^n + x^n)} - \sqrt{a^n}}{\sqrt{(a^n + x^n)} + \sqrt{a^n}} \right\} \end{aligned}$$

Example 13. Integrate $1/x \sqrt{x^6 + 1}$.

Sol. Put $x^6 + 1 = t^2$, so that $6x^5 dx = 2t dt$ or $3x^5 dx = t dt$.

$$\therefore \int \frac{dx}{x \sqrt{x^6 + 1}} = \frac{1}{3} \int \frac{3x^5 dx}{x^6 \sqrt{x^6 + 1}} = \frac{1}{3} \int \frac{t dt}{(t^2 - 1)t}$$

$$= \frac{1}{3} \int \frac{dt}{t^2 - 1} = \frac{1}{3} \cdot \frac{1}{2} \log \left(\frac{t-1}{t+1} \right) = \frac{1}{6} \log \left[\frac{\sqrt{x^6 + 1} - 1}{\sqrt{x^6 + 1} + 1} \right]$$

Example 14. Integrate $1/\sqrt{2e^x - 1}$.

Sol. Put $2e^x - 1 = t^2$, so that $2e^x dx = 2t dt$,

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{2e^x - 1}} &= \int \frac{2e^x dx}{2e^x \sqrt{2e^x - 1}}, \\ &\quad (\text{by multiplying num. and denom. by } 2e^x) \\ &= \int \frac{2t dt}{(t^2 + 1)t} = 2 \int \frac{dt}{t^2 + 1} = 2 \tan^{-1} t = 2 \tan^{-1} \sqrt{2e^x - 1}. \end{aligned}$$

Example 15. Putting $x = 1/t$, show that $\int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx = 6$.

$$\begin{aligned} \text{Sol. } \int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx &= \int_3^1 \frac{[(1/t) - (1/t^3)]^{1/3} (-1/t^2) dt}{(1/t)^4}, \\ &\quad (\text{by putting } x = 1/t) \end{aligned}$$

$$= - \int_3^1 (t^2 - 1)^{1/3} t dt, \text{ on simplifying}$$

$$= - \int_2^0 (u^3)^{1/3} \cdot \frac{3}{2} u^2 du, \text{ (by putting } t^2 - 1 = u^3 \text{ or } 2t dt = 3u^2 du)$$

$$= - \frac{3}{2} \int_2^0 u^3 du = - \frac{3}{2} \left[\frac{1}{4} u^4 \right]_2^0 = - \frac{3}{2} \left[-\frac{1}{4} (2)^4 \right] = 6.$$

Henced Proved.

Example 16. Evaluate $\int_1^3 \left(x + \frac{1}{x} \right)^{3/2} \left(\frac{x^2 - 1}{x^2} \right) dx$.

$$\begin{aligned} \text{Sol. } \int_1^3 \left(x + \frac{1}{x} \right)^{3/2} \left(\frac{x^2 - 1}{x^2} \right) dx &= \int_1^3 \left(x + \frac{1}{x} \right)^{3/2} \left(1 - \frac{1}{x^2} \right) dx \\ &= \int_2^{10/3} t^{3/2} dt, \text{ (by putting } x + \frac{1}{x} = t, \text{ so that } \left(1 - \frac{1}{x^2} \right) dx = dt) \\ &= \left[\frac{2}{5} t^{5/2} \right]_2^{10/3} = \frac{2}{5} \left[\left(\frac{10}{3} \right)^{5/2} - (2)^{5/2} \right] \\ &= \frac{2}{5} (2)^{5/2} \left[\left(\frac{5}{3} \right)^{5/2} - 1 \right] = \frac{8\sqrt{2}}{5} \left[\frac{25}{9} \left(\frac{5}{3} \right)^{1/2} - 1 \right] \end{aligned}$$

Solved Examples on Integration by Trigonometric Substitution

Example 17. Evaluate $\int_a^\infty \frac{dx}{x^4(a^2+x^2)^{1/2}}$.

$$\begin{aligned} \int_a^\infty \frac{dx}{x^4(a^2+x^2)^{1/2}} &= \int_{\theta=\pi/4}^{\pi/2} \frac{a \sec^2 \theta d\theta}{a^4 \tan^4 \theta \cdot a \sec \theta} \\ &= \frac{1}{a^4} \int_{\pi/4}^{\pi/2} \frac{\cos^3 \theta d\theta}{\sin^4 \theta} = \frac{1}{a^4} \int_{\pi/4}^{\pi/2} \frac{(1 - \sin^2 \theta) \cos \theta d\theta}{\sin^4 \theta} \\ &= \frac{1}{a^4} \int_{1/\sqrt{2}}^1 \frac{(1 - t^2) dt}{t^4}, \quad (\text{by putting } \sin \theta = t) \\ &= \frac{1}{a^4} \int_{1/\sqrt{2}}^1 (t^{-4} - t^{-2}) dt = \frac{1}{a^4} \left[-\frac{1}{3} t^{-3} + t^{-1} \right]_{1/\sqrt{2}}^1 \\ &= \frac{1}{a^4} \left[-\frac{1}{3} + 1 + \frac{2}{3} \sqrt{2} - \sqrt{2} \right] = \frac{1}{a^4} \left[\frac{2}{3} - \frac{1}{3} \sqrt{2} \right] = (2 - \sqrt{2})/3a^4. \end{aligned}$$

Example 18. Evaluate $\int_0^a x \sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)} dx$.

Sol. Put $x^2 = a^2 \cos 2\theta$, so that $2x dx = -2a^2 \sin 2\theta d\theta$

Also when $x = 0$, $\cos 2\theta = 1$ or $2\theta = \frac{1}{2}\pi$ or $\theta = \frac{1}{4}\pi$

and when $x = a$, $\cos 2\theta = 0$ or $2\theta = 0$ or $\theta = 0$

$$\begin{aligned} \therefore \int_0^a x \sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)} dx &= -a^2 \int_{\pi/4}^0 \sqrt{\left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)} \sin 2\theta d\theta \\ &= -a^2 \int_{\pi/4}^0 \sqrt{\left(\frac{2\sin^2 \theta}{2\cos^2 \theta}\right)} 2\sin \theta \cos \theta d\theta = -a^2 \int_{\pi/4}^0 2\sin^2 \theta d\theta \\ &= a^2 \int_0^{\pi/4} (1 - \cos 2\theta) d\theta = a^2 \left(\theta - \frac{1}{2} \sin 2\theta \right)_0^{\pi/4} \\ &= a^2 \left[\frac{\pi}{4} - \frac{1}{2} \sin \frac{1}{2}\pi \right] = a^2 \left[\frac{\pi}{4} - \frac{1}{2} \right] = \frac{1}{4} a^2 (\pi - 2). \end{aligned}$$

Example 19. Integrate $1/x^2 \sqrt{x^2 - a^2}$.

Sol. Put $x = a \sec \theta$, so that $dx = a \sec \theta \tan \theta d\theta$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} &= \int \frac{a \sec \theta \tan \theta d\theta}{a^2 \sec^2 \theta \sqrt{a^2 \sec^2 \theta - a^2}} \\ &= \frac{1}{a^2} \int \frac{d\theta}{\sec \theta} = \frac{1}{a^2} \int \cos \theta d\theta = \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \sqrt{1 - \cos^2 \theta} \end{aligned}$$

(GKP 2016)

$$= \frac{1}{a^2} \sqrt{\left(1 - \frac{a^2}{x^2}\right)} = \frac{\sqrt{x^2 - a^2}}{a^2 x}$$

Example 20. Integrate $1/(2ax + x^2)^{3/2}$.

$$\text{Sol. } 2ax + x^2 = (x + a)^2 - a^2.$$

Put $x + a = a \sec \theta$, so that $dx = a \sec \theta \tan \theta d\theta$

$$\text{and } 2ax + x^2 = (x + a)^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 \tan^2 \theta.$$

$$\begin{aligned} \therefore \int \frac{dx}{(2ax + x^2)^{3/2}} &= \int \frac{a \sec \theta t \tan \theta d\theta}{(a^2 \tan^2 \theta)^{3/2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a^3 \tan^3 \theta} \\ &= \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = -\frac{1}{a^2} \cdot \frac{1}{\sin \theta} = -\frac{1}{a^2 \sqrt{1 - \cos^2 \theta}} \\ &= -\frac{1}{a^2 \sqrt{[1 - \{a^2/(x+a)^2\}]}} = -\frac{(x+a)}{a^2 \sqrt{2ax+x^2}}. \end{aligned}$$

Example 21. Integrate $\log [x + \sqrt{x^2 + a^2}]$ with respect to x .

$$\text{Sol. } \int \log \{x + \sqrt{x^2 + a^2}\} dx$$

$$= [\log \{x + \sqrt{x^2 + a^2}\}] \cdot x - \int \frac{1 + \frac{1}{2} \cdot 2x \cdot (x^2 + a^2)^{-1/2}}{x + \sqrt{x^2 + a^2}} \cdot x dx,$$

integrating by parts taking 1 as 2nd function.

$$= x \log \{x + \sqrt{x^2 + a^2}\} - \int \frac{x dx}{\sqrt{x^2 + a^2}}, \text{ on simplifying}$$

$$= x \log \{x + \sqrt{x^2 + a^2}\} - \int \frac{t dt}{t}, \text{ where } x^2 + a^2 = t^2$$

$$= x \log \{x + \sqrt{x^2 + a^2}\} - t, \text{ where } t = \sqrt{x^2 + a^2}$$

$$= x \log \{x + \sqrt{x^2 + a^2}\} - \sqrt{x^2 + a^2}.$$

Example 22. Prove that $\int_0^\infty \frac{dx}{[x + \sqrt{x^2 + 1}]^n} = \frac{n}{n^2 - 1}$ ($n > 1$).

(Gorakhpur 2007, 09)

$$\text{Sol. Let } x + \sqrt{x^2 + 1} = t$$

...(1)

$$\text{so that } \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) dx = dt$$

$$\text{or } \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} dx = dt$$

$$\text{or } \frac{t}{\sqrt{x^2 + 1}} dx = dt \text{ or } dx = \sqrt{x^2 + 1} \frac{dt}{t}$$

...(2)

Also $\frac{1}{t} = \frac{1}{x + \sqrt{x^2 + 1}} = \frac{\sqrt{x^2 + 1} - 2x}{(\sqrt{x^2 + 1} + x)(\sqrt{x^2 + 1} - x)}$
 or $\frac{1}{t} = \sqrt{x^2 - 1} - x$

Hence from (1) and (3), we have

$$\therefore t + \frac{1}{t} = 2\sqrt{x^2 - 1}$$

Therefore from (2) we have

$$dx = \frac{1}{2} \frac{1}{t} \left(t + \frac{1}{t} \right) dt$$

or $dx = \frac{t^2 + 1}{2t^2} dt$

when $x = 0, t = 1$ and when $x = \infty, t = \infty$.

$$\begin{aligned}\therefore I &= \int_0^\infty \frac{dx}{[x + \sqrt{x^2 + 1}]^n} = \int_1^\infty \frac{1}{t^n} \frac{t^2 + 1}{2t^2} dt \\ &= \frac{1}{2} \int_0^\infty \left(\frac{1}{t^n} + \frac{1}{t^{n+2}} \right) dt \\ &= \frac{1}{2} \left[\frac{t^{-n+1}}{-n+1} + \frac{t^{-n-1}}{-n-1} \right]_1^\infty \\ &= \frac{1}{2} \left[\frac{-1}{(n-1)t^{n+1}} - \frac{1}{(n+1)t^{n+1}} \right]_1^\infty \\ &= \frac{1}{2} \left[0 - 0 - \left\{ \frac{-1}{n-1} - \frac{1}{n+1} \right\} \right] = \frac{n}{n^2 - 1}\end{aligned}$$

Example 23. Evaluate $\int x \sqrt{\frac{1+x}{1-x}} dx$

Sol. Let $I = \int x \sqrt{\frac{1+x}{1-x}} dx$

Multiplying the numerator and denominator by $\sqrt{1+x}$ we get

$$\begin{aligned}I &= \int \frac{x(1+x)}{\sqrt{1-x^2}} dx \\ &= \int \frac{x dx}{\sqrt{1-x^2}} + \int \frac{x^2 dx}{\sqrt{1-x^2}} \\ &= -\frac{1}{2} \int \frac{-2x}{\sqrt{(1-x^2)}} dx - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx\end{aligned}$$

Now $\int \frac{-2x}{\sqrt{1-x^2}} dx = \int \frac{2t dt}{t}$

where

$$\begin{aligned}1 - x^2 &= t^2 \\&= 2 \int dt = 2t \\&= 2\sqrt{1 - x^2}\end{aligned}$$

and

$$\begin{aligned}\int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx &= \int \sqrt{1-x^2} - \int \frac{dx}{\sqrt{1-x^2}} \\&= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \sin^{-1}x \\&= \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x\end{aligned} \quad \dots(iii)$$

then from (i), (ii) & (iii)

$$I = -\sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x$$

EXERCISE 1.7

1. Integrate $1/\{\sqrt{x+2} + \sqrt{x+1}\}$.
2. Integrate $\sqrt{\frac{x+1}{x-1}}$.
3. Integrate $\sqrt[\{]{(5-x)/(2-x)}}$.
4. Integrate $x^3(1+x^2)^{1/3}$.
5. Show that $\int_0^a \frac{x dx}{\sqrt{a^2 - x^2}} = a$.
6. Show that $\int \frac{(x^2 - 2) dx}{\sqrt{3-x^2}} = -\frac{1}{2} \left[x\sqrt{3-x^2} + \sin^{-1}\left(\frac{x}{\sqrt{3}}\right) \right]$
7. Evaluate $\int \frac{(1-x^2) dx}{(1+x^2)\sqrt{1+x^4}}$.
8. Integrate $(1+x^2)/[(1-x^2)\sqrt{1-3x^2+x^4}]$.
9. Integrate $(1-x)/[(1+x)\sqrt{x+x^2+x^3}]$.
10. Integrate $x^{3/4}(1+x^{2/3})^3$.
11. Integrate $x^{2/3}(1+x^{6/5})^3$.
12. Integrate $x(1+x^2)^{1/2}$.
13. Integrate $x(1+x^2)^{2/3}$.
14. Integrate $x^3(1+x^2)^{1/3}$.
15. Evaluate $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)} dx$.
16. Evaluate $\int \frac{dx}{\sqrt{(x-a)(x-b)}}$.

17. Evaluate $\int_{\beta}^x \frac{dx}{\sqrt{(\alpha-x)(x-\beta)}}.$
18. Evaluate $\int \frac{dx}{(x-\beta)\sqrt{(x-\alpha)(\beta-x)}}.$
19. Evaluate $\int_0^a x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} dx.$
20. Evaluate the integral $\int \frac{x^{1/2}}{1+x^{1/3}} dx.$

(Gorakhpur 2006)

§ 1.12. Integration of $1/(a + b \cos x).$

We wrote

$$\begin{aligned}
 a + b \cos x &= a \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) + b \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) \\
 &= (a+b) \cos^2 \frac{x}{2} + (a-b) \sin^2 \frac{x}{2} \\
 \therefore I &= \int \frac{dx}{a + b \cos x} = \int \frac{dx}{(a+b) \cos^2 \frac{x}{2} + (a-b) \sin^2 \frac{x}{2}} \\
 &= \int \frac{\sec^2 \frac{x}{2} dx}{(a+b) + (a-b) \tan^2 \frac{x}{2}} \\
 &= 2 \int \frac{dt}{(a+b) + (a-b)t^2} \text{ where } t = \tan \frac{x}{2} \\
 &= \frac{2}{a-b} \int \frac{dt}{\frac{a+b}{a-b} + t^2}
 \end{aligned}$$

Now three cases arise.

Case 1. When $a > b$, $\frac{a+b}{a-b}$ is positive.

In this case

$$\begin{aligned}
 I &= \frac{2}{a-b} \cdot \sqrt{\left(\frac{a-b}{a+b}\right)} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} t \right) \\
 &= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left\{ \sqrt{\left(\frac{a-b}{a+b}\right)} \tan \frac{x}{2} \right\}
 \end{aligned}$$

Case 2. When $x < b$, $\frac{a+b}{a-b}$ is negative.

In this case

$$\begin{aligned} I &= \frac{2}{a-b} \int \frac{dt}{t^2 - \frac{b+a}{b-a}} \\ &= -\frac{2}{b-a} \frac{1}{2} \sqrt{\left(\frac{b-a}{b+a}\right)} \log \left(\frac{t - \sqrt{\frac{b+a}{b-a}}}{t + \sqrt{\frac{b+a}{b-a}}} \right) \\ &= \frac{-1}{\sqrt{b^2 - a^2}} \log \left(\frac{t \sqrt{(b-a)} - \sqrt{(b+a)}}{t \sqrt{(b-a)} + \sqrt{(b+a)}} \right) \\ &= \frac{-1}{\sqrt{(b^2 - a^2)}} \log \left(\frac{\sqrt{(b-a)} \tan \frac{x}{2} - \sqrt{(b+a)}}{\sqrt{(b-a)} \tan \frac{x}{2} + \sqrt{(b+a)}} \right) \\ &= \frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{(b-a)} \tan \frac{x}{2} + \sqrt{(b+a)}}{\sqrt{(b-a)} \tan \frac{x}{2} - \sqrt{(b+a)}} \end{aligned}$$

Case 3. When $a = b$, the integral

$$\begin{aligned} I &= \int \frac{dx}{a + a \cos x} = \frac{1}{a} \int \frac{dx}{1 + \cos x} = \frac{1}{2a} \int \frac{dx}{\cos^2 \frac{x}{2}} \\ &= \frac{1}{2a} \int \sec^2 \frac{x}{2} dx = \frac{1}{a} \tan \frac{x}{2}. \end{aligned}$$

1.13. Integration of $1/(a + b \sin x)$.

We write

$$\begin{aligned} a + b \sin x &= a \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) + 2b \sin \frac{x}{2} \cos \frac{x}{2} \\ &= \cos^2 \frac{x}{2} \left(a + 2b \tan \frac{x}{2} + a \tan^2 \frac{x}{2} \right) \end{aligned}$$

$$\begin{aligned} \therefore I &= \int \frac{dx}{a + b \sin x} = \int \frac{\sec^2 \frac{x}{2} dx}{a + 2b \tan \frac{x}{2} + a \tan^2 \frac{x}{2}} \\ &= 2 \int \frac{dt}{a + 2bt + t^2} \text{ where } t = \tan \frac{x}{2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{a} \int \frac{dt}{t^2 + \frac{2b}{a}t + 1} \\
 &= \frac{2}{a} \int \frac{dt}{\left(t + \frac{b}{a}\right)^2 + 1 - \frac{b^2}{a^2}} \\
 &= \frac{2}{a} \int \frac{dt}{\left(t + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}
 \end{aligned}$$

Now three cases arise.

Case 1. When $a > b$, $a^2 - b^2$ is positive. In this case

$$\begin{aligned}
 I &= \frac{2}{a} \cdot \sqrt{\left(\frac{a^2}{a^2 - b^2}\right)} \tan^{-1} \left(\frac{t + \frac{b}{a}}{\sqrt{\left(\frac{a^2 - b^2}{a^2}\right)}} \right) \\
 &= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\frac{at + b}{a^2 - b^2} \right) = \frac{2}{(a^2 - b^2)} \tan^{-1} \left(\frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \right)
 \end{aligned}$$

Case 2. When $a < b$, $a^2 - b^2$ is negative. In this case

$$\begin{aligned}
 I &= \frac{2}{a} \int \frac{dt}{\left(t + \frac{b}{a}\right)^2 - \frac{b^2 - a^2}{a^2}} \\
 &= \frac{2}{a} \cdot \frac{1}{2} \sqrt{\left(\frac{a^2}{b^2 - a^2}\right)} \log \frac{\left(t + \frac{b}{a}\right) - \sqrt{\left(\frac{b^2 - a^2}{a^2}\right)}}{\left(t + \frac{b}{a}\right) + \sqrt{\left(\frac{b^2 - a^2}{a^2}\right)}} \\
 &= \frac{1}{\sqrt{b^2 - a^2}} \log \left(\frac{at + b - \sqrt{(b^2 - a^2)}}{at + b + \sqrt{(b^2 - a^2)}} \right) \\
 &= \frac{1}{\sqrt{b^2 - a^2}} \log \left(\frac{a \tan \frac{x}{2} + b - \sqrt{(b^2 - a^2)}}{a \tan \frac{x}{2} + b + \sqrt{(b^2 - a^2)}} \right)
 \end{aligned}$$

Case 3. When $a = b$, $a^2 - b^2 = 0$. In this case

$$I = \frac{2}{a} \int \frac{dt}{(t+1)^2} = \frac{-2}{a(t+1)}$$

$$= -\frac{2}{a \left(1 + \tan \frac{x}{2}\right)}$$

Example 1. Prove that :

$$\begin{aligned}
 & \int_0^\alpha \frac{d\theta}{\cos \alpha + \cos \theta} = \operatorname{cosec} \alpha \log (\sec \alpha) \\
 \text{Sol. } I &= \int_0^\alpha \frac{d\theta}{\cos \alpha + \cos \theta} \\
 &= \int_0^\alpha \frac{d\theta}{\cos \alpha \left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right) + \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right)} \\
 &= \int_0^\alpha \frac{d\theta}{(1 + \cos \alpha) \cos^2 \frac{\theta}{2} - (1 - \cos \alpha) \sin^2 \frac{\theta}{2}} \\
 &= \int_0^\alpha \frac{\sec^2 \frac{\theta}{2} d\theta}{(1 + \cos \alpha) - (1 - \cos \alpha) \tan^2 \frac{\theta}{2}} \\
 &= \frac{2}{1 - \cos \alpha} \int_{\theta=0}^{\theta=\alpha} \frac{dt}{\frac{1 + \cos \alpha}{1 - \cos \alpha} - t^2} \quad \left(\text{by putting } t = \tan \frac{\theta}{2}\right) \\
 &= \frac{2}{2 \sin^2 \frac{\alpha}{2}} \int_{\theta=0}^{\theta=\alpha} \frac{dt}{\frac{2 \cos^2 \frac{\alpha}{2}}{2 \sin^2 \frac{\alpha}{2}} - t^2} \\
 &= \frac{1}{\sin^2 \frac{\alpha}{2}} \int_{\theta=0}^{\theta=\alpha} \frac{dt}{\cot^2 \frac{\alpha}{2} - t^2} \\
 &= \frac{1}{\sin^2 \frac{\alpha}{2}} \cdot \frac{1}{2 \cot \frac{\alpha}{2}} \left[\log \left(\frac{\cot \frac{\alpha}{2} + t}{\cot \frac{\alpha}{2} - t} \right) \right]_{\theta=0}^{\theta=\alpha} \\
 &= \frac{1}{\sin \alpha} \left[\log \left(\frac{\cot \frac{\alpha}{2} + \tan \frac{\alpha}{2}}{\cot \frac{\alpha}{2} - \tan \frac{\alpha}{2}} \right) \right]_{\theta=0}^{\theta=\alpha} \\
 &= \frac{1}{\sin \alpha} \left[\log \left(\frac{\cot \frac{\alpha}{2} + \tan \frac{\alpha}{2}}{\cot \frac{\alpha}{2} - \tan \frac{\alpha}{2}} \right) - \log 1 \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sin \alpha} \log \frac{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}} \\
 &= \operatorname{cosec} \alpha \log \frac{1}{\cos \alpha} \\
 &= \operatorname{cosec} \alpha \log \sec \alpha.
 \end{aligned}$$

Example 2. Prove that :

$$\int_0^\pi \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \text{ or } \frac{\pi}{a^2 - 1}$$

according as $a <$ or > 1 .

Sol. Let $I = \int_0^\pi \frac{dx}{1 - 2a \cos x + a^2}$

$$\begin{aligned}
 \text{Then } I &= \int_0^\pi \frac{dx}{(1 + a^2) \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) - 2a \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)} \\
 &= \int_0^\pi \frac{dx}{(1 + a^2 - 2a) \cos^2 \frac{x}{2} + (1 + a^2 + 2a) \sin^2 \frac{x}{2}} \\
 &= \int_0^\pi \frac{\sec^2 \frac{x}{2} dx}{(1 - a)^2 + (1 + a)^2 \tan^2 \frac{x}{2}} \\
 &= 2 \int_0^\infty \frac{dt}{(1 - a)^2 + (1 + a)^2 t^2} \text{ where } \tan \frac{x}{2} = t \\
 &= \frac{2}{(1 + a)^2} \int_0^\infty \frac{dt}{\left(\frac{1-a}{1+a}\right)^2 + t^2}
 \end{aligned}$$

If $a < 1$, then $\frac{1-a}{1+a}$ is positive, therefore

$$\begin{aligned}
 I &= \frac{2}{(1 + a)^2} \frac{1 + a}{1 - a} \tan^{-1} \left(\frac{(1 + a)t}{1 - a} \right)_0^\infty \\
 &= \frac{2}{1 - a^2} (\tan^{-1} \infty - \tan^{-1} 0) = \frac{2}{1 - a^2} \frac{\pi}{2} \\
 &= \frac{\pi}{1 - a^2}
 \end{aligned}$$

If $a > 1$ then $\frac{1-a}{1+a}$ is negative i.e. $\frac{a-1}{a+1}$ is positive

$$\begin{aligned}\therefore I &= \frac{2}{(1+a)^2} \frac{a+1}{a-1} \left\{ \tan^{-1} \frac{(a+1)t}{a-1} \right\}_0^\infty \\ &= \frac{2}{a^2-1} (\tan^{-1} \infty - \tan^{-1} 0) \\ &= \frac{2}{a^2-1} \cdot \frac{\pi}{2} = \frac{\pi}{a^2-1}\end{aligned}$$

Example 3. Evaluate $\int_0^\pi \frac{dx}{\cos x + 2 \sin x + 3}$.

$$\text{Sol. Let } I = \int_0^\pi \frac{dx}{\cos x + 2 \sin x + 3}$$

$$\begin{aligned}\text{Then } I &= \int_0^\pi \frac{dx}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} + 4 \sin \frac{x}{2} \cos \frac{x}{2} + 3 \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right)} \\ &= \int_0^\pi \frac{dx}{4 \cos^2 \frac{x}{2} + 4 \sin \frac{x}{2} \cos \frac{x}{2} + 2 \sin^2 \frac{x}{2}} \\ &= \frac{1}{2} \int_0^\pi \frac{\sec^2 \frac{x}{2} dx}{2 + 2 \tan \frac{x}{2} + \tan^2 \frac{x}{2}} \\ &= \int_0^\infty \frac{dt}{2 + 2t + t^2} \text{ where } \tan \frac{x}{2} = t \\ &= \int_0^\infty \frac{dt}{1 + (t+1)^2} = [\tan^{-1}(t+1)]_0^\infty \\ &= \tan^{-1} \infty - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

Example 4. Evaluate $\int_0^{\pi/2} \sqrt{\sec x + 1} dx$.

$$\begin{aligned}\text{Sol. } I &= \int_0^{\pi/2} \sqrt{(\sec x + 1)} dx = \int_0^{\pi/2} \frac{\sqrt{(1 + \cos x)}}{\sqrt{\cos x}} dx \\ &= \int_0^{\pi/2} \frac{\sqrt{2 \cos^2 \frac{x}{2}}}{\sqrt{\left(1 - 2 \sin^2 \frac{x}{2}\right)}} dx = \int_0^{\pi/2} \frac{\sqrt{2} \cos \frac{x}{2}}{\sqrt{\left(1 - 2 \sin^2 \frac{x}{2}\right)}} dx\end{aligned}$$

Put $\sqrt{2} \sin \frac{x}{2} = t$ so that $\frac{\sqrt{2}}{2} \cos \frac{x}{2} dx = dt$

or $\sqrt{2} \cos \frac{x}{2} dx = 2dt$

\therefore When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$$\begin{aligned}\therefore I &= \int_0^1 \frac{2dt}{\sqrt{1-t^2}} = [2 \sin^{-1} t]_0^1 \\ &= 2 [\sin^{-1} 1 - \sin^{-1} 0] \\ &= 2 \frac{\pi}{2} = \pi.\end{aligned}$$

§ 1.14. Integration of $\frac{a + b \cos x + c \sin x}{l + m \cos x + n \sin x}$.

We express the numerator of the integrand as

$$a + b \cos x + c \sin x$$

$$= A(l + m \cos x + n \sin x) + B[-m \sin x + n \cos x] + C$$

where coefficient of B is the differential coefficient of denominator.

Comparing the coefficients of $\sin x, \cos x$ and constant term, we obtain the value of A, B and C . Then the given integral can be split into the sum of three integrals as

$$\begin{aligned}&\int \frac{a + b \cos x + c \sin x}{l + m \cos x + n \sin x} dx \\ &= A \int dx + B \int \frac{-m \sin x + n \cos x}{l + m \cos x + n \sin x} dx + C \int \frac{dx}{l + m \cos x + n \sin x} \\ &= Ax + B \log(l + m \cos x + n \sin x) + C \int \frac{dx}{l + m \cos x + n \sin x}\end{aligned}$$

The last integral can be evaluated as explained in the solved example 3 of § 1.13.

§ 1.15. Integration of $\frac{p \cos x + q \sin x}{a \cos x + b \sin x}$.

In this case we express the numerator of the integral as

$$p \cos x + q \sin x = A[a \cos x + b \sin x] + B(d.c. of Dr.)$$

where A and B are constants to be determined by comparing the coefficients of $\sin x$ and $\cos x$. Thus the given integral is

$$\begin{aligned}\int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx &= A \int dx + B \int \frac{(-a \sin x + b \cos x)}{a \cos x + b \sin x} dx \\ &= Ax + B \log(a \cos x + b \sin x)\end{aligned}$$

Example 5. Prove that :

$$\int \frac{3 + 4 \sin x + 2 \cos x}{3 + 2 \sin x + \cos x} dx = 2x - 3 \tan^{-1} \left(1 + \tan \frac{x}{2} \right)$$

Sol. Let

$$3 + 4 \sin x + 2 \cos x = A(3 + 2 \sin x + \cos x)$$

$$+ B(2 \cos x - \sin x) + C$$

Comparing the coefficients of $\sin x$, $\cos x$ and constant term, we get

$$2A - B = 4, A + 2B = 2, 3A + C = 3$$

which gives $A = 2$, $B = 0$, $C = -3$

Therefore the given integral is

$$\begin{aligned} & \int \frac{3 + 4 \sin x + 2 \cos x}{3 + 2 \sin x + \cos x} dx \\ &= \int 2 dx - 3 \int \frac{dx}{3 + 2 \sin x + \cos x} \\ &= 2x - 3 \int \frac{dx}{3 \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) + 4 \sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \\ &= 2x - 3 \int \frac{dx}{4 \cos^2 \frac{x}{2} + 4 \sin \frac{x}{2} \cos \frac{x}{2} + 2 \sin^2 \frac{x}{2}} \\ &= 2x - \frac{3}{2} \int \frac{\sec^2 \frac{x}{2} dx}{2 + 2 \tan \frac{x}{2} + \tan^2 \frac{x}{2}} \\ &= 2x - 3 \int \frac{dt}{2 + 2t + t^2} \text{ where } t = \tan \frac{x}{2} \\ &= 2x - 3 \int \frac{dt}{1 + (t+1)^2} \\ &= 2x - 3 \tan^{-1}(1+t) \\ &= 2x - 3 \tan^{-1} \left(1 + \tan \frac{x}{2} \right) \end{aligned}$$

Example 6. Evaluate $\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx$.

(GKP 2013, 17)

Sol. Here we put

$$2 \sin x + 3 \cos x = A (3 \sin x + 4 \cos x) + B (3 \cos x - 4 \sin x).$$

Comparing the coefficients of $\sin x$, $\cos x$, from the two sides, we have

$$2 = 3A - 4B \text{ and } 3 = 4A + 3B$$

$$\therefore A = \frac{18}{25}, B = \frac{1}{25}$$

$$\begin{aligned} \therefore \int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx &= \frac{18}{25} \int dx + \frac{1}{25} \int \frac{3 \cos x - 4 \sin x}{3 \sin x + 4 \cos x} dx \\ &= \frac{18}{25}x + \frac{1}{25} \log(3 \sin x + 4 \cos x) \end{aligned}$$

Example 7. : $\int \frac{\sin x}{\sin(x - \alpha)} dx$ (GKP 2007, Sid. 2017)

Sol. Let $I = \int \frac{\sin x}{\sin(x - \alpha)} dx$... (1)

Let $t = x - \alpha$

Differentiating above, we get

$$dt = dx$$

Now from equation (1)

$$\begin{aligned} I &= \int \frac{\sin(t + \alpha)}{\sin t} dt \\ &= \int \frac{(\sin t \cos \alpha + \cos t \cdot \sin \alpha)}{\sin t} dt \end{aligned}$$

$[\sin(A + B) = \sin A \cos B + \cos A \cdot \sin B]$

$$\begin{aligned} \text{Now, } I &= \int (\cos \alpha + \cot t \sin \alpha) dt \\ &= \int \cos \alpha dt + \int \cot t \sin \alpha dt \\ &= \cos \alpha \int dt + \sin \alpha \int \cot t dt \end{aligned}$$

$$I = \cos \alpha \cdot t + \sin \alpha \log \sin t$$

Now putting the value of t in equation, we get

$$I = (x - \alpha) \cos \alpha + \sin \alpha \log \sin(x - \alpha)$$

EXERCISE 1.8

1. Evaluate $\int_0^{\pi/2} \frac{dx}{4 + 5 \sin x}$ (Purvanchal 2005)
2. Integrate $\frac{\sin x + \cos x}{\sqrt{\sin 2x}}$
3. Evaluate $\int_0^{\pi/2} \frac{dx}{1 + 2 \cos x}$
4. Evaluate $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$
5. Evaluate $\int \frac{\sin 2x dx}{(\sin x + \cos x)^2}$
6. Evaluate $\int \frac{\sin x dx}{\sqrt{1 + \sin x}}$
7. Evaluate $\int \frac{2 + 3 \cos x + 4 \sin x}{1 + 2 \cos x + 3 \sin x} dx$
8. Evaluate $\int \frac{dx}{(a \sin x + b \cos x)^2}$ (Purvanchal 2004)

