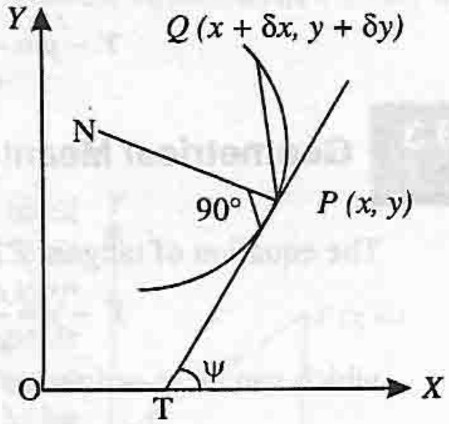
3 Tangents and Normals

3.1 Definition

Tangent : Let $P(x, y)$ be any given point on a curve and $Q(x + \delta x, y + \delta y)$ any other point on it. As Q tends to P, the straight line PQ tends, in general, to a definite straight line (weather Q be taken on one side of P or the other). This straight line is called the tangent to the curve at P.



Normal : Normal at P is a line through P at right angles to the tangent at P. Thus PT is tangent and PN is normal at P.

3.2 Equation of the Tangent

If $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two points on the curve $y = f(x)$ then equation of chord PQ is

$$Y - y = \frac{(y + \delta y) - y}{(x + \delta x) - x}(X - x)$$

(where X and Y are current coordinates)

or

$$Y - y = \frac{\delta y}{\delta x}(X - x).$$

Now as Q tends to P then $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$ and chord PQ becomes tangent to the curve at $P(x, y)$. Therefore equation of tangent at $P(x, y)$ is

$$\boldsymbol{Y - y = \frac{dy}{dx}(X - x).}$$

Note 1. The equation of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

$$Y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(X - x_1)$$

where $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the value of $\frac{dy}{dx}$ at (x_1, y_1).

Note 2. If the curve is given in the parametric form say $x = f(t)$ and $y = \phi(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)},$$

and equation of tangent at any point 't' is given by

$$Y - \phi(t) = \frac{\phi'(t)}{f'(t)}[x - f(t)]$$

3.3 Equation of the Normal

Normal being a straight line perpendicular to the tangent, its slope, m is given by

$$m\frac{dy}{dx}=-1$$

$$\Rightarrow \qquad m=-\frac{dx}{dy}.$$

Hence equation of normal to the curve $y=f(x)$ at $P\ (x,\ y)$ is

$$Y-y=-\frac{dx}{dy}(X-x).$$

3.3 Geometrical Meaning of $\frac{dy}{dx}$

The equation of tangent PT to the curve at $P\ (x,\ y)$ is

$$Y-y=\frac{dy}{dx}(X-x)$$

which can be re-written as

$$Y=\left(\frac{dy}{dx}\right)X+\left(y-x\frac{dy}{dx}\right) \qquad ...(1)$$

which is of the form

$$Y=mX+C \qquad ...(2)$$

which represents a straight line whose gradient m is equal to $\tan\psi$, where ψ is t angle which tangent at P makes with the positive direction of x-axis. Here ψ is measured anticlock wise direction.

Comparing (1) and (2), we get

$$\frac{dy}{dx}=\tan\psi,$$

i.e. the differential coefficient $\frac{dy}{dx}$ at the point $P\ (x,\ y)$ of a curve $y=f(x)$ is equal to $\tan\psi$.

Corollary : Tangents parallel to the co-ordinate axis.

If a tangent is parallel to the axis of x, then $\psi=0$ i.e. $\tan\psi=0$ and so we have $dy/dx=$ *at that point.*

If a tangent is parallel to axis of y, then $\psi=\pi/2$ and $\tan\psi=\tan\pi/2=\infty$. So we hav $dy/dx=\infty$ *or* $dx/dy=0$ *at that point.*

Working Rule : In order to find the equations of tangent and normal to a given curv at a given point, we may use the following method :

(i) Find $\frac{dy}{dx}$ from the given equation $y=f(x)$.

(ii) Find the value of $\frac{dy}{dx}$ at the given point $P\ (x_1,\ y_1)$.

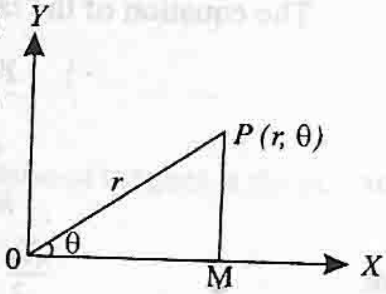
(iii) If $\left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}$ is a non-zero finite number, then obtain the equation of tangent and normal at $(x_1,\ y_1)$ by using the formulae

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}(x-x_1) \text{ and } y-y_1=-\left(\frac{dx}{dy}\right)_{(x_1,\ y_1)}(x-x_1),$$

respectively. Otherwise go to step (iv).

(iv) If $\left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} = 0$, then the equations of the tangent and normal at (x_1, y_1) are $y - y_1 = 0$ and $x - x_1 = 0$ respectively. If $\left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} = \pm\infty$, then the equations of the tangent and normal at (x_1, y_1) are $(x - x_1) = 0$ and $y - y_1 = 0$ resp.

3.4. Polar Co-ordinates



Let O be a fixed point called the pole and OX, a fixed straight line called the initial line. The position of a point 'P' is determined relative to O if θ, the magnitude of the angle XOP and r, the length of OP are given. Then (r, θ) are said to be polar co-ordinates of P; r and θ are called the radius vector and vectorial angle respectively. The angle θ, is considered to be positive when measured in the anti clockwise direction and r is said to be positive when measured away from O along the line bounding the vectorial angle. If the co-ordinates (r, θ) of a point satisfy a given equation different positions of that point can be found by assigning values to θ and then calculating the corresponding values of r, then locus is a curve of which the given equation is called the polar equation.

From the figure it is clear that ΔOPM gives the relation between cartesian and polar co-ordinates of a point P.

So, $\quad x = r\cos\theta,\ y = r\sin\theta$

Conversely if x and y be given, we have

$$x^2 + y^2 = r^2 \quad \text{and} \quad \tan\theta = y/x.$$

If the positive value of r be taken then the equation

$$\cos\theta = \frac{x}{r},\quad \sin\theta = \frac{y}{r}$$

give one value of θ between $-\pi$ and π or between 0 and 2π. The pairs of co-ordinates (r, θ), $(r, -2\pi + \theta)$, $(-r, \pi + \theta)$, $(-r, -\pi + \theta), \ldots$ all represent the same point.

ILLUSTRATIVE EXAMPLES

Example 1. Find the tangent at (1,2) to the curve $y = x^3 + 1$.

Solution. The given curve is $y = x^3 + 1$

$$\therefore \quad \frac{dy}{dx} = 3x^2$$

and $$\left(\frac{dy}{dx}\right)_{(1,2)} = 3.1^2 = 3.$$

Hence the required tangent is

$$y - 2 = 3(x - 1).$$

Example 2. Find the equation of the tangent at (x, y) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. The equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \qquad \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

The equation of the tangent at (x, y) is, therefore

$$Y-y=-\frac{b^2x}{a^2y}(X-x),$$

or $$\frac{y}{b^2}(Y-y)+\frac{x}{a^2}(X-x)=0,$$

or $$\frac{Xx}{a^2}+\frac{Yy}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Hence the tangent at (x, y) to $x^2/a^2+y^2/b^2=1$ is

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1.$$

Example 3. Find the equation of tangent and normal at $\theta=\pi/2$ to the curve $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$.

Solution. The given curve is

$$x=a(\theta+\sin\theta), \qquad y=a(1+\cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta}=a(1+\cos\theta),\ \frac{dy}{d\theta}=-a\sin\theta.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\tan\theta/2.$$

Hence $$\left(\frac{dy}{dx}\right)_{\theta=\pi/2}=-\tan\frac{\pi}{4}=-1.$$

Also for $\theta=\pi/2$, the point on the curve is $\left\{a\left(\frac{\pi}{2}+1\right), a\right\}$

Then the equation of tangent at $\theta=\pi/2$ is

$$y-a=(-1)\left\{x-a\left(\frac{\pi}{2}+1\right)\right\}$$

or $$x+y=a\left(\frac{\pi}{2}+2\right)$$

Equation of the normal at $\theta=\pi/2$ is

$$(y-a)(-1)+x-a\left(\frac{\pi}{2}+1\right)=0$$

or $$x-y=a\frac{\pi}{2}.$$

Example 4. Find the equations of tangent and normal to the curve $y^2 = 3x^2 + 1$ at (1, 2).

Solution. Given that

$$y^2 = 3x^2 + 1 \qquad \text{...(i)}$$

Now, differentiating (i) with respect to x

$$2y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{3x}{y}$$

At (1,2), $$\left[\frac{dy}{dx}\right]_{(1,\,2)} = \frac{3\cdot 1}{2} = \frac{3}{2} \qquad \text{...(ii)}$$

Thus, the slope of tangent = 3/2 and therefore the equation of tangent at the point (12) is given by

$$y - 2 = \frac{3}{2}(x-1)$$

$$\Rightarrow \qquad 3x - 2y + 1 = 0 \qquad \text{...(iii)}$$

Again, the slope of normal $= -2/3$ and therefore, the equation of normal at the point (1, 2) is given by

$$(y-2) = -2/3\,(x-1)$$

$$\Rightarrow \qquad 2x + 3y - 8 = 0 \qquad \text{...(iv)}$$

Example 5. Show that the abscissae of the points on the curve $y = x(x-2)(x-4)$ where the tangents are parallel to the axis of x are given by $x = 2 \pm (2/\sqrt{3})$.

Solution : The given curve is

$$y = x(x-2)(x-4)$$

$$= x^3 - 6x^2 + 8x \qquad \text{...(i)}$$

Differentiating (i) with respect to x

$$\frac{dy}{dx} = 3x^2 - 12x + 8$$

The tangent to the given curve at the point (x, y) is parallel to the axis of x, if $\frac{dy}{dx} = 0$

i.e., $3x^2 - 12x + 8 = 0$

$$\Rightarrow \qquad x = (12 \pm \sqrt{144-96})/6$$

$$= 2 \pm (2/\sqrt{3})$$

Hence the abscissae of the required points are $2 \pm \left(\frac{2}{\sqrt{3}}\right)$.

Example 6. Find the equation of the tangent to the curve $xy^2 = 4(4-x)$ at the point where it is cut by the straight line $y = x$.

Sol. The given equation of curve is

$$xy^2 = 4(4-x) \qquad \text{...(i)}$$

Intersect to the straight line

$$y = x \qquad \text{...(ii)}$$

From (i) & (ii)

$$x^3 = 4(4 - x)$$

$$\Rightarrow \quad x^3 = 16 - 4x$$

$$\Rightarrow \quad x^3 + 4x - 16 = 0$$

$$\Rightarrow \quad (x-2)(x^2 + 2x + 8) = 0$$

$$\Rightarrow \quad x = 2 \;\&\; x = \text{imaginary}$$

For $x = 2$ & equation (ii), we get

(ii) intersect (i) at (2, 2)

We have to find the tangent of (i) at (2, 2), For this differentiate (i) w.r. to x

$$2xy\frac{dy}{dx} + y^2 = 0 - 4$$

$$\frac{dy}{dx} = \frac{-4 - y^2}{2xy}$$

At (2, 2)

$$\left[\frac{dy}{dx}\right]_{(2,\,2)} = \frac{-4-4}{8} = -1$$

Then the tangent at (2, 2)

$$y - 2 = (-1)(x - 2)$$

$$\Rightarrow \quad y - 2 = -x + 2$$

$$\Rightarrow \quad x + y - 4 = 0$$

EXERCISE 3.1

1. Find the equation of the tangent at the point (x, y) on each of the following curves :

   (i) $y^2 = 4ax$ (ii) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

   (iii) $y = a \log \sin x$

2. Find the equation of the tangent at the point t on each of the following curves :

   (i) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

   (ii) $x = a\sin^3 t$, $y = b\cos^3 t$

3. Find the points at which the tangent to each of the following curves is (a) parallel to and (b) perpendicular to the axis of x :

   (i) $y = x^{2/3}(x + a)^{1/3}$, (ii) $ax^2 + 2hxy + by^2 = 1$

4. Show that the condition that the curves

   $$ax^2 + by^2 = 1 \text{ and } a'x^2 + b'y^2 = 1 \qquad \textit{(GKP 2016)}$$

   should intersect orthogonally is that

   $$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}. \qquad \textit{(SU 2016)}$$

5. In the curve $x^m y^n = a^{m+n}$, prove that the portion of the tangent intercepted between the axes, is divided at its point of contact into segments which are in a constant ratio.

6. Prove that the curve $x/a + y/b = 1$ touches the curve $y = be^{-xa}$ at the point where the curve crosses the axis of y.

7. Prove that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b), whatever be the value of n.

8. Find the equations of the tangent and normal to the curve
$$y(x-2)(x-3) - x + 7 = 0$$
at the point where it cuts the axis of x.

9. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axes of x, show that its equation is
$$y\cos\phi - x\sin\phi = a\cos 2\phi.$$ ***(GKP 2016, SU 2016, SID 2017)***

10. In the catenary $y = a\cos h\,(x/a)$, prove that the length of the portion of the normal intercepted between the curve and the axis of x is y^2/a.

11. If $x\cos\alpha + y\sin\alpha = p$ touches the curve $b^2x^2 + a^2y^2 = a^2b^2$ then show that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

12. Show that for the curve $y = x^{2/3}$, the tangent at the origin is $x = 0$ although $\dfrac{dy}{dx}$ does not exist.

13. Show that the normal at the point $\theta = \pi/2$ on the curve $x = 3\cos\theta - \cos^3\theta$ $y = 3\sin\theta - \sin^3\theta$ passes through the origin.

ANSWERS

1. (i) $yY = 2a(X + x)$;

(ii) $\{2y(x^2 + y^2) + a^2y\}Y + \{2x(x^2 + y^2) - a^2x\}X = a^2(x^2 - y^2)$;

(iii) $Y - y = a\cot x\,(X - x)$.

2. (i) $Y = (X - at)\tan t/2$;

(ii) $X/a\sin t + Y/b\cos t = 1$.

3. (i) (a) at $(-2a/3, 4^{1/3}\,a/3)$; (b) at $(0, 0), (-a, 0)$

(ii) (a) where $ax + hy = 0$ intersects the curve;

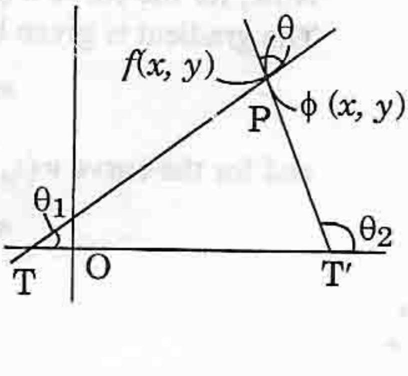
(b) where $by + hx = 0$ intersects the curve.

8. $20y - x + 7 = 0,\ y + 20x - 140 = 0$.

3.6 Angle of intersection of two Cartesian curve

The angle of intersection of two curves is defined as the angle between their tangents at their point of intersection.

Let the equation of the two curves be
$$f(x, y) = 0 \qquad \ldots(i)$$
and
$$\phi(x, y) = 0 \qquad \ldots(ii)$$
Let $(dy/dx)_1$ stand for the dy/dx of the curve (i) and $(dy/dx)_2$ for the dy/dx of the curve (ii).



Suppose (x_1, y_1) is a point of intersection of (i) and (ii). Let m_1 and m_2 be the gradients (or slopes) of the tangents at the point (x_1, y_1) to curves (i) and (ii) respectively. Then,

$$m_1 = (dy/dx)_1 \text{ at the point } (x_1, y_1)$$

and $$m_2 = (dy/dx)_2 \text{ at the point } (x_1, y_1)$$

Suppose θ is the angle of intersection of (i) and (ii) at the point (x_1, y_1)

then $$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Case 1 : If $m_1 = m_2$, the angle of intersection θ is $0°$. In this case the two curves have the same tangent at the point (x_1, y_1) and thus the two curves touch each-other at the point (x_1, y_1).

Case 2 : If $m = \infty$, $m_2 = 0$ or if $m_1 = 0$, $m_2 = \infty$, the *n* the angle of intersection θ is $90°$.

Case 3 : If $m_1 m_2 = -1$, again the angle of intersection is $90°$.

3.7 Orthogonal Families of Curves

Let $u = u(x, y)$ and $v = v(x, y)$ be the functions of x and y such that

$$u + iv = f(x + iy) \qquad \ldots\text{(i)}$$

$$i = \sqrt{-1}$$

Then, the curves $u(x, y)$ = constant and $v(x, y)$ = constant from two families of curves such that each member of one family intersects each member of the other family orthogonally.

Differentiating (i) partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(x + iy) \qquad \ldots\text{(ii)}$$

and $$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'(x + iy) \qquad \ldots\text{(iii)}$$

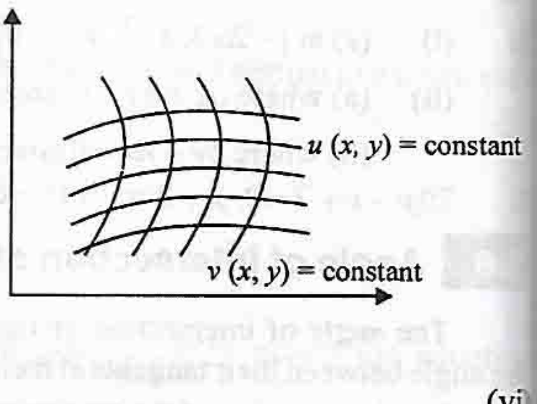
Putting the value of $f'(x + iy)$ from (ii) we have

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \qquad \ldots\text{(iv)}$$

Equating real and imaginary parts from (iv) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial v}{\partial x} = -\frac{\partial x}{\partial y} \qquad \ldots\text{(v)}$$



Now, for the curve $u(x, y)$ = constant

The gradient is given by

$$m_1 = -\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y} \qquad \ldots\text{(vi)}$$

and for the curve $v(x, y)$ = constant, the gradient is given by

$$m_2 = -\frac{\partial v}{\partial x}\bigg/\frac{\partial u}{\partial y} \qquad \ldots\text{(vii)}$$

From (vi), & (vii)

$$\therefore \qquad m_1 m_2 = \left(\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y}\right)\left(-\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}$$

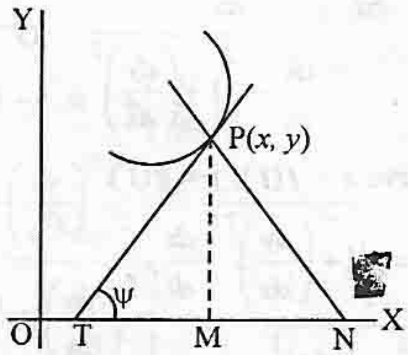
Using (v), we get

$$m_1 m_2 = \frac{\partial v}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\bigg/\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -1 \qquad \ldots(8)$$

Hence the two families of curves $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$ intersect orthogonally.

3.8 Cartesian Subtangent and Sub normal

Let the tangent and normal and any point P on a curve meet the x-axis in T and N respectively. Let PM be ordinate of P. Then TM is called subtangent and MN is called the subnormal.



If the angle which the tangent makes with the x-axis be ψ then we have $\tan\psi = dy/dx$ and $MP = y$.

Thus, $\quad$ **Subtangent** $= TM = y\cot\psi = y\Big/\left(\dfrac{dy}{dx}\right) = y\dfrac{dx}{dy}$.

Subnormal $= MN = PM\tan\psi = y\dfrac{dy}{dx}$.

Length of tangent $= PT = PM \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi}$

$$= y\cdot\frac{\sqrt{1+\tan^2\psi}}{\tan\psi}.$$

$$= y\cdot\frac{\sqrt{1+(dy/dx)^2}}{(dy/dx)}.$$

Length of the normal $= PN = PM\sec\psi$

$$= y\sqrt{1+\tan^2\psi}$$

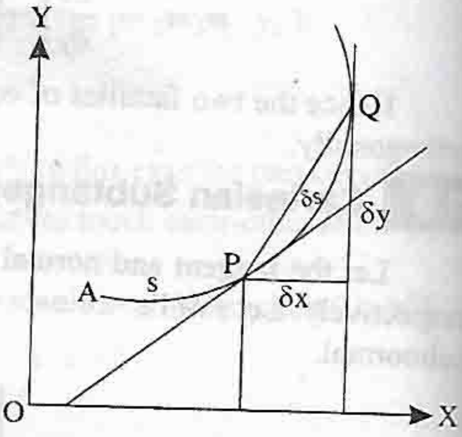
$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

3.9 (a) Differential coefficient of the Length of Arc : (Cartesian Form)

Suppose that A be a fixed point on the curve and $P(x, y)$ be an arbitrary point on the curve such that arc $AP = s$. Let $Q\ (x + \delta x,\ y + \delta y)$ be another point on the curve such that arc $AQ = s + \delta s$.

When $Q \to P$, $\delta x \to 0$ and

$$\lim_{\delta x \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$



$$\therefore \quad \lim_{\delta x \to 0} \frac{\text{chord } PQ}{\partial x} \cdot \frac{\partial x}{\delta x} = 1$$

or $$\lim_{\delta x \to 0} \frac{\text{chord } PQ}{\delta x} = \lim_{\delta x \to 0} \frac{\delta s}{\delta x}$$

or $$\lim_{\delta x \to 0} \frac{\sqrt{\delta x^2 + \delta y^2}}{\delta x} = \frac{ds}{dx}$$

or $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

In case of parametric curve $x = f(t)$, $y = g(t)$,

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

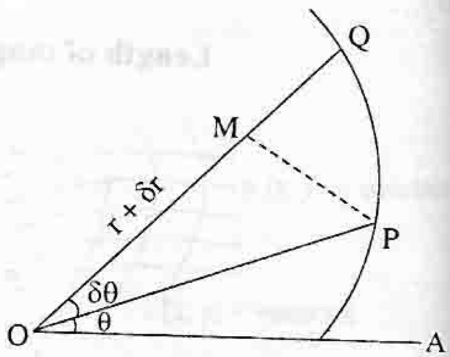
3.9 (b) Differential Coefficient of the Length of Arc (Polar Form)

Let $P\ (r, \theta)$ and $Q\ (r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve $r = f(\theta)$. Let the length of the arc. AP, measured from the fixed point A, be s.

Then $$AQ = s + \delta s.$$

$$\therefore \quad \text{arc } PQ = \delta s.$$

Let δc denote the length of the chord PQ. , w perpendicular PM from P on OQ.



Now $\quad PM = r \sin \delta\theta$ and $OM = r \cos \delta\theta$.

Again $\quad MQ = OQ - OM$

$$= r + \delta r - r \cos \delta\theta.$$

Now from the right angled triangle PMQ

$$PQ^2 = PM^2 + MQ^2$$

or $$(\delta c)^2 = (r \sin \delta\theta)^2 + (r + \delta r - r \cos \delta\theta)^2$$

$$= r^2 \sin^2 \delta\theta + r^2 + (\delta r)^2 + r^2 \cos^2 \delta\theta + 2r\,\delta r - 2r^2 \cos \delta\theta$$

$$- 2r\,\delta r \cos \delta\theta.$$

$$= r^2 + r^2 + (\delta r)^2 + 2r\delta r - 2r^2 \cos \delta\theta - 2r\,\delta r \cos \delta\theta$$

$$= 2r^2\ (1 - \cos \delta\theta) + 2r\,\delta r\ (1 - \cos \delta\theta) + (\delta r)^2$$

$$= 2r^2 . 2\sin^2 \frac{1}{2}\delta\theta + 2r\,\delta r . 2\sin^2 \frac{\delta\theta}{2} + (\delta r)^2.$$

$$= 4r^2 \sin^2 \frac{\delta\theta}{2} + 4r . \delta r . \sin^2 \frac{\delta\theta}{2} + (\delta r)^2. \qquad ...(1)$$

Dividing both sides of equation (1) by $(\delta\theta)^2$, we get

$$\left(\frac{\delta c}{\delta\theta}\right)^2 = 4r^2 \cdot \frac{\sin^2 (\delta\theta/2)}{(\delta\theta)^2} + 4r \cdot \delta r \frac{\sin^2 (\delta\theta/2)}{(\delta\theta)^2} + \left(\frac{\delta r}{\delta\theta}\right)^2$$

or
$$\left(\frac{\delta c}{\delta s}\right)^2 \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 \left(\frac{\sin (\delta\theta/2)}{\delta\theta/2}\right)^2 + r\,\delta r \left(\frac{\sin (\delta\theta/2)}{\delta\theta/2}\right)^2 + \left(\frac{\delta r}{\delta\theta}\right)^2 \qquad ...(2)$$

Proceeding to the limit as $Q \to P$, *i.e.* as $\delta\theta \to 0$, we have

$$\frac{\delta c}{\delta s} \to 1;\ \frac{\delta s}{\delta\theta} \to \frac{ds}{d\theta};\ \frac{\sin (\delta\theta/2)}{(\delta\theta/2)} \to 1 \text{ and } \frac{\delta r}{\delta\theta} \to \frac{dr}{d\theta}.$$

Therefore (2) becomes

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 . 1 + r . 0 . 1 + \left(\frac{dr}{d\theta}\right)^2.$$

or
$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

or
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

Similarly dividing each term of equation (1) by δr^2 and proceeding of the limit as $Q \to P$. We can have

$$\frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]}$$

3.10 Other Formulae

We know that

$$\tan\phi = r\frac{d\theta}{dr}.$$

$$\therefore \qquad \cos\phi = \frac{1}{\sqrt{(1+\tan^2\phi)}}.$$

$$= \frac{1}{\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]^{1/2}} = \frac{dr/d\theta}{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}}$$

$$= \frac{dr/d\theta}{ds/d\theta}$$

i.e.
$$\cos\phi = \frac{dr}{ds}.$$

Again $$\sin\phi = \frac{\tan\phi}{\sqrt{(1+\tan^2\phi)}}$$

$$= \frac{r\,(d\theta/dr)}{\sqrt{(1+r\,d\theta/dr)^2}}$$

$$= \frac{r\,[(d\theta/dr)\,(dr/d\theta)]}{\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}}$$

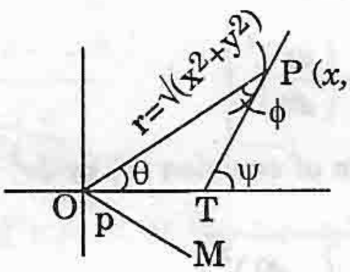
$$= \frac{r}{\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}}$$

$$= \frac{r}{dsd\theta}$$ [from § 3·9]

i.e., $$\sin\phi = r\,\frac{d\theta}{ds}.$$

Pedal Equation

A relation between p and r for a given curve is called its **pedal equation.**



To find the pedal equation from Cartesian equation : Let the equation of the curve be $y = f(x)$. The tangent to the curve at (x, y) is

$$Y - y = (X - x)\frac{dy}{dx} \qquad ...(1)$$

Thus the length p of the perpendicular on it from origin $(X = 0, Y = 0)$ is given by

$$p = \frac{x\dfrac{dy}{dx} - y}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}. \qquad ...(2)$$

Also $$r^2 = x^2 + y^2.$$

After eliminating x and y from the equation (1), (2) and (3) we get the required pedal equation.

ILLUSTRATIVE EXAMPLE

Example 7. Calculate $ds/d\theta$ for the curve $r = \log\sin 3\theta$.

Solution. Here $r = \log\sin 3\theta$. ...(1)

Differentiating, we get

$$\frac{dr}{d\theta} = 3\cot 3\theta.$$

Now $$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \quad ...(2)$$

$$= \sqrt{[r^2 + (3\cot 3\theta)^2]} = \sqrt{(r^2 + 9\cot^2 3\theta)} \quad \text{[from (2)]}$$

Example 8. For the curve $r = ae^{\theta \cot \alpha}$, prove that $\frac{s}{r}$ = constant, s being measured from the origin.

Solution. Here $$r = ae^{\theta \cot \alpha}. \quad ...(1)$$

Differentiating (1), we get

$$\frac{dr}{d\theta} = ae^{\theta \cot \alpha}.\cot \alpha = r\cot \alpha \quad ...(2)$$

Now, $$\frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]}$$

$$= \sqrt{\left[1 + \left(\frac{r}{r\cot \alpha}\right)^2\right]}$$

$$= \sqrt{(1 + \tan^2 \alpha)} = \sec \alpha.$$

i.e. $ds = \sec \alpha \,.\, dr.$

Integrating, we get

$$s = r\sec \alpha + c. \quad ...(3)$$

Initially at the origin $r = 0$ and $s = 0$, we get

$$0 = 0 + c \quad \text{from (3)}$$

or $$c = 0$$

$\therefore$ $$s = r\sec \alpha \quad \text{by (3)}$$

or $$\frac{s}{r} = \sec \alpha = \text{constant}.$$

Example 9. Prove that

$$\sin^2 \phi \frac{d\phi}{d\theta} + r\frac{d^2 r}{ds^2} = 0.$$

Solution. We know that

$$\frac{dr}{ds} = \cos \phi$$

$\therefore$ $$\frac{d^2 r}{ds^2} = -\sin \phi \cdot \frac{d\phi}{ds}$$

$\therefore$ $$\sin \phi \frac{d\phi}{ds} + \frac{d^2 r}{ds^2} = 0$$

or $$r\sin \phi.\frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds} + r\frac{d^2 r}{ds^2} = 0 \quad \text{(multiplying by } r\text{)}$$

or $$\sin^2\phi\frac{d\phi}{d\theta}+r\frac{d^2r}{ds^2}=0. \qquad \left[\because \sin\phi = r\frac{d}{d}\right.$$

Example 10. Find the pedal equation of the parabola $y^2=4a\,(x+a)$.

Solution. Here $y^2=4a\,(x+a)$. ...(1)

Differentiating (1), we get

$$2y\frac{dy}{dx}=4a$$

or $$\frac{dy}{dx}=\frac{2a}{y}.$$

Hence the tangent at (x, y) is

$$Y-y=\left(\frac{2a}{y}\right)(X-x).$$

Therefore $$p=\frac{x\,(2ay)-y}{\sqrt{[1+4a^2/y^2]}}=\frac{2ax-y^2}{\sqrt{(y^2+4a^2)}}$$

$$=\frac{2ax-4a\,(x+a)}{\sqrt{(4ax+4a^2+4a^2)}}$$

$$=\frac{-2a\,(x+2a)\,(x+2a)}{\sqrt{\{4a\,(x+2a)\}}}$$

$$=-\sqrt{\{a\,(x+2a)\}}$$

Also, $$r^2=x^2+y^2$$

$$=x^2+4a\,(x+a)=(x+2a)^2 \quad ...$$

Therefore $p^2=ar$. by (1) and (2),

which is the required pedal equation.

Example 11. For the cycloid $x=a\,(1-\cos t)$, $y=a\,(t+\sin t)$, find ds/dt, ds/dx a ds/dy.

Solution. Here $x=a\,(1-\cos t),\ y=a\,(t+\sin t)$ ...

Differentiating, we get

$$\frac{dx}{dt}=a\sin t,\ \frac{dy}{dt}=a\,(1+\cos t).$$

Therefore, $$\frac{ds}{dt}=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}$$

$$=a\,\{\sin^2 t+1+2\cos t\ \cos^2 t\}^{1/2}$$

$$=a\,\{2+(1+\cos t)\}^{1/2}$$

$$=a.[4\cos^2\ (t/2)]^{1/2}$$

$$=2\cos\,(t/2)$$

$$\frac{ds}{dx}=\frac{ds}{dt}\cdot\frac{dt}{dx}=2a\cos\,(t/2)/a\sin t$$

$$= \operatorname{cosec}(t/2),$$

and $$\frac{ds}{dy} = \frac{ds}{dt}\cdot\frac{dt}{dy}$$

$$= 2a\cos(t/2)/2a\cos^2(t/2)$$

$$= \sec(t/2)$$

Example 12. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Solution. The given curves are

$$y = 4 - x^2 \qquad ...(1)$$

and $$y = x^2 \qquad ...(2)$$

Subtracting equation (2) from equation (1) we find that abscissa of the point of intersection is given by

$$4 - 2x^2 = 0$$

or $$x = \sqrt{2}$$

From (1), $$\frac{dy}{dx} = -2x = -2\sqrt{2}, \text{ at } x = \sqrt{2}$$

from (2), $$\frac{dy}{dx} = 2x = 2\sqrt{2}, \text{ at } x = \sqrt{2}.$$

Hence, if θ is the required angle of intersection, then

$$\tan\theta = -\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 4.2} = \frac{4\sqrt{2}}{7}.$$

Therefore $$\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$$

EXERCISE 3.2

1. For the ellipse $x = a\cos t$, $y = b\sin t$, prove that

$$\frac{ds}{dt} = 1(1 - e^2 \cos^2 t)^{1/2}$$

2. In the curve $y = a\log\sec(x/a)$, prove that

$$\frac{d^2x}{ds^2} = -\frac{1}{2a}\sin\frac{2x}{a}.$$

3. For the parabola $y^2 = 4ax$, prove that

$$\frac{ds}{dx} = \left(1 + \frac{a}{x}\right)^{1/2}.$$

4. Calculate ds/dt for the following curves :

(i) $x = t^2$, $y = t - 1$,

(ii) $x = a\sec t$, $y = b\tan t$

5. Calculate $ds/d\theta$ for the curve : $r = \frac{1}{2}\sec^2\theta$.

6. Show that the pedal equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

7. Show that the pedal equation of the ellipse $\frac{l}{r} = 1 + e\cos\theta$ is

$$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right).$$

8. In the curve $y = a\log\sec(x/a)$, prove that

$$\frac{d^2x}{dt^2} = -\frac{1}{2a}\sin\left(\frac{2x}{a}\right).$$

9. In the curve $r^m = a^m \cos m\theta$, prove that

$$\frac{ds}{d\theta} = a\sec^{(m-1)/m} m\theta.$$

10. Show that for any pedal curve $p = f(r)$, $\frac{ds}{dr} = \frac{r}{\sqrt{(r^2 - p^2)}}$.

11. Find the points of intersection and the angle of intersection of the curves $x^2 - y^2 = a^2$ and $x^2 + y^2 = z^2\sqrt{z}$

12. Find $\frac{ds}{dx}$ for the following curves :

(i) $y^2 = 4ax$ (ii) $y = a\log\sec(x/a)$

13. Find $\frac{ds}{dt}$ for the parametric curves $x = t^2$, $y = t - 1$.

14. Find the pedal equation of the following curves

(i) $y^2 = 4ax$ (ii) $x^2 - y^2 = a^2$

(iii) $x^{2/3} + y^{2/3} = a^{2/3}$ (iv) $c^2(x^2 + y^2) = x^2y^2$

15. Find the length of subtangent and sub-normal at any point 'θ' on curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.

16. Show that for the curve $x^{m+n} = a^{m-n}y^{2n}$, the mth power of subtangent varies as the nth power of sub-normal.

17. Prove that in the ellipse $x^2/a^2 + y^2/b^2 = 1$ the length of the normal varies inversely as the perpendicular from the origin on the tangent.

18. For any curve show that $\frac{\text{Subnormal}}{\text{Subtangent}} = \left(\frac{\text{Length of normal}}{\text{Length of tangent}}\right)^2$

ANSWERS

4. (i) $\sqrt{(1+4t^2)}$;

 (ii) $(a^2 \sin^2 t + b^2)^{1/2} \sec^2 t$.

5. $\frac{1}{2}(1+4\tan^2\theta)^{1/2} \sec^2\theta$.

11. $\left(\pm\frac{a}{\sqrt{2}}\sqrt{(\sqrt{2}+1)} \pm \frac{a}{\sqrt{2}}\sqrt{(\sqrt{2}-1)}\right); \frac{\pi}{4}$

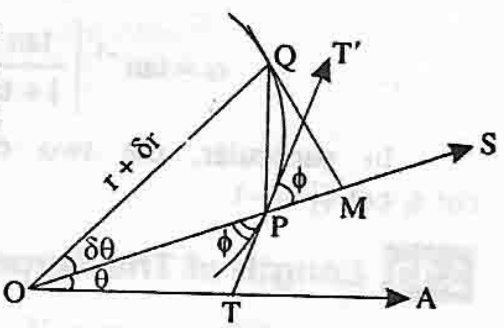
12. (i0 $\sqrt{1+a/x}$, (ii) $\sec(x/a)$

13. $\sqrt{1+4t^2}$, $\sqrt{2}a e^t$

14. (i) $p^2 p^2 (r^2+4a^2)(p^2+4a^2) = a^2 (p^2-r^2)^2$

 (ii) $pr = a^2$ (iii) $r^2 = a^2 - 3p^2$ (iv) $\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$

3.11 Angle Between Radius Vector and Tangent



Let $P(r, \theta)$ be any point of the curve $r = f(\theta)$. Take any other point $Q(r+\delta r, \theta+\delta\theta)$ in the neighbourhood of P on the curve. Produce OP to S. Let PT be the tangent at P and let $\angle SPT' = \phi$.

Let $\angle SPQ = \alpha$ so that $\angle SPT' = \phi$ is the limit of α as $Q \to P$.

We have $\angle QPO = \pi - \alpha$,

$\angle PQO = \angle SPQ - \angle QOP = \alpha - \delta\theta$.

Now by the applications of sine formula to the ΔQPO we get

$$\frac{OQ}{OP} = \frac{\sin \angle OPQ}{\sin \angle PQO}$$

or

$$\frac{r+\delta r}{r} = \frac{\sin(\pi-\alpha)}{\sin(\alpha-\delta\theta)} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)}$$

or

$$1 + \frac{\delta r}{r} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)}$$

or

$$\frac{\delta r}{r} = \frac{\sin\alpha}{\sin(\alpha-\delta\theta)} - 1$$

$$= \frac{\sin\alpha - \sin(\alpha-\delta\theta)}{\sin(\alpha-\delta\theta)}$$

$$= \frac{2\cos\left\{\frac{(2\alpha-\delta\theta)}{2}\right\}\sin\frac{\delta\theta}{2}}{\sin(\alpha-\delta\theta)}$$

or $$\frac{1}{r}\frac{\delta r}{\delta\theta}=\frac{\cos(\alpha-\delta\theta/2)}{\sin(\alpha-\delta\theta)}\cdot\frac{\sin(\delta\theta/2)}{\delta\theta/2}.$$

As $Q\to P$, then $\delta\theta\to 0, \delta r\to 0$ and $\alpha\to\phi$

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\cos\phi}{\sin\phi}\cdot 1,$$

$\therefore$ $$\tan\phi=r\frac{d\theta}{dr}.$$

Remark : For a point P of the curve $r=f(\theta)$, ϕ is defined to be the angle throu which the positive direction of the radius vector has to rotate to coincide with the directi of the tangent in which θ increases. The direction of the tangent for which θ increases taken as the positive direction of the tangent.

3.12 Angle of intersection of two polar curves

Let the two curve $r=f(\theta)$ and $r=f(\theta)$ intersect at P & let the value of ϕ at the poin for the two curves be ϕ_1 and ϕ_2 respectively. Then the angle of intersection of the two curv at P (*i.e.*, the angle between the tangents to the curves at P) is evidently equal to $|\phi_1-\phi_2$ $\phi_1\sim\phi_2$

If α is the acute angle of intersection of two curves at P, we have

$$\tan\alpha=|\tan(\phi_1-\phi_2|=\left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$$

$\therefore$ $$\alpha=\tan^{-1}\left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$$

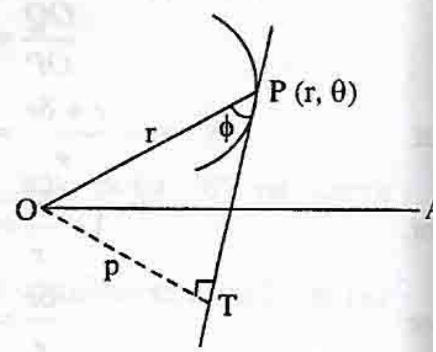
In particular, the two curves cut orthogonally if $\tan\phi_1\tan\phi_2=-1$ or $\cot\phi_1\cot\phi_2=-1$.

3.13 Length of The Perpendicular From Pole to the Tangent

Let $r=f(\theta)$ be a curve. If p be the length of the perpendicular OT from the pole O the tangent at any point P on the curve, then, from the figure, in ΔOPT, we have

$$\frac{p}{r}=\sin\phi$$

or $$p=r\sin\phi.$$



If we want the result in terms of r and θ, we have only to substitute for $\sin\phi$ from the equation $\tan\phi=r\frac{d\theta}{dr}$.

Thus, $$\frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$=\frac{1}{r^2}(\cot^2\phi+1)$$

$$=\frac{1}{r^2}\left\{\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2+1\right\}$$

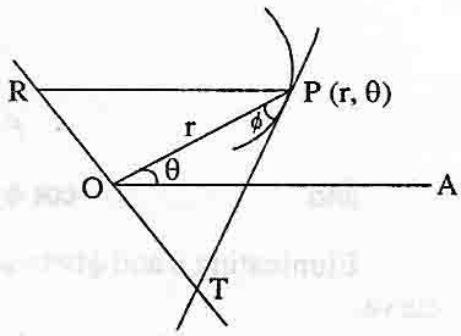
$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Sometimes u is used to denote $1/r$ and thus the above formula becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

3.14 Lengths of Polar Subtangent and Polar Subnormal

Let $P(r, \theta)$ be any point on a curve $r = f(\theta)$ and let the tangent and normal at P meet the straight line through the pole at right angles to the radius vector OP in T and R respectively.



Then OT and OR are respectively called the **polar subtangent and polar subnormal** at P.

In ΔOPT, we have

$$\frac{OT}{OP} = \tan\phi, \quad \text{or} \quad OT = OP\tan\phi = r^2\frac{d\theta}{dr}.$$

Hence

polar subtangent $= r^2\dfrac{d\theta}{dr} = -\dfrac{du}{d\theta},$

Since $\quad u = 1/r.$

In ΔOPR, we have

$$OR = OP\tan\angle OPN = r\cot\phi$$
$$= r\cdot\frac{1}{r}\frac{dr}{d\theta} = \frac{dr}{d\theta}.$$

Therefore

$$\textbf{Polar subnormal} = \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \qquad [\because \quad u = 1/r]$$

Note. Sometimes the lengths PT and PR are called the length of the polar tangent and polar normal.

So we have

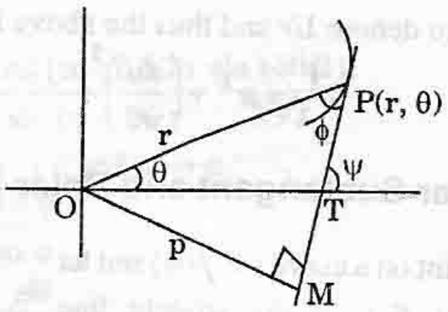
$$\textbf{Polar tangent} = PT = OP\sec\phi = r\sqrt{(1+\tan^2\phi)}$$
$$= r\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}.$$

$$\textbf{Polar normal} = PR = OP\operatorname{cosec}\phi = r\sqrt{(1+\cot^2\phi)}$$
$$= \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}$$

3.15 To form the pedal equation of a curve whose polar equation is given

Let $f(r, \theta) = 0$ $\qquad \ldots(1)$

be the given polar equation of curve we have



$$P = r\sin\phi \quad \text{...}$$

and $$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} \quad \text{...(}$$

Eliminating θ and ϕ between (i), (ii) and (iii) we get the required pedal equation of curve.

Note : Sometimes we do not get the value of ϕ from equation (iii) in a conveni form. Then instead of using the relation (ii) & (iii) we can use the single relation

$$\frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{...(}$$

Now eliminating θ between (i) & (iv) we get the required pedal equation.

ILLUSTRATIVE EXAMPLES

Example 14. Find the angle between the radius vector and the tangent at a point on the cardioid $r = a\,(1 - \cos\theta)$.

Solution. Here $\quad r = a\,(1-\cos\theta)$

$$\therefore \quad \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \tan\phi = r\frac{d\theta}{dr} = \frac{a\,(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{1-1+2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\theta/2$$

Hence $\quad \phi = \theta/2.$

Example 15. Find the angle at which the radius vector cuts the cur $\frac{l}{r} = 1 + e\cos\theta$.

Solution. The curve is $\frac{l}{r} = 1 + e\cos\theta$.

Differentiating logarithmically, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$$

or $$r\frac{d\theta}{dr} = \tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

$$\therefore \quad \phi = \tan^{-1}\frac{1+e\cos\theta}{e\sin\theta}.$$

Example 16. Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.

Solution. Here

$$r^n = a^n \cos n\theta \quad ...(1)$$

$$r^n = b^n \sin n\theta \quad ...(2)$$

Differentiating (1) logarithmically,

$$\frac{n}{r}\frac{dr}{d\theta} = -n \tan n\theta$$

or $$r\frac{d\theta}{dr} = \tan \phi_1 = -\cot n\theta. \quad ...(3)$$

Differentiating (2) logarithmically,

$$\frac{n}{r}\frac{dr}{d\theta} = n \cot n\theta$$

or $$r\frac{d\theta}{dr} = \tan \phi_2 = \tan n\theta. \quad ...(4)$$

From (3) and (4)

$$\tan \phi_1 \tan \phi_2 = -\cot n\theta \tan n\theta = -1.$$

Hence the curves cut orthogonally.

Example 17. Find the angle ϕ for the curve

$$a\theta = \sqrt{(r^2 - a^2)} - a \cos^{-1} (a/r).$$

Solution. Here

$$a\theta = \sqrt{(r^2 - a^2)} - a \cos^{-1} (a/r). \quad ...(1)$$

Differentiating (1) with respect to r, we get

$$a\frac{d\theta}{dr} = \frac{1}{2\sqrt{(r^2 - a^2)}} \cdot 2r + \frac{a}{\sqrt{[1-(a/r)^2]}} \cdot \left(-\frac{a}{r^2}\right).$$

or $$a\frac{d\theta}{dr} = \frac{r}{\sqrt{(r^2 - a^2)}} - \frac{a^2}{r\sqrt{(r^2 - a^2)}}$$

$$= \frac{r^2 - a^2}{r\sqrt{(r^2 - a^2)}}$$

$\therefore$ $$r\frac{d\theta}{dr} = \frac{1}{a}\sqrt{(r^2 - a^2)}$$

or $$\tan \phi = \frac{\sqrt{(r^2 - a^2)}}{a}$$

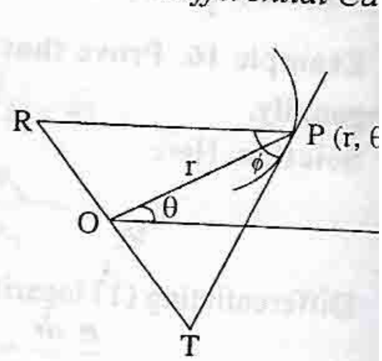
$\therefore$ $$\cos \phi = \frac{a}{r}$$

or $$\phi = \cos^{-1} (a/r).$$

Example 18. If ϕ be the angle between the tangent to a curve and the radius vector from the origin of co-ordinates to the point of contact, prove that

$$\tan \phi = \left\{ x\frac{dy}{dx} - y \right\} \bigg/ \left\{ x + y\frac{dy}{dx} \right\}.$$

Solution. From the figure



$$\psi = \theta + \phi,$$

$$\frac{y}{x} = \tan\theta$$

and $$\frac{dy}{dx} = \tan\psi$$

$\therefore$ $$\phi = \psi - \theta$$

and $$\tan\phi = \tan(\psi - \theta)$$

$$= \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$$

$$= \frac{(dy/dx) - (y/x)}{1 + \left(\dfrac{dy}{dx}\right)\cdot(y/x)}$$

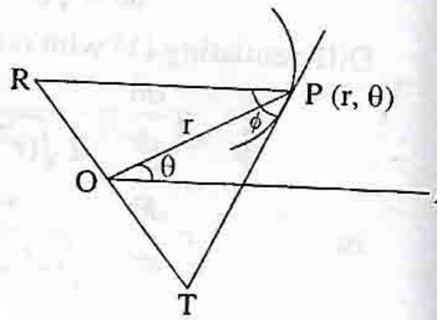
$$\tan\phi = \frac{x\,(dy/dx) - y}{x + y\,(dy/dx)}$$

Hence the result is proved.

Example 19. Show that the locus of extremity of polar subnormal of the cu $r = f(\theta)$ is $r = f'(\theta - \pi/2)$.

Solution. The equation of the curve is $r = f(\theta)$. If polar coordinate of the extremit of polar sub-normal be (r_1, θ_1), then $OR = r_1$, $\angle ROA = \theta_1$.

Also $\angle ROA = \angle ROP + \angle POA$.



$\therefore$ $$\theta_1 = \frac{\pi}{2} + \theta$$

or $$\theta = \theta_1 - \pi/2.$$

In $\Delta ROP, OR = r_1 = OP\cot\phi$

$$= r\cdot\frac{dr}{r d\theta} = f'(\theta).$$

from given equation we get

$$\frac{dr}{d\theta} = f'(\theta).$$

Therefore $\quad r_1 = f'(\theta) = f'(\theta_1 - \pi/2).$

$\therefore$ Locus of R is $r = f'(\theta - \pi/2)$.

Example 20. Find the angle of intersection between the pair of curves $r = 6\cos$ and $r = 2(1 + \cos\theta)$.

Solution : The given curves are

$$r = 6\cos\theta \qquad \text{...(1)}$$

and $$r = 2(1 + \cos\theta) \qquad \text{...(2)}$$

From (1) on taking logarithm, we get

$$\log r = \log 6 + \log\cos\theta$$

$\therefore$ $$\cot\phi_1 = \frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$=\cot\left(\frac{1}{2}\pi+\phi\right)$$

$\Rightarrow$ $$\phi_1=\frac{1}{2}\pi+\theta$$

Again from (2) on taking logarithm, we get

$$\log r=\log 2+\log(1+\cos\theta)$$

$\therefore$ $$\cot\phi_2=\frac{1}{r}\frac{dr}{d\theta}=\frac{-\sin\theta}{1+\cos\theta}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$=-\tan\frac{\theta}{2}=\cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$\Rightarrow$ $$\phi_2=\frac{1}{2}\pi+\frac{1}{2}\theta$$

Now the angle of intersection of (1) & (2)

$$\phi_1\sim\phi_2=\left(\frac{1}{2}\pi+\theta\right)-\left(\frac{1}{2}\pi+\frac{\theta}{2}\right)$$

$$=\frac{\theta}{2}$$

where θ is the vectorial angle of the point of intersection of (1) & (2). Now to get θ for the point of intersection of (1) & (2), we have on eliminating r between (1) & (2)

$$6\cos\theta=2(1+\cos\theta)$$

$\Rightarrow$ $$2\cos\theta=1$$

$\Rightarrow$ $$\cos\theta=1/2$$

$\Rightarrow$ $$\theta=\pi/3$$

$\therefore$ The required angle of intersection $\frac{1}{2}\left(\frac{\pi}{3}\right)=\frac{\pi}{6}$

Example 21. Find the pedal equation of the curve

$$r^m\cos m\theta=a^m$$

Solution : The given curve is $r^m\cos m\theta=a^m$

Taking log, we get

$$m\log r+\log\cos m\theta=m\log a$$

Differentiating with respect to θ, we have

$$\frac{m}{r}\frac{dr}{d\theta}=\frac{m\sin m\theta}{\cos m\theta}=0$$

$\therefore$ $$\cot\phi=\frac{1}{r}\left(\frac{dr}{d\theta}\right)=\tan m\theta=\cot\left(\frac{1}{2}\pi-m\theta\right)$$

so that $$\theta=\frac{1}{2}\pi-m\theta$$

Now $$p=r\sin\theta$$

$$= r\sin\left(\frac{1}{2}\pi - m\theta\right)$$

$$= r\cos m\theta$$

$$= r\,(a^m / r^m) \;\Rightarrow\; pr^{m-1} = a^m$$

Example 22. Find the polar subtangent of $r = a\,(1 + \cos\theta)$.

Solution : Given equation of curve $r = a\,(1 + \cos\theta)$

Differenting with respect to θ,

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\text{Polar subtangent} = r^2\frac{d\theta}{dr}$$

$$= a^2\,(1+\cos\theta)^2\cdot\left(-\frac{1}{a\sin\theta}\right)$$

$$= \frac{-a\left(2\cos^2\frac{\theta}{2}\right)^2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= -2a\cos^2\frac{\theta}{2}\cot\frac{\theta}{2}$$

$$= 2a\cos^2\frac{\theta}{2}\cot\frac{1}{2}\theta$$

EXERCISE 3.3

1. Find the angle of intersection of the following curves :
   (i) $r\,(1+\cos\theta) = a,\ r\,(1-\cos\theta) = b$
   (ii) $r = a\,(1+\cos\theta),\ r = b\,(1-\cos\theta)$
   (iii) $r^2 = 16\sin 2\theta,\ r^2\sin 2\theta = 4$
   (iv) $r = a\cos\theta,\ 2r = a$
2. Find the polar subtangent of the following curves :
   (i) $r = a(1-\cos\theta)$ (ii) $l/r = 1 + e\cos\theta$
3. Find the pedal equation of the following curves
   (i) $r^n = a^n\sin n\theta$ (ii) $r^2 = a^2\cos 2\theta$
   (iii) $r = a\,\text{cosec}\, n\theta$ (iv) $r = a(1+\cos\theta)$
   (v) $r = ae^{\theta\cot\alpha}$
5. For the curve $r^n = a^n\cos n\theta$, prove that
   (i) $\dfrac{ds}{d\theta} = a\,(\sec n\theta)^{(n-1)/n}$ (ii) $\phi = \dfrac{\pi}{2} + n\theta$
   (iii) $a^{2n}\left(\dfrac{d^2r}{ds^2}\right) + nr^{2n-1} = 0$ (iv) $r^{n+1} = pa^n$
6. For any curve, prove that :

(i) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$

(ii) $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2 - p^2}}$

7. If $\dfrac{2a}{r} = 1 + \cos\theta$, then with usual notation, show that

$$\frac{ds}{d\phi} = \frac{2a}{\sin^3 \psi}$$

8. Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = a^n \sin n\theta$ intersect orthogonally.

ANSWERS

1. (i) $\pi/2$ (ii) $\frac{1}{2}\pi$ (iii) $2\pi/3$ (iv) $\pi/3$

2. (i) $2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}$ (ii) $\frac{l}{e}\sin\theta$

3. (i) $r^{n+1} = pa^n$ (iii) $pa^2 = r^3$ (iv) $\dfrac{r^2}{p^2} = 1 + n^2(r^2 - a^2)$

(v) $r^3 = 2ap^2$ (vi) $p = r\sin\alpha$

1.16 Equation of Tangent in Polar Form

Here, we shall find the equation of tangent to the curve, whose polar equation is $r = f(\theta)$ r being the radius vector. Sometimes u is used to denote $\frac{1}{r}$, therefore, $u = f(\theta)$, where $u = \frac{1}{r}$, is also used as the polar equation in place of $r = f(\theta)$.

Let (r_1, θ_1) *i.e.*, $\left(\dfrac{1}{u_1}, \theta\right)_1$ be the coordinates of the point of contact. Also let us take the polar equation of a straight line as :

$$x = A\cos(\theta - \theta_1) + B\sin(\theta - \theta_1) \qquad \ldots\text{(i)}$$

where A and B are arbitrary constants.

If (i) represents the tangents to the curve $u = f(0)$, then by differentiating (i) w.r.t. θ, we obtain

$$\frac{du}{d\theta} = -A\sin(\theta - \theta_1) + B\cos(\theta - \theta_1) \qquad \ldots\text{(ii)}$$

Since the tangent touches the curves of therefore, the value of $\dfrac{du}{d\theta}$ at the point of contact $\left(\dfrac{1}{u_1}, \theta_1\right)$ for the curve and for the tangent will be the same. Hence Putting $\theta = \theta_1$ in (i) and (ii) we obtain

$$[x]_\theta = \theta_1 = A \quad \text{and} \quad \left[\frac{du}{d\theta}\right]_{\theta = \theta_1} = B = 3$$

Substituting these values of A and B in (i) the equation of tangent to the curve $x = f$ at the point $\left(\frac{1}{u_1}, \theta_1\right)$ is given by $x = u\cos(\theta - \theta_1) + \left[\frac{du}{d\theta}\right]_{\theta=\theta_1} \sin(\theta - \theta_1)$ ...(

3.17 Equation of Normal in Polar form

Since the normal is perpendicular to the tangent at the point of contract, therefore shall find the equation of the normal with the help of the equation of the tangent obtain above.

The equation of a straight line perpendicular to the tangent to the curve $x = f(0)$ at the point $\left(\frac{1}{u_1}, \theta_1\right)$ is given by,

$$C_u = u_1 \cos\left[\left(\theta + \frac{\pi}{2}\right) - \theta_1\right] + \left[\frac{dy}{d\theta}\right]_{\theta=\theta_1} \sin\left[\left(\theta + \frac{\pi}{2}\right) - \theta_1\right]$$

or $$C_u = u_1 \sin(\theta - \theta_1) + \left[\frac{du}{d\theta}\right]_{\theta=\theta_1} \cos(\theta - \theta_1) \quad ...($$

where C is an arbitrary constant

Now using $x = u_1$ when $\theta = \theta_1$ in (1) we get

$$C_{u_1} = \left[\frac{du}{d\theta}\right]_{\theta=\theta_3} \quad \text{or} \quad C = \frac{1}{u_1}\left[\frac{du}{d\theta}\right]_{\theta=\theta_1}$$

Therefore, the equation of the normal to the curve $u = f(\theta)$ at the point of contra $\left(\frac{1}{u_1}, \theta_1\right)$ is given by

$$\frac{u}{u_1}\left[\frac{du}{d\theta}\right]_{\theta=\theta_1} = -u\sin(\theta - \theta_1) + \left[\frac{du}{d\theta}\right]_{\theta=\theta_1} \cos(\theta - \theta_1) \quad ...($$

which can also be put in following form

$$x_1 \sin(\theta - \theta_1) + \left\{\frac{u}{u_1} - \cos(\theta - \theta_1)\right\}\left[\frac{du}{d\theta}\right]_{\theta=\theta_1} = 0 \quad ...(4$$

EXERCISE 3.4

1. Find the equation of tangent of normal to the cure $r = a\theta$ at $\theta = \alpha$.
2. Find the equation & tangent to the cure $r\theta = a$ at $\theta = \alpha$.

ANSWERS

1. $r^{-1} = r_1^{-1}\cos(\theta - \alpha) - (a\alpha^2)^{-1}\sin(\theta - \alpha)$

   $(r_1/r)(a\alpha^2)^{-1} = r^{-1}\sin(\theta - \alpha) + (a\alpha^2)^{-1}\cos(\theta - \alpha)$

2. $r^{-1} = r_1^{-1}\cos(\theta - \alpha) - a^{-1}\sin(\theta - \alpha)$

   $r_1(ar)^{-1} = -r_1^{-1}\sin(\theta - \alpha) + a^{-1}\cos(\theta - \alpha)$

□