Chapter 3

DEFINITE INTEGRALS

§ 3·1. Definition

We know that $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is the integral of $f(x)$ is said to be a definite integral. a and b are called the lower and upper limits of the integral and the interval (a, b) is called the range of integration. In this chapter we are giving certain properties of definite integrals which are helpful in the speedy evaluation of the integral.

§ 3·2. Properties of definite integrals.

For present purposes we shall suppose that

$$\int f(x)\,dx = F(x).$$

Property 1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$$\text{L.H.S'} = \left[F(x)\right]_a^b = F(b) - F(a)$$

$$\text{R.H.S.} = \left[F(t)\right]_a^b = F(b) - F(a)$$

Property 2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\text{R.H.S.} = -\left[F(x)\right]_b^a = -[F(a) - F(b)]$$

$$= [F(b) - F(a)] = \text{L.H.S.}$$

Thus if the limits are interchanged the value changes in sign.

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where c is any point in the interval (a, b).

$$\text{R.H.S.} = \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) = \text{L.H.S.}$$

Remark. This property can be geeralised as

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

Proof of this follows easily.

Property 4. $\int_0^a \mathbf{f(x)\,dx} = \int_0^a \mathbf{f(a-x)\,dx}$

Proof. R.H.S. $= \int_0^a f(a-x)\,dx$

Putting $a - x = t$; $-dx = dt$

$$= -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt \qquad \text{by property 2}$$

$$= \int_0^a f(x)\,dx \qquad \text{(property 1)}$$

= L.H.S.

Property 5. $\int_{-a}^{a} \mathbf{f(x)\,dx = 0}$ or $2\int_0^a \mathbf{f(x)\,dx}$

according as $f(x)$ is an odd or even function of x.

Odd or even function *(definition). A function $f(x)$ is said to be odd if*

$$f(-x) = -f(x)$$

and even if $f(-x) = f(x)$.

For instance $\sin x$ is an odd function because

$\sin(-x) = -\sin x$ where as x^2 or $\cos x$ are examples of even functions

Proof. $\int_{-a}^{a} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^a f(x)\,dx$ by property 3

In the first integral on the right put $x = -t$, then

$$\int_{-a}^{a} f(x)\,dx = -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-t)\,dt + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \qquad \ldots(1)$$

Case I. If $f(x)$ is odd then $f(-x) = -f(x)$ then from (1), we have

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 0.$$

Case II. If $f(x)$ is even $f(-x) = f(x)$ then from (1), we get

$$\int_{-a}^{a} f(x)\,dx = +\int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx.$$

Property 6. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

or 0 if $f(2a-x) = -f(x)$

Proof. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ by property 3

In the second integral on R.H.S. put $x = 2a - t$

then $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^0 -f(2a-t)\,dt$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \quad ...(1)$$

by property 2 and 1.

Case I. If $f(2a-x) = f(x)$, then from (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx$$

Case II. If $f(2a-x) = -f(x)$, then from (1) again we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Following examples will illustrate the application and usefulness of these properties.

Example 1. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\sin\theta\,d\theta}{\cos\theta + \sin\theta}$

(GKP 1998, P.U. 2006)

Solution. Let $I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

Putting $x = a\cos\theta$,

$dx = -a\sin\theta\,d\theta$

$$\therefore\ I = \int_{\pi/2}^0 \frac{a\sin\theta\,d\theta}{a\cos\theta + \sqrt{a^2 - a^2\cos^2\theta}}$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\,d\theta \quad ...(i)$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta}\,d\theta \quad ...(ii)$$

[by property (4)]

Adding (i)& (ii) we get

$$2I = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

Example 2. *Evaluate* $\int_0^{\pi} \frac{x \sin x \, dx}{(1 + \cos^2 x)}$ *(Gorkhpur 85; Avadh 95)*

Solution. Let $I = \int_0^{\pi} \frac{x \sin x \, dx}{(1 + \cos^2 x)}$ ...(i)

$= \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x) \, dx}{\{1 + \cos^2 (\pi - x)\}}$ by property 4

or $I = \int_0^{\pi} \frac{(\pi - x) \sin x \, dx}{(1 + \cos^2 x)}$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x \, dx}{(1 + \cos^2 x)} = 2 \int_0^{\pi/2} \frac{\pi \sin x \, dx}{1 + \cos^2 x}$$

by property of definite integral.

$$= -2\pi \int_1^0 \frac{dt}{1 + t^2} \quad \text{by putting } \cos x = t$$

$$= -2\pi \left[\tan^{-1} t\right]_1^0 = 2\pi \left[\tan^{-1} t\right]_0^1$$

$$= 2\pi \left[\frac{\pi}{4}\right] = \frac{\pi^2}{2}$$

$\therefore$ $I = \frac{\pi^2}{4}$.

Remark. In order to evaluate the integrals of the type $\int_0^a x f(x) \, dx$, we apply property 4 provided $f(a - x) = f(x)$ and adding this with the given integral we can remove x and solve the integral.

Example 3. *Evaluate* $\int_0^{\pi/2} \log \sin x \, dx$

(GKP 82, 2008, 12, 13; Purvanchal 89, 2003, 2005; Avadh 91, 96, Sid. 2017)

Solution. Let $I = \int_0^{\pi/2} \log \sin x \, dx$ ...(i)

$= \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x\right) dx$ by property 4

$= \int_0^{\pi/2} \log \cos x \, dx$ ...(ii)

Additing (i) and (ii) we get

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\pi/2} \log (\sin x \cos x) \, dx$$

$$= \int_0^{\pi/2} \log \left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log_e 2 \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \log_e 2 \left[x\right]_0^{\pi/2}$$

(on putting $2x = t$ in the first integral)

$$= \int_0^{\pi/2} \log \sin t \, dt - \frac{\pi}{2}\log_e 2 \qquad \text{(property 6)}$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2}\log_e 2$$

$$= I - \frac{\pi}{2}\log_e 2 \quad \text{On transposition we get}$$

$$I = -\frac{\pi}{2}\log_e 2$$

$$\therefore \quad \int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log_e 2.$$

Note— This result is useful in solving a good number of problems. Students are advised to go through the proof carefully and remember the result.

Example 4. *Show that* $\int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{(1+x^2)} = \pi \log_e 2$

(Gorakhpur 85, 96; Avadh 82)

Solution. Substituting $x = \tan\theta$; $dx = \sec^2\theta \, d\theta$ we have

$$\int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{1+x^2} = \int_0^{\pi/2} \log(\tan\theta + \cot\theta)\frac{\sec^2\theta \, d\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta \, d\theta - \int_0^{\pi/2} \log\cos\theta \, d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta \, d\theta - \int_0^{\pi/2} \log\cos\left(\frac{\pi}{2} - \theta\right) d\theta \qquad \text{property 4}$$

$$= -2\int_0^{\pi/2} \log\sin\theta \, d\theta$$

$$= -2\left(-\frac{\pi}{2}\log_e 2\right) = \pi \log_e 2$$

Hence proved.

Example 5. *Show that* $\int_0^{\infty} \frac{x \, dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

Solution. Put $x = \tan 6$; $dx = \sec^2\theta \, d\theta$

then the given integral which may be denoted by I is expressed as

$$I = \int_0^{\pi/2} \frac{\tan\theta \sec^2\theta \, d\theta}{(1+\tan\theta)(1+\tan^2\theta)}$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta)} d\theta$$

Thus $$I = \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta + \cos\theta} \quad ...(i)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-\theta\right) d\theta}{\sin\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-\theta\right)}$$

by property of definite integral

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\cos\theta + \sin\theta} \quad ...(ii)$$

Adding (i) and (ii) we get

$$2I = \int_0^{\pi/2} \left(\frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta}\right) d\theta = \int_0^{\pi/2} d\theta$$

$$\therefore \quad 2I = \left[\theta\right]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

Example 6. *Show that* $\int_0^{\pi} \frac{x \, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$. *(Gorakhpur 2006)*

Solution. Let $$I = \int_0^{\pi} \frac{x \, dx}{a^2\cos^2 x + b^2\sin^2 x} \quad ...(i)$$

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2\cos^2 x + b^2\sin^2 x} \quad ...(ii)$$

by property 4 of definite integrals.

Adding (i) and (ii) we get

$$2I = \int_0^{\pi} \frac{\pi \, dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

by property 6 of definite integrals

$$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 + b^2\tan^2 x}$$

Put $b\tan x = t \quad \therefore \; b\sec^2 x \, dx = dt$

$$= \frac{2\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{2\pi}{b}\left[\frac{1}{a}\tan^{-1}\frac{t}{a}\right]_0^{\infty}$$

$$= \frac{2\pi}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{2\pi}{ab}\left[\frac{\pi}{2}\right]$$

$\therefore \quad I = \dfrac{\pi^2}{2ab}$. Prov

Example 7. *Solve* $\displaystyle\int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x+\sin 2x}$.

Solution. Let $I = \displaystyle\int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x+\sin 2x}$

Putting $2x = t$

$$dx = \frac{dt}{2}$$

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi/2} \frac{t/2\,dt}{1+\cos t+\sin t}$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{t\,dt}{1+\cos t+\sin t}$$

$$I = \frac{1}{4}\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-t\right)dt}{1+\cos t+\sin t}$$

[by property (4

Adding (i) & (ii) we get

$$2I = \frac{1}{4}\times\frac{\pi}{2}\int_0^{\pi/2} \frac{dt}{1+\cos t+\sin t}$$

$$2I = \frac{\pi}{8}\int_0^{\pi/2} \frac{dt}{\cos^2\frac{t}{2}+\sin^2\frac{t}{2}+\cos^2\frac{t}{2}-\sin^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{\pi}{8}\int_0^{\pi/2} \frac{dt}{2\cos^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{\pi}{16}\int_0^{\pi/2} \frac{\sec^2\frac{t}{2}\,dt}{1+\tan\frac{t}{2}}$$

Putting $\tan\frac{t}{2} = u$

$$\frac{1}{2}\sec^2\frac{t}{2}\,dt = du$$

$$2I = \frac{\pi}{16}\int_0^1 \frac{2\,du}{1+u}$$

$$I = \frac{\pi}{16}\left[\log(1+u)\right]_0^1$$

$$I = \frac{\pi}{16}\log 2.$$

EXERCISES 3·1

Show that

1. $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} = \frac{\pi}{4}$ *(Gorakhpur 81, 2017)*

3. $\int_0^{\pi/2} \frac{dx}{1 + \tan x} = \frac{\pi}{4}$

4. $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$ *(Gorakhpur 2002; Avadh 92)*

5. $\int_0^{\pi} \frac{x \tan x\, dx}{\sec x + \tan x} = \pi \left(\frac{\pi}{2} - 1\right)$ *(Purvanchal 2008; Gorakhpur 86, 2016)*

6. (a) $\int_0^{\pi/2} \log \tan x\, dx = 0,$ (b) $\int_0^{\pi/2} \log \cot x\, dx = 0$ *(Gorakhpur 2003)*

Evaluate

7. $\int_0^{\pi} \theta \sin^3 \theta\, d\theta$ **8.** $\int_0^{\pi} \frac{x\, dx}{a^2 - \cos^2 x}, a > 1$

9. Prove that $\int_0^{\pi} \log(1 + \cos x)\, dx = \pi \log\left(\frac{1}{2}\right)$ *(Gorakhpur 2005)*

10. Show that $\int_0^1 \frac{\sin^{-1} x\, dx}{x} = \frac{\pi}{2} \log_e 2$ *(Gorakhpur 92, Avadh 90)*

11. Show that $\int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta = \frac{\pi}{8} \log_e 2.$
(Purvanchal 90, 2004; Gorakhpur 87, 92)

12. Show that $\int_0^1 \frac{\log(1 + x)\, dx}{1 + x^2} = \frac{\pi}{8} \log_e 2.$ *(Gorakhpur 2008, 04, 14; Avadh 94)*

13. Show that $\int_0^{\pi/2} \phi(\sin 2x) \sin x\, dx = \int_0^{\pi/2} \phi(\sin 2x) \cos x\, dx$

14. Show that $\int_0^{\pi/2} \frac{\sqrt{\cot x}\, dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$

15. Prove that $\int_0^{\pi/2} x^2 \operatorname{cosec}^2 x\, dx = \pi \log_e 2$

[Hint : Integrate by parts twice]

16. Prove that $\int_0^{2a} \phi(x)\, dx = \int_0^{a} \{\phi(x) + \phi(2a - x)\}\, dx$

17. Prove that $\int_0^{\pi} x \log(\sin x)\, dx = \frac{\pi^2}{2} \log\left(\frac{1}{2}\right)$ *(Gorakhpur 98)*

18. Prove that $\int_0^{\pi} \log(1 - 2a\cos x + a^2)\, dx = 0$ or $2\pi \log a$ according as $a < 1$ and $a > 1$. *(Gorakhpu*

19. Show that $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx = \frac{\pi}{4} \log\left(\frac{1}{2}\right)$ *(Purvanchal 2*

20. Evaluate $\int_0^{\pi} \frac{x\, dx}{1 + \sin x}$ *(Gorakhpur 2*

21. Evaluate $\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)(1 + \cos x)^2\, dx$ *(Gorakhpur 2*

22. Show that $\int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\, dx = \pi(\sqrt{2} - 1)$

$$\int_0^{\infty} \frac{1}{1 + x^4}\, dx = \frac{\pi}{4}\sqrt{2}$$

23. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a - b\cos\theta}$, $a > b > 0$.

24. Prove that $\int_0^{\pi} \log(1 - 2a\cos x + a^2)\, dx = 0$

or $2\pi \log a$ according as $a < 1$ and $a > 1$.

§ 3·3. Definite integral as the limit of a sum.

So far we have considered integration as the reverse of differentiation. N we can see that it is possible to express a definite integral and hence also an inde integral (because an indefinite integral can be taken as the definite inte $\int_a^x f(x)\, dx$ as the limit of a sum of the series when the number of terms ten infinity while each term of the series tends to zero. Thus if $f(x)$ is a contin function in the interval (a, b) and $a < b$ then we define the integral $\int_a^b f(x)\, dx$ a

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} h\,[f(a) + f(a + h) + f(a + 2h) + \ldots + f(a + \overline{n-1}$$

where $b - a = nh$.

Here $n \to \infty \Rightarrow h \to 0$.

It is worth while to mention here that integrals were primarily introduc measure areas, volumes etc. by dividing them into enumerable sub-regions. T ultimately resulted in summing a series. The sign of integration $\int$ being the first le of the word sum also justifies it.

Example 8. *Evaluate* $\int_a^b x^2\, dx$ *from the definition of the integral as the li of a sum.* *(GKP 2*

Solution. $\int_a^b x^2\, dx = \lim_{n \to \infty} h\left[a^2 + (a + h)^2 + \ldots (a + \overline{n-1}\,h)^2\right]$

where $b - a = nh$

$$= \lim_{n \to \infty} h\,[na^2 + 2ah\,\{1 + 2 + 3 + \ldots + (n-1)\} + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{n \to \infty} h\left[na^2 + 2ah \cdot \frac{n}{2}(n-1) + \frac{h^2}{6}(n-1)(2n-1)(n)\right]$$

$$= \lim_{n \to \infty} \left[nha^2 + anh\,(nh - h) + \frac{1}{6}(nh - h)(2nh - h)\,nh\right]$$

$$= (b - a)\,a^2 + a\,(b-a)^2 + \tfrac{1}{3}(b-a)^3$$

$$= \frac{(b-a)}{3}\{3a^2 + 3\,(b-a)\,a + (b-a)^2\}$$

$$= \tfrac{1}{3}(b-a)(a^2 + ab + b^2) = \tfrac{1}{3}(b^3 - a^3).$$

Example 9. *From the definition of a definite integral as the limit of a sum find the value of* $\int_\alpha^\beta \sin\theta\, d\theta$ *(Purvanchal 2003; Gorakhpur 2002)*

Solution. Here $f(\theta) = \sin\theta$

$$f(\alpha) = \sin\alpha\,;\; f(\alpha + h) = \sin(\alpha + h) \text{ etc.}$$

$$\therefore \int_\alpha^\beta \sin\theta\, d\theta = \lim_{n \to \infty} h\,[\sin\alpha + \sin(\alpha + h) + \sin(\alpha + 2h) + \ldots + \sin(\alpha + \overline{n-1}\,h)]$$

where $hn = \beta - \alpha$.

$$= \lim_{n \to \infty} \frac{h\left[\sin\left\{\alpha + \dfrac{(n-1)\,h}{2}\right\}\sin\dfrac{nh}{2}\right]}{\sin\dfrac{h}{2}} \text{ (on summing the series)}$$

(See any book of Trigonometry)

$$= \lim_{n \to \infty} 2\left\{\frac{h/2}{\sin h/2}\right\}\sin\left(\frac{2\alpha + \beta - \alpha - h}{2}\right)\sin\left(\frac{\beta - \alpha}{2}\right)$$

$$= 2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\beta - \alpha}{2}\right)$$

$$= \cos\alpha - \cos\beta.$$

§ 3·4. Summation of Series.

In the previous section we have seen that

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h\,[f(a) + f(a + h) + \ldots. f(a + \overline{n-1}\,h)]$$

where $b - a = nh$

$$= \lim_{n \to \infty} \sum_{r=0}^{n-1} h f(a + rh)$$

In particular if we put $a = 0$ and $b = 1$, we get

$$\int_0^1 f(x)\,dx = \lim_{n \to \infty} h \sum_{r=0}^{n-1} f(rh)$$

now $\quad nh = 1 - 0 = 1 \qquad$ so that $h = \dfrac{1}{n}$.

Therefore $\displaystyle \lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$

Remark. In order to sum a series by this formula it must have the following properties :

(1) A series can be expressed in the form of the integral as given above if it is possible to write down the terms in the form

$$\frac{1}{n} f\left(\frac{r}{h}\right)$$

(2) The number of terms in the series shall be n, but as each term tends to zero the addition or omission of any finite number of terms will not change the limit i.e.

$$\lim_{n \to \infty} \sum_{r=k}^{r=n+l} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

where k and l are independent of n.

Working Rule. Find out the rth term of the series and write down it in the form $\dfrac{1}{n} f\left(\dfrac{r}{h}\right)$. Then express the given series as $\displaystyle \lim_{n \to \infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$ assigning proper values to r.

Now replace $\dfrac{r}{n}$ by x, $\dfrac{1}{n}$ by dx $\displaystyle \lim_{n \to \infty} \Sigma$ by $\int$ to get the corresponding definite integral. Limits of this integral will be the values of $\dfrac{r}{n}$ as $n \to \infty$ corresponding to lower and upper values of r. Example given below will help to understand the process.

Example 10. *Prove that*

$$\lim_{n \to \infty} \left[\frac{1}{(3n+1)} + \frac{1}{(3n+2)} + \ldots + \frac{1}{(3n+n)}\right] = \log \frac{4}{3}.$$

Solution. rth term of the series $= \dfrac{1}{3n+r} = \dfrac{1}{n\{3 + (r/n)\}}$

Hence the given series $= \displaystyle \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n\{3 + (r/n)\}}$

Now for the corresponding definite integral

the lower limit $\quad = \displaystyle \mathop{\mathrm{Lt}}_{n \to \infty} \frac{1}{n} = 0$

and upper limit $= \underset{n \to \infty}{\text{Lt}} \dfrac{n}{n} = 1$

Therefore given series $= \displaystyle\int_0^1 \frac{dx}{(3+x)}$

$= \Big[\log(x+3)\Big]_0^1$

$= \log 4 - \log 3 = \log \dfrac{4}{3}$.

Example 11. *Prove that*

$$\lim_{n \to \infty} \left[n \left\{ \frac{1}{n^2} + \frac{1}{(n^2+2^2)} + \ldots + \frac{1}{n^2+(2n-2)^2} \right\} \right] = \frac{1}{2} \tan^{-1} 2$$

(Gorakhpur 86)

Solution. $(r+1)$th term $= \dfrac{n}{n^2+(2r)^2}$

$= \dfrac{n}{n^2\{1+(4r^2/n^2)\}} = \dfrac{1}{n\{1+(4r^2/n^2)\}}$

Hence required limit $= \displaystyle\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n\{1+(4r^2/n^2)\}}$

$= \displaystyle\int_0^1 \frac{dx}{(1+4x^2)}$

$= \displaystyle\frac{1}{4}\int_0^1 \frac{dx}{(x^2+\frac{1}{4})}$

$= \dfrac{1}{4} \cdot 2 \left[\tan^{-1} 2x\right]_0^1$

$= \dfrac{1}{2}[\tan^{-1} 2 - \tan^{-1} 0] = \dfrac{1}{2} \tan^{-1} 2.$ **Proved.**

Example 12. *Evaluate* $\displaystyle\underset{n \to \infty}{\text{Lt}} \sum_{r=1}^{n} \frac{n^2}{(n^2+r^2)^{3/2}}$

Solution. Given limit $= \displaystyle\lim_{n \to \infty} \sum_{r=1}^{n} \frac{(1/n)}{\{1+(r^2/n^2)\}^{3/2}}$

Now for the corresponding integral

lower limit $= \underset{n \to \infty}{\text{Lt}} \dfrac{1}{n} = 0$

upper limit $= \underset{n \to \infty}{\text{Lt}} \dfrac{n}{n} = 1$

$\therefore$ Given sum $= \int_0^1 \frac{dx}{(1+x^2)^{3/2}}$ Put $x = \tan\theta$; $dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$$= \int_0^{\pi/4} \cos\theta\, d\theta \qquad \text{(after simplificati...}$$

$$= \left[\sin\theta\right]_0^{\pi/4} = \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}.$$

Example 13. *Find the limit as $n \to \infty$ of the product*

$$\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots. + \left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$$

(Gorakhpur ...

Solution. Let us suppose that the given limit is A

Thus $A = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots.\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

$$\log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots.+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 \log(1+x^2)\,dx$$

$$= \left\{x\log(1+x^2)\right\}_0^1 - \int_0^1\left(\frac{2x}{(1+x^2)}\right)x\,dx$$

(on integration by parts taking unity as second function)

$$= \log 2 - 2\int_0^1 \frac{(1+x^2-1)\,dx}{1+x^2}$$

$$= \log 2 - 2\left[\int_0^1 dx - \int_0^1 \frac{dx}{1+x^2}\right]$$

$$= \log 2 - 2\left[x - \tan^{-1}x\right]_0^1 = \log 2 - 2\left(1-\frac{\pi}{4}\right)$$

$$\therefore \log A - \log 2 = \left(\frac{\pi}{4}-1\right)2 = \frac{\pi-4}{2}$$

$$\therefore \qquad A = 2e^{(\pi-4)/2}$$

Example 14. *Apply the definition of a definite integral as the limit of a sum to evaluate*

$$\lim_{n \to \infty} \left(\frac{n!}{n^n}\right)^{1/n}$$ *(Gorakhpur 2006)*

Solution. Let us suppose that the limit is A

i.e. $$A = \lim_{n \to \infty} \left(\frac{n!}{n^n}\right)^{1/n}$$

$$= \lim_{n \to \infty} \left(\frac{1.2.3\, n}{n^n}\right)^{1/n}$$

$$= \lim_{n \to \infty} \left\{\left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \dots \left(\frac{n}{n}\right)\right\}^{1/n}$$

$\therefore$ $$\log A = \lim_{n \to \infty} \frac{1}{n}\left\{\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \dots + \log\left(\frac{n}{n}\right)\right\}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log x \, dx$$

$$= \Big(x \log x\Big)_0^1 - \int_0^1 \left(\frac{1}{x}\right) x \, dx \quad \text{(on integration by parts)}$$

$$= 0 - \Big[x\Big]_0^1 = -1.$$

$\therefore$ $$A = e^{-1} = 1/e.$$

EXERCISE 3·2

From the definition of a definite integral as the limit of a sum evaluate :

1. $\int_a^b x \, dx$ 2. $\int_0^{\pi/2} \sin\theta \, d\theta$

3. $\int_a^b e^x \, dx$

4. (a) $\int_0^1 \frac{1}{\sqrt{x}} dx$ (b) $\int_a^b \frac{1}{x^2} dx$ *(Gorakhpur 2003)*

5. $\int_\alpha^\beta \cos\theta \, d\theta$

Find the limit when $n \to \infty$ of the following series :

6. $\frac{1}{n^3}\{1 + 4 + 9 + \dots n^2\}$

7. $\left\{\frac{1}{(1^3+n^3)}+\frac{2^2}{(2^3+n^3)}+....+\frac{r^2}{(r^3+n^3)}+....+\frac{1}{2n}\right\}$

(Avadh 93, Gorakhpur 2010)

8. $\left[\frac{n}{n^2}+\frac{n}{(n^2+1^2)}+\frac{n}{(n^2+2^2)}+....+\frac{n}{n^2+(n+1)^2}\right]$

9. $\left\{\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\frac{3}{n^2}\sec^2\frac{9}{n^2}+.....+\frac{1}{n}\sec^2 1\right\}$

10. Evaluate $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{(n^2-r^2)}}$

11. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^3}{(r^4+n^4)}\right)$

12. (a) Show that $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+....\frac{1}{2n}\right]=\log 2$

12. (b) Show that the limit of the sum

$$\frac{1}{n}+\frac{1}{(n+1)}+\frac{1}{(n+2)}+....+\frac{1}{3n}$$

when $n\to\infty$ is log 3. *(Gorakhpur 2007; Avadh 95)*

13. Find by integration the limit to which the sum

$$\frac{n^{1/2}}{n^{3/2}}+\frac{n^{1/2}}{(n+3)^{3/2}}+\frac{n^{1/2}}{(n+6)^{3/2}}+....+\frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}$$

tends when n is indefinitely increased

14. Find the limit when $n\to\infty$ of the product

$$\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right).......\left(1+\frac{n}{n}\right)\right\}^{1/n}$$

(Gorakhpur 87, 2003)

15. Evaluate

$$\lim_{n\to\infty}\left[\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\sin\frac{3\pi}{2n}......\sin\frac{n\pi}{2n}\right]^{1/n}$$

16. Show by means of integration that

$$\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\tan\frac{3\pi}{2n}......\tan\frac{n\pi}{2n}\right]^{1/n}=1$$

17. Find the limit when n tends to infinity of the product

$$\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}.......\left(1+\frac{n}{n}\right)^{1/n}\right\}$$

[**Hint** : Take the limit as A, then

$$\log A = \int_0^1 \frac{1}{x} \log(1+x)\, dx = \left[x - \frac{x^2}{4} + \frac{x^2}{9} - \ldots\ldots \right]_0^1$$

on expanding log $(1+x)$ and integrating term by term

$$= \frac{\pi^2}{12} \text{ from trigonometry.}$$

18. Evaluate

$$\lim_{n \to \infty} \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} \right]$$ *(Gorakhpur 2002)*

§ 3.5. Differentiation under integral sign.

The functions which are not easily integrable can be integrated by differentiating the integrand of the definite integral with respect to a quantity which is independent of the limits as well as the variable with respect to which integration is performed. The following examples will illustrate the method.

Example 15. *Evaluate* $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx.$ *(Gorakhpur 2007, 2010)*

Sol. Let $I = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}$ ...(1)

$$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{1+a^2x^2} \cdot dx = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \int_0^\infty \frac{1}{1-a^2} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} \right] dx \quad \text{(by purtial fraction)}$$

$$= \frac{1}{1-a^2} [\tan^{-1} x - a \tan^{1} ax]_0^\infty$$

$$= \frac{1}{1-a^2} \left[\frac{\pi}{2} - \frac{a\pi}{2} \right] = \frac{\pi}{2(1+a)}$$

$$\therefore \quad I = \int \frac{\pi}{2(1+a)} da + c$$

$$= \frac{\pi}{2} \log(1+a) + c$$ ...(ii)

But, when $a = 0$, $I = 0$ from (i)

Hence from (ii), we get $c = 0$.

$$\therefore \quad I = \frac{\pi}{2} \log(1+a)$$

Example 16. *Evaluate* $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2} dx.$ *(Gorakhpur 2002)*

Sol. Let $I = \int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2} dx$ ...(i)

$$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(1+b^2x^2)} dx$$

$$= \frac{2a}{b^2 - a^2} \int \left[\frac{1}{1 + a^2x^2} - \frac{1}{1 + b^2x^2} \right] dx \text{ (by partial fraction)}$$

$$= \frac{2a}{b^2 - a^2} \left[\frac{1}{a} \tan^{-1} ax - \frac{1}{b} \tan^{-1} bx \right]_0^\infty$$

$$= \frac{2a}{b^2 - a^2} \left[\frac{1}{a} \cdot \frac{\pi}{2} - \frac{1}{b} \frac{\pi}{2} \right]$$

$$= \frac{\pi}{b(a + b)}$$

$$\therefore \quad I = \int \frac{\pi}{b(a + b)} da + c$$

$$= \frac{\pi}{b} \log (a + b) + c \qquad \text{...(ii)}$$

But when $a = 0$, $I = 0$ from (i), Hence from (ii) $c = -\frac{\pi}{b} \log b$.

$$\therefore \quad I = \frac{\pi}{b} \log (a + b) - \frac{\pi}{b} \log b = \frac{\pi}{b} \log \left(\frac{a + b}{b} \right)$$

Example 17. *Evaluate* $\int_0^{\pi/2} \log (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) \, d\theta$.

Sol. Let $I = \int_0^{\pi/2} \log (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) \, d\theta$

$$\therefore \quad \frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{2\alpha \cos^2 \theta \, d\theta}{\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta}$$

$$= 2\alpha \int_0^{\pi/2} \frac{d\theta}{\alpha^2 + \beta^2 \tan^2 \theta}$$

Put $\tan \theta = t$, so that $\sec^2 \theta \, d\theta = dt$ *i.e.*

$$d\theta = \frac{dt}{\sec^2 \theta} = \frac{dt}{1 + \tan^2 \theta} = \frac{dt}{1 + t^2}$$

$$\therefore \quad \frac{dI}{d\alpha} = 2\alpha \int_0^\infty \frac{dt}{(\alpha^2 + \beta^2 t^2)(1 + t^2)}$$

$$= \frac{2\alpha}{\beta^2 - \alpha^2} \int_0^\infty \left[\frac{\beta^2}{\alpha^2 + \beta^2 t^2} - \frac{1}{1 + t^2} \right] dt \text{ (by partial fraction)}$$

$$= \frac{2\alpha}{\beta^2 - \alpha^2} \left[\frac{\beta}{\alpha} \tan^{-1} \frac{\beta t}{\alpha} - \tan^{-1} t \right]_0^\infty$$

$$= \frac{2\alpha}{\beta^2 - \alpha^2} \left[\frac{\beta}{\alpha} \cdot \frac{\pi}{2} - \frac{\pi}{2} \right] = \frac{\pi}{\alpha + \beta}$$

$$\therefore \quad I = \int \frac{\pi}{\alpha + \beta} d\alpha + c = \pi \log (\alpha + \beta) + c \qquad \text{...(i)}$$

But when $\alpha = \beta$, $I = \int_0^{\pi/2} \log \alpha^2 \, d\theta = 2 \log \alpha \cdot \frac{\pi}{2}$

$= \pi \log \alpha$

$\therefore$ from (1), we get

$$\pi \log \alpha = \pi \log (\alpha + \alpha) + c$$
$$= \pi \log 2\alpha + c$$
$$= \pi \log 2 + \pi \log \alpha + c$$

$\therefore \qquad c = -\pi \log 2$

Putting this value of c in (i) we get

$$I = \pi \log (\alpha + \beta) - \pi \log 2$$
$$= \pi \log \left(\frac{\alpha + \beta}{2}\right)$$

Example 18. *Prove that* $\int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1} \frac{b}{a}$. *Hence deduce that* $\int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2}$.

(GKP 2009)

Sol. Let $I = \int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx$ ...(i)

$$\therefore \quad \frac{dI}{db} = \int_0^{\infty} \frac{e^{-ax} \cos bx}{x} \cdot x \, dx$$

$$= \int_0^{\infty} e^{-ax} \cos bx \, dx$$

$$= \left[e^{-ax} \frac{\sin bx}{b}\right]_0^{\infty} - \int_0^{\infty} (-a) e^{-ax} \frac{\sin bx}{b} dx$$

$$= 0 - 0 + \frac{a}{b} \int_0^{\infty} e^{-ax} \sin bx \, dx$$

$$= \frac{a}{b} \left[\left\{e^{-ax} \frac{(-\cos bx)}{b}\right\}_0^{\infty} - \int_0^{\infty} (-a) e^{-ax} \left(\frac{-\cos bx}{b}\right) dx\right]$$

$$= \frac{a}{b} \left[\frac{1}{b} - \frac{a}{b} \int_0^{\infty} e^{-ax} \cos bx \, dx\right]$$

$$= \frac{a}{b^2} - \frac{a^2}{b^2} \frac{dI}{db}$$

$$\therefore \quad \left(1 + \frac{a^2}{b^2}\right) \frac{dI}{db} = \frac{a}{b^2}$$

or $$\frac{dI}{db} = \frac{a}{a^2 + b^2}$$

Integrating with respect to b, we get

$$I = \int \frac{a}{a^2 + b^2}\, db + e = a \cdot \frac{1}{a} \tan^{-1} \frac{b}{a} + c$$

or $\quad I = \tan^{-1} \dfrac{b}{a} + c$

But when $b = 0$, $I = 0$ from (i). Hence from (ii) we get $c = 0$.

$\therefore \quad I = \tan^{-1} \dfrac{b}{a}$

Deduction. Putting $a = 0$ in $\displaystyle\int_0^\infty \frac{e^{-ax} \sin bx}{x}\, dx = \tan^{-1} \frac{b}{a}$, we get

$$\int_0^\infty \frac{\sin bx}{x} = \tan^{-1} \infty = \frac{\pi}{2}.$$

Example 19. *Prove that*

$$\int_0^{\pi/2} \frac{\log(1 + \cos\alpha \cos x)}{\cos x}\, dx = \frac{1}{2}\left(\frac{\pi^2}{4} - \alpha^2\right)$$

Sol. Let $I = \displaystyle\int_0^{\pi/2} \frac{\log(1 + \cos\alpha \cos x)}{\cos x}\, dx$

Then $\displaystyle\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{-\sin\alpha \cos x}{(1 + \cos\alpha \cos x)\cos x}\, dx$

$$= -\sin\alpha \int_0^{\pi/2} \frac{dx}{1 + \cos\alpha \cos x}$$

$$= -\sin\alpha \int_0^{\pi/2} \frac{dx}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \cos\alpha\left[\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right]}$$

$$= -\sin\alpha \int_0^{\pi/2} \frac{dx}{(1 + \cos\alpha)\cos^2\frac{x}{2} + (1 - \cos\alpha)\sin^2\frac{x}{2}}$$

$$= -\sin\alpha \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\, dx}{(1 + \cos\alpha)\,(1 - \cos\alpha)\tan^2\frac{x}{2}}$$

we put $\tan\dfrac{x}{2} = t$ so that $\sec^2\dfrac{x}{2}\, dx = 2dt$

$$\therefore \quad \frac{dI}{d\alpha} = -\sin\alpha \int_0^1 \frac{2dt}{(1 + \cos\alpha) + (1 - \cos\alpha)t^2}$$

$$= -\frac{2\sin\alpha}{1 - \cos\alpha} \int_0^1 \frac{dt}{\dfrac{1 + \cos\alpha}{1 - \cos\alpha} + t^2}$$

$$= -\frac{2\sin\alpha}{1 - \cos\alpha} \sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} \left[\tan^{-1} \frac{t\sqrt{1 - \cos\alpha}}{\sqrt{1 + \cos\alpha}}\right]_0^1$$

$$= -\frac{2\sin\alpha}{\sqrt{1-\cos^2\alpha}}\left[\tan^{-1}\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}-0\right]$$

$$= -2\tan^{-1}\left(\tan\frac{\alpha}{2}\right)$$

$$\therefore \quad \frac{dI}{d\alpha} = -\alpha$$

Integrating with respect to α we get

$$I = -\frac{\alpha^2}{2} + c \qquad \text{...(ii)}$$

when $\alpha = \frac{\pi}{2}$, $I = 0$ from (i). Hence from (ii)

$$0 = -\frac{\pi^2}{8} + c \text{ or } c = \frac{\pi^2}{8}$$

$$\therefore \quad I = -\frac{\alpha^2}{2} + \frac{\pi^2}{8} = \frac{1}{2}\left[\frac{\pi^2}{4} - \alpha^2\right]$$

§ 3.6. Integration under the sign of integration.

Some functions which are not easily integrable can be integrated by integrating the integrand of the definite integral with respect to a quantity which is independent of the limits as well as the variable with respect to which the integration is performed. The following examples will illustrate the method.

Example 20. *Evaluate* $\int_0^1 \frac{x^{a-1} - x^{b-1}}{\log x}\, dx.$

Sol. We know that

$$\int_0^1 x^{n-1}\, dx = \left[\frac{x^n}{n}\right]_0^1 = \frac{1}{n}$$

Integrating both sides with respect to n, between the llimits b and a, we get

$$\int_0^1 \left[\int_b^a x^{n-1}\, dn\right] dn = \int_b^a \frac{1}{n}\, dn$$

or

$$\int_0^1 \left[\frac{x^{n-1}}{\log x}\right]_{n=b}^{n=a} dx = [\log n]_b^a$$

or

$$\int_0^1 \frac{x^{a-1} - x^{b-1}}{\log x}\, dx = \log a - \log b$$

$$= \log\frac{a}{b}$$

Example 21. *Prove that*

(i) $\int_0^\infty e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}$

(ii) $\int_0^\infty e^{-a^2x^2} \cos bx\, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}$ *(Gorakhpur 2000)*

Sol. (i) Put $a^2x^2 = t$ *i.e.* $x = \dfrac{t^{1/2}}{a}$

so that $dx = \dfrac{1}{2a} t^{\frac{1}{2}-1}\, dt$

$$\therefore \int_0^\infty e^{-a^2x^2}\, dx = \int_0^\infty e^{-t} \cdot \frac{1}{2a} t^{\frac{1}{2}-1}\, dt$$

$$= \frac{1}{2a}\int_0^\infty e^{-t} t^{\frac{1}{2}-1}\, dt$$

$$= \frac{1}{2a}\Gamma\left(\frac{1}{2}\right) \qquad \text{(see § 4.1 of chapter 4)}$$

$$= \frac{\sqrt{\pi}}{2a} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

(ii) Let $I = \displaystyle\int_0^\infty e^{-a^2x^2} \cos bx\, dx$

Differentiating it with respect to b, we get

$$\frac{dI}{db} = -\int_0^\infty e^{-a^2x^2} x \sin bx\, dx$$

$$= -\left[\left\{\frac{-e^{-a^2x^2}}{2a^2}\sin bx\right\}_0^\infty - \frac{b}{2a^2}\int_0^\infty (-e^{-a^2x^2}) \cos bx\, dx\right]$$

(Integration by part, taking $\sin bx$ as first function).

$$\therefore \quad \frac{dI}{db} = -\frac{b}{2a^2} I \qquad \text{(Integrand part being zero on both the limits)}$$

or $$\frac{dI}{I} = -\frac{b}{2a^2}\, db$$

$$\therefore \quad \int \frac{dI}{I} = -\int \frac{b\,db}{2a^2} + \log c \qquad (c \text{ is constant})$$

$$\log I = -\frac{b^2}{4a^2} + \log c$$

$$\therefore \quad I = ce^{-b^2/4a^2} \qquad \text{...(1)}$$

when $b = 0, I = \displaystyle\int_0^\infty e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}$

(by first part of this question)

$\therefore$ From (1), $\dfrac{\sqrt{\pi}}{2a} = c$

Hence we have

$$I = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}$$

EXERCISE 3·3

1. Prove that $\displaystyle\int_0^{\infty} \frac{1 - \cos xy}{x} e^{-x}\, dx = \frac{1}{2} \log (1 + y^2)$

2. Prove that $\displaystyle\int_0^{\pi} \frac{\log (1 + \sin \alpha \cos x)}{\cos x}\, dx = \pi$

3. Prove that if $-1 < a < 1$ and $-\dfrac{\pi}{2} < \sin^{-1} a < \dfrac{\pi}{2}$

$$\int_0^{\pi} \frac{\log (1 + a \cos x)}{\cos x}\, dx = \pi \sin^{-1} a$$

4. Prove that $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi (a^2 + b^2)}{4a^3 b^3}$

5. Prove that $\displaystyle\int_0^{\pi/2} \frac{\log (1 + y \sin^2 x)}{\sin^2 x}\, dx = \pi[\sqrt{(1 + y)} - 1], y > 1$

6. Prove that $\displaystyle\int_0^{\pi/2} \log \left(\frac{a + b \sin \theta}{a - b \sin \theta}\right) \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \frac{b}{a}$

7. Prove that $\displaystyle\int_0^{1} \frac{x^n - 1}{\log x}\, dx = \log (n + 1)$

8. Prove that $\displaystyle\int_0^{1} \frac{x^{1/a} - x^{-1/a}}{\log x}\, dx = \log \frac{a + 1}{a - 1}$

9. From the integral $\displaystyle\int_0^{\infty} e^{-ax}\, dx = \frac{1}{a}$,

prove that $\displaystyle\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log \frac{b}{a}$

10. Prove that

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin mx\, dx = \tan^{-1} \frac{b}{m} - \tan^{-1} \frac{a}{m}$$

and hence deduce that $\displaystyle\int_0^{\infty} \frac{\sin mx}{x}\, dx = \frac{\pi}{2}$.

[**Hint : Integrate** $\displaystyle\int_0^{\infty} e^{-\alpha x} \sin mx\, dx = \frac{m}{\alpha^2 + m^2}$ w.r. to α between the limit a and b]