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SINGULAR SOLUTION AND ORTHOGONAL TRAJECTORIES

1. Geometrical significance of solutions of first order differential equations.

This first order differential equation is

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad .....(1)$$

Its general solution involving one arbitrary constant is

$$g(x,y,c) = 0 \qquad .....(2)$$

Equation (2) represents a curve in xy–plane for a given value of c. If we treat c as parameter and allow it to have all real values, equation (2) yields a family of infinitely many curves. These are called **integral curves**. Thus the general solution of a first order differential equation (1) geometrically represents an infinite set of curves in the xy–plane.

2. Singular solutions.

Let us take c as parameter in the equation

$$g(x,y,c) = 0 \qquad ....(1)$$

which is the solution of first order differential equation.

Differentiating g(x,y,c) = 0 partially with respect to parameter c, we get

$$\frac{\partial g}{\partial c} = 0 \qquad (2)$$

Eliminating c from equations (1) and (2), we get an equation involving only x and y in the form $\phi(x,y) = 0$. This solution is the **singular solution** of the differential equation

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

In differential calculus, we know that the c – eliminant of the equation of a family of curves given by g(x,y,c) = 0 and its partial derivative equation

$$\frac{\partial}{\partial c}[g(x,y,c)] = 0$$

represent a curve Γ which touches every **member of the family C** of curves represented by equation g(x,y,c) = 0. The curve Γ is called the envelope of the family of curves C. Consequently. the singular solution of a differential equation

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

represents the envelope of the family of curves represented by the general solution g(x,y,c) = 0 of the differential equation.

(i) **c–discriminant.** It is obtained by eliminating c between the equation

$$g(x,y,c) = 0 \text{ and } \frac{\partial g}{\partial c} = 0$$

Thus c – discriminant is the locus of each point of which g(x,y,c) = 0 has equal values of c, and this contains the envelope of the family g(x,y,c) = 0.

(ii) **p–discriminant.** It is obtained by eliminating p between the equation

$$f(x,y,p) = 0 \text{ and } \frac{\partial f}{\partial p} = 0$$

Thus p – discriminant is the locus of each point of which f(x,y,p) = 0 has equal values of p, and this contains the envelope of f(x,y,p) = 0, where p is treated as parameter.

Remark:

The c discriminant of g(x, y, c) = 0 and $\frac{\partial g}{\partial c} = 0$ does not exist when $\frac{\partial g}{\partial c}$ is independent of c. In this case **singular solution** does not exist. The singular solution if it exists, represents the envelope of the family of integral curves of the given differential equation

It is noted that the locus obtained from c – discriminant contains the envelope as a factor once, the nodal-locus twice and the cusp-locus thrice whereas the locus obtained from p-discriminant contains the envelope as a factor once, the cusp-locus once and tac-locus twice.

Thus the singular solution of the differential equation is obtained by c–discriminant and p–discriminant.

Exp. 1. Reduce $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$ to Clairaut's form and find its singular solution.

Sol. Let us put $u = x^2$, $v = y^2$, $\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$

so that $p = \frac{x}{y}P$ where $\frac{dv}{du} = P$.

On substituting these values in the given differential equation, we have

$$x^2P^2 - \left(x^2 + y^2 - 1\right)P + y^2 = 0$$

or $$uP^2 - (u + v - 1)P + v = 0$$

or $$v = uP + \frac{P}{P-1}$$

This is of Clairaut's form. Its solution is

$$v = cu + \frac{c}{c-1}$$

where c is arbitrary constant. The general solution is

$$y^2 = cx^2 + \frac{c}{c-1}$$

or $$c^2x^2 - c\left(x^2 + y^2 - 1\right) + y^2 = 0$$

The c – discri min ant is

$$\left(x^2 + y^2 - 1\right)^2 = 4x^2y^2$$

or $$\left[x^2 + y^2 - 1 - 2xy\right]\left[x^2 + y^2 - 1 + 2xy\right] = 0$$

or $$\left[(x-y)^2 - 1\right]\left[(x+y)^2 - 1\right] = 0$$

p – discriminant is also the same. Hence it is singular solution.

Exp. 2. Final the general and singular solutions of the differential equation

$$x^2p^2 + yp(2x + y) + y^2 = 0$$

Sol. Let us put $u = xy,\ v = y$ so that $x = u/v$ (1)

then $dy = dv,\ dx = \dfrac{vdu - udv}{v^2}$

Therefore, $p = \dfrac{dy}{dx} = \dfrac{v^2 dv}{vdu - udv} = \dfrac{v^2 P}{v - uP}$, where $\dfrac{dv}{du} = P$

The given equation is written as

$$(xp + y)^2 + y^2 p = 0$$

or $$\left(\frac{x}{y}p + 1\right) + p = 0$$

Substituting the values of x, y, and p in the above equation we get

$$\left(\frac{u}{v^2}.\frac{v^2 P}{v - uP} + 1\right)^2 + \frac{v^2 P}{v - uP} = 0$$

or $$\left(\frac{uP + v - uP}{v - uP}\right)^2 + \frac{v^2 P}{v - uP} = 0$$

or $$v = uP - \frac{1}{P}$$

which is in Clairaut's form. Its general solution is

$$v = uc - \frac{1}{c}$$

i.e. $$c^2 xy - cy - 1 = 0.$$

Thus c–discriminant is $y^2 = -4xy$ i.e. $y(y + 4x) = 0$ and p–discriminant is $y^2(2x + y)^2 = 4x^2y^2$ i.e. $y^3(4x + y) = 0$. Hence the singu' solution is

$$y(y + 4x) = 0.$$

PROBLEMS

Find the general and singular solutions of

1. $(y - px)(p - 1) = p$
2. $y^2 - 2xyp + p^2(x^2 - 1) = a^2$
3. $x^2(y - px) = yp^2$
4. $9p^2(2 - y)^2 = 4(3 - y)$
5. $(px - y)^2 = p^2 - 1$
6. $p = \log(xp - y)$
7. $x^3p^2 + x^2yp + a^3 = 0$

8. $p^2 \cos^2 y + p \sin x \cos x \cos y = \sin y \cos^2 x$
[Hint: Put $u = \sin x$, $v = \sin y$].

ANSWERS

1. $(y - cx)(c - 1) = c$; $(y - x)^2 - 2(x + y) + 1 = 0$
2. $c^2(1 - x^2) + 2cxy + a^2 - y^2 = 0$; $y^2 + a^2x^2 = a^2$
3. $y^2 = cx^2 + c^2$; $4y^2 + x^4 = 0$
4. $y^2(3 - y) = (x + c)^2$; $y = 3$
5. $(cx - y)^2 = c^2 - 1$; $x^2 - y^2 = 1$
6. $c = \log(cx - y)$; $y = x(\log x - 1)$
7. $c^2 + cxy + a^3x = 0$; $(xy^2 - 4a^3)x = 0$
8. $\sin y = c \sin x + c^2$; $4 \sin y - \sin^2 x = 0$.

3. Trajectories.

Trajectory is a curve which cuts every member of the family of curves according to a given law. Mostly, we consider a system of curves to be obtained intersecting a given system at a constant angle. In case this angle is a right angle, the trajectory is called an Orthogonal trajectory, otherwise it is called oblique.

4. Orthogonal trajectory for curves in Cartesian coordinates.

Suppose differential equation of a given family of curves be

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \text{.....(1)}$$

and differential equation of another family of curves be

$$F\left(X, Y, \frac{dY}{dX}\right) = 0 \qquad \text{.....(2)}$$

The slope of the curves of two families be $\frac{dy}{dx}$ and $\frac{dY}{dX}$ and member of family (1) cuts members of family (2) at an angle α, we have

$$\tan\alpha = \frac{\frac{dY}{dX} - \frac{dy}{dx}}{1 + \left(\frac{dY}{dX}\right)\left(\frac{dy}{dx}\right)} \quad .....(3)$$

Relation (3) gives

$$\frac{dy}{dx} = \frac{\frac{dY}{dX} + \tan\alpha}{1 - \left(\frac{dY}{dX}\right)\tan\alpha} \quad .....(4)$$

If $\alpha = 90°$, then from (3),

$$\frac{dY}{dX}\left(\frac{dy}{dx}\right) = -1$$

i.e. $$\frac{dY}{dX} = \frac{-1}{\left(\frac{dy}{dx}\right)}$$

Rule. To find the different equation of orthogonal trajectories, we replace dy/dx by −1/(dy/dx) in the differential equation of original family of curves.

Exp. 1. Find the orthogonal trajectories of the family of parabolas $y^2 = cx$.

Sol. The given equation of family of curves is

$$y^2 = cx \quad .....(1)$$

Differentiating (1), with respect to x, we get

$$2y\frac{dy}{dx} = c \quad .....(2)$$

Eliminating arbitrary constant c from (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{2x} \quad .....(3)$$

The family of orthogonal trajectories of (1) is given by the differential equation

$$-\frac{dx}{dy} = \frac{y}{2x}$$

or $$2x\,dx + y\,dy = 0$$

On integration, we have

$$x^2 + \frac{y^2}{2} = a\ . \quad \text{a is arbitrary constant}$$

or $$\frac{x^2}{(\sqrt{a})^2} + \frac{y^2}{(\sqrt{2a})^2} = 1$$

This is a family of ellipses as the orthogonal trajectories of the family of parabolas.

Exp. 2. Show that the family of ellipses

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad \text{(GKP,2004)} \qquad .....(1)$$

(where a, b are constant and λ is parameter) is self orthogonal.

Sol. Differentiating (1) with respect to x, we get

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$$

or $$\frac{x}{a^2+\lambda} = \frac{-y}{b^2+\lambda}\left(\frac{dy}{dx}\right) = \frac{x + y\,{}^{dy}/_{dx}}{a^2-b^2} \qquad .....(2)$$

Eliminating λ between (1) and (2), we have

$$\frac{x\left(x + y\,{}^{dy}/_{dx}\right)}{a^2-b^2} - \frac{y\left(x + y\,{}^{dy}/_{dx}\right)}{(a^2-b^2)\left({}^{dy}/_{dx}\right)} = 1$$

or $$\left(x + y\frac{dy}{dx}\right)\left(x - {}^{y}/_{dy\ dx}\right) = a^2 - b^2 \qquad .....(3)$$

which is the differential equation of the given family of ellipses. The differential equation of the orthogonal trajectories is given by

$$\left[x - \frac{y}{(dy/dx)}\right]\left[x + y\frac{dy}{dx}\right] = a^2 - b^2 \qquad .....(4)$$

But, we observe that equations (3) and (4) are identical. Therefore, the equation of the family of orthogonal trajectories is the same as that of the given curves. Hence the given family of ellipses is **'self orthogonal'**.

5. Orthogonal trajectories in polar form.

Suppose Φ and ϕ be the angles between the tangent and radius vector of two family of curves, then

$$\tan\Phi = \frac{Rd\Theta}{dR} \quad \text{and} \quad \tan\phi = \frac{rd\theta}{dr}$$

If α is the angle between the tangents to the two family of curves, then

$$\alpha = \Phi \sim \phi$$

and

$$\tan\alpha = \frac{\tan\Phi - \tan\phi}{1 + \tan\Phi\tan\phi}$$

It $\alpha = 90^\circ$, $\quad 1 + \tan\Phi\tan\phi = 0$

Therefore, $\left(R\frac{d\Theta}{dR}\right)\left(\frac{rd\theta}{dr}\right) = -1 \quad \text{or} \quad \frac{Rd\Theta}{dR} = -1/(rd\theta/dr)$

Thus we see that the differential equation of the orthogonal trajectories is obtained from the differential equation (polar form) of a given family of curves by replacing

$$\frac{rd\theta}{dr} \quad \text{by} \quad -1/(rd\theta/dr)$$

Exp. 1. Find the orthogonal trajectories of the family of cardioids

$$r = a(1 + \cos\theta) \qquad .....(1)$$

where a is parameter. (GKP,2005)

Sol. Differentiating (1) with respect to θ, we have

$$\frac{dr}{d\theta} = -a\sin\theta \qquad .....(2)$$

Eliminating a from (1) and (2), we have

$$\frac{rd\theta}{dr} = \frac{1+\cos\theta}{-\sin\theta} \qquad .....(3)$$

which is differential equation of the family of cardioids.

Now replacing $(rd\theta/dr)$ by $-1/(rd\theta/dr)$ in (3), we get

$$\frac{-1}{(rd\theta/dr)} = \frac{1+\cos\theta}{-\sin\theta}$$

or $$\frac{rd\theta}{dr} = \frac{\sin\theta}{1+\cos\theta} \qquad .....(4)$$

which is the differential equation of the orthogonal trajectories. By separating the variables, we have

$$\frac{dr}{r} = \frac{(1+\cos\theta)}{\sin\theta}d\theta$$

Integrating, we have

$$\log r = \log\tan\left(\theta/2\right) + \log\sin\theta + \log c$$

or $$r = c\tan\left(\theta/2\right)\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 2c\sin^2\frac{\theta}{2}$$

or $$r = c(1-\cos\theta) \qquad .....(5)$$

Equation (5) represents another family of cardioids.

Exp. 2. Find the orthogonal trajectories of system of curves

$$r^n \sin n\theta = a^n$$

(GKP., 2006, 2009) .(1)

where a is parameter.

Sol. Taking log of both sides

$$\log r^n + \log\sin n\theta = \log a^n$$

and differentiating with respect to θ, we have

or $$\frac{ndr}{rd\theta} + n\cot n\theta = 0$$

or $$\frac{dr}{rd\theta} = -\cot n\theta$$

or $$\frac{1}{(rd\theta/dr)} = -\cot n\theta.$$

For the orthogonal trajectories, we have

$$\frac{rd\theta}{dr} = \cot n\theta$$

or $\frac{dr}{r} = \tan n\theta d\theta$

Integrating, we have

$$\log r = \frac{1}{n}\log \sec n\theta + \log c$$

or $r^n = c^n \sec n\theta$

or $r^n \cos n\theta = c^n$.

which is the equation of the system of orthogonal trajectories.

PROBLEMS

1. Find the orthogonal trajectories of the family of semi - cubical parabolas $ay^2 = x^3$, where a is parameter. **(GKP., 2007)**
2. Find the orthogonal trajectories of the family of circles $(x-1)^2 + y^2 = 2\alpha x$, where α is parameter. **(GKP., 2008)**
3. Show that the orthogonal trajectories of the system of parabolas $y^2 = 4a(x+a)$ belong to the system itself.
4. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.
5. Find the orthogonal trajectories of the family of conics $y^2 - x^2 + 4xy - 2cx = 0$
6. A system of rectangular hyperbolas with centre on the origin, passes through the fixed points $(\pm a, 0)$. Show that its orthogonal trajectories are given by

$$\left(x^2 + y^2\right)^2 = 2a^2\left(x^2 - y^2\right) + c$$

[Hint. Equation of rect. hyperbolas with centre at origin is $bx^2 + 2\lambda xy - by^2 = 1$]

7. Find the orthogonal trajectories of

(a) $r = a/(1+\cos\theta)$ (b) $r^n = a^n \sin n\theta$

(c) $A = r^2 \cos\theta$ (d) $r = c(\cos\theta + \sin\theta)$

8. Find the orthogonal trajectories of a family of circles which touch a given line at given point.

ANSWERS

1. $2x^2 + 3y^3 = c$
2. $x^2 + (y - c)^2 = 1 + c^2$
4. $x^2 + y^2 - 2a^2 \log x = c$
5. $y^3 + 4x^3 + 3x^2 y = 0$
7. (a) $r = b/(1 - \cos\theta)$
   (b) $r^n = c^n \cos n\theta$
   (c) $B = r \sin^2 \theta$
   (d) $r = a(\cos\theta - \sin\theta)$
8. Circles with centres on the given line and passing through the gives point.