Chapter

BETA AND GAMMA FUNCTIO

§ 4·1. Gamma Functions

We define Gamma functions $\Gamma(n)$ (read as Gamma n) by the integral

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\, dt, \qquad n > 0$$

These are also known as Eulerian Integrals of second kind after the Mathematician Euler who introduced them in 1729.

§ 4·2. An Important Property.

To prove that $\Gamma(n+1) = n\,\Gamma n$ *for all values of* n.

(Gorakhpur 83, 2006,

We have $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\, dt$

Integrating by parts taking t^{n-1} as first function, we have

$$\Gamma(n) = {}_0^\infty\left\{-e^{-t} t^{n-1}\right\} - \int_0^\infty (n-1)\, t^{n-2}\,(-e^{-t})\, dt$$

$$= 0 + (n-1)\int_0^\infty .e^{-t} t^{n-2}\, dt$$

$$= (n-1)\int_0^\infty e^{-t} t^{n-2}\, dt$$

$$= (n-1)\,\Gamma(n-1)$$

replacing n by $(n+1)$, we get

$\Gamma(n+1) = n\,\Gamma(n)$ as required.

Cor I. $\Gamma(1) = 1$

by definition $\Gamma(1) = \int_0^\infty e^{-t} t^{t-1}\, dt$

$$= \int_0^\infty e^{-t}\, dt = {}_0^\infty\left[-e^{-t}\right]$$

$$\Gamma(1) = 1$$

Cor II. If n is a positive integral $\Gamma(n+1) = n\,!$.

We know that $\Gamma(n+1) = n\,\Gamma(n)$ for all $n > 0$, applying this result successively positive integer n, we have

$$\Gamma(n+1) = n(n-1)(n-2) \ldots\ldots\ldots 3.2.1\,\Gamma(1)$$
$$= n\,! \text{ as } \Gamma(1) = 1.$$

Cor III. $\overline{|0} = \infty$

Proof. : By the recurrence formula for Gamma function, we have

$$\overline{|(n)} = \frac{\overline{|(n+1)}}{n}$$

Putting $n = 0$ in this, we obtain $\overline{|(0)} = \dfrac{\overline{|(1)}}{0}$

$$= \frac{1}{0} = \infty$$

Cor IV. $\overline{|(n)} = \infty$, where n is a negative integer.

Prof. Let $n = -p$, where p is a +ve integer then by $\overline{|(n)} = \dfrac{\overline{|(n+1)}}{n}$, we obtain

$$\overline{|(-p)} = \frac{\overline{|(-p+1)}}{-P} = \frac{\underline{|(-p)}}{-p} = \infty$$

(by using $(-P)\,! = \infty$)

We know that $^nP_r = \dfrac{n\,!}{n-r\,!}$ ...(1)

Putting $n = 0$ in (1) we obtain $^0P_r = \dfrac{0\,!}{\underline{|(r)}\,!}$

But $^nP_r = n(n-1) \ldots\ldots (n-r+1)$

So that $^0P_r = 0$

Hence we have $0 = \dfrac{0\,!}{(-r)\,!}$ ...(2)

Now putting $r = n$ in (1), we obtain

$$^nP_n = \frac{n\,!}{(n-n)\,!} = \frac{n\,!}{0\,!}$$

or $\quad n\,! = \dfrac{n\,!}{0\,!}$ (by using $^nP_n = n(n-1)(n-2 \ldots\ldots 3 \,.\, 2 \,.\, 1 = n\,!)$

or $\quad n\,! = \dfrac{n\,!}{1} = n\,!$ *i.e.* $0\,! = 1$ ...(3)

Putting $0\,! = 1$ in (2), we obtain

$0 = \dfrac{1}{(-r)\,!}$ which gives

$(-r)\,! = \dfrac{1}{0} = \infty$ ...(4)

§ 4·3. Transformation of Gamma Functions.

We have $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\, dt$ ...(1

I. Put $t = \lambda x$; so that $dt = \lambda\, dx$

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-\lambda x} (\lambda x)^{n-1} \lambda\, dx$$

$$= \lambda^n \int_0^\infty e^{-\lambda x} x^{n-1}\, dx$$

$$\therefore \quad \int_0^\infty e^{-\lambda x} x^{n-1}\, dx = \lambda^{-n}\, \Gamma(n).$$

(Gorakhpur 82

II. Put $t^n = x$ in (1) so that $nt^{n-1}\, dt = dx$

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-x^{1/n}} \left(\frac{1}{n}\, dx\right)$$

$$= \frac{1}{n} \int_0^\infty e^{-x^{1/n}}\, dx$$

or $$\int_0^\infty e^{-x^{1/n}}\, dx = n\, \Gamma\, n$$

III. In (1) put $e^{-t} = x$; so that $-e^{-t}\, dt = dx$

$$\therefore \quad \Gamma(n) = \int_1^0 \{-\log x\}^{n-1} (-dx)$$

$$= (-1)^{n-1} \int_0^1 (\log x)^{n-1}\, dx$$

or $$\Gamma(n) = (-1)^{n-1} \int_0^1 (\log x)^{n-1}\, dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

§ 4·4. Product of two Integrals.

We shall prove that,

$$\int_a^b f(x)\, dx \times \int_a^b g(y)\, dy = \int_a^b \int_c^d f(x)\, g(y)\, dx\, dy.$$

Proof. Let us suppose that $\int g(y)\, dy = G(y)$

then $$\int_a^b \int_c^d f(x)\, g(y)\, dx\, dy = \int_a^b f(x) \left[G(y)\right]_c^d dx$$

$$= \int_a^b f(x)\, [G(d) - G(c)]\, dx$$

$$= \int_a^b f(x)\, dx \times \{G(d) - G(c)\}$$

$$= \int_a^b f(x)\, dx \times \int_c^d g(y)\, dy$$ **Hence proved.**

§ 4·5. Value of Γ (1/2). *(Purvanchal 90, 97)*

We know that $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\, dt$

If we take $n = 1/2$. we get

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2}\, dt$$

putting $t = x^2$ or $t^{1/2} = x$
so that $t^{-1/2}\, dt = 2\, dx$

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\, dx \quad \text{...(1)}$$

$$= 2\int_0^\infty e^{-y^2}\, dy \quad \text{...(2)}$$

Property 1 of definite integrals

Multiplying (1) and (2) we get

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \quad \text{from article 4·4.}$$

On transposing to polars, we get

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, d\theta\, dr$$

$$= -2\int_0^{\pi/2} \left[e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta$$

$$= 2\left(\theta\right)_0^{\pi/2} = \pi \qquad \therefore\ \Gamma(1/2) = \sqrt{\pi}.$$

Remark. If n is positive integer than the formula $\Gamma(m+1) = m\Gamma(m)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ gives the formula

$$\Gamma\left(\frac{n}{2}\right) = \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \ldots \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$$

In particular $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$

§ 4·6. Beta functions.

Beta functions are denoted by $B(m, n)$, read as Beta m, n and is defined by the integral

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad m, n > 0$$

These are also known as the Eulerian integral of first kind.

§ 4·7. Properties of Beta Functions

(a) Symmetry property of Beta functions.

Here we shall prove the $B(m,n) = B(n,m)$. *(Gorakhpur 80, 2011)*

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ from definition.

Putting $x = 1 - y$, we get

$$B(m,n) = -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = B(n,m).$$ **Proved.**

(b) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$ *(GKP 2017)*

Proof. Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]\,dx$$

$$= \int_0^1 [x^m(1-x)^{n-1} + x^{m-1}(1-x)^n]\,dx$$

$$= \int_0^1 x^m(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^n\,dx$$

$$= \beta(m+1,n) + \beta(m,n+1)$$

(c) $\dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n)}{m+n}$

Proof. We have

$$\frac{\beta(m,n+1)}{n} = \frac{\Gamma m\,\Gamma(n+1)}{n\,\Gamma(m+n+1)} = \frac{\Gamma m\cdot n\,\Gamma n}{n(m+n)\,\Gamma(m+n)}$$

$$= \frac{\Gamma m\, n\,\Gamma n}{(m+n)\,\Gamma(m+n)} = \frac{\beta(m,n)}{m+n}$$

Hence, $\dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m,n)}{m+n}$

Similarly, $\dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n)}{m+n}$

§ 4·8. An useful transformation.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad ...(1)$$

Putting $x = \dfrac{1}{1+y}$; so that $dx = -\dfrac{1}{(1+y)^2}$

and $1 - x = \dfrac{y}{1+y}$, we get

$$B(m,n) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{n-1}\,dy}{(1+y)^{m+n}} \qquad ...(2)$$

(Gorakhpur 81)

Remark 1. Since $B(m,n) = B(n,m)$ hence by interchanging m and n in (2), e have

$$B(m,n) = \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}} \qquad ....(3)$$

Remark 2. Relation (2) or (3) provide alternative definitions of Beta functions.

4·9. Relation between Beta and Gamma functions.

To prove that $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(Gorakhpur 87, 98, 2003, 04, 10; Purv. 96 98)

We know that $\dfrac{\Gamma(n)}{z^n} = \displaystyle\int_0^\infty e^{-zt}\,t^{n-1}\,dt$ ...(1)

where z is a constant

$$\therefore \qquad \Gamma(n) = \int_0^\infty z^n\,e^{-zt}\,t^{n-1}\,dt$$

Multiplying both sides by $e^{-z}\,z^{m-1}$, we get

$$\Gamma(n)\,e^{-z}\,z^{m-1} = \int_0^\infty z^{(n+m-1)}\,e^{-z(t+1)}\,t^{n-1}\,dt$$

Now integrating both sides with respect to z within the limits 0 and ∞ and sing the definition

$\displaystyle\int_0^\infty e^{-z}\,z^{m-1}\,dz = \Gamma(m)$, we have

$$\Gamma(n)\,\Gamma(m) = \int_0^\infty \left[\int_0^\infty e^{-z(t+1)}\,z^{(m+n-1)}\,dz\right] t^{n-1}\,dt \qquad ...(2)$$

But $\displaystyle\int_0^\infty e^{-z(t+1)}\,z^{m+n-1}\,dz = \frac{\Gamma(m+n)}{(1+t)^{m+n}}$ (See § 4·3)

Therefore from (2), we get

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)\,t^{n-1}\,dt}{(1+t)^{m+n}}$$

$$= \Gamma(m+n)\int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \Gamma(m+n)\,B(m,n)$$

or

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

§ 4·10. Reflection Formula for Gamma Function

This is also known as Complement formula given by

$\lceil\overline{n}\,\lceil\overline{(1-n)} = \dfrac{\pi}{\sin n\pi}$ where $0 < n < 1$.

Proof. : Since $\beta\,(m, n) = \dfrac{\lceil\overline{(m)}\,\lceil\overline{(n)}}{\lceil\overline{(m+n)}}$

and where $\displaystyle\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} = \frac{\lceil\overline{(m)}\,\lceil\overline{(n)}}{\lceil\overline{(m+n)}}$

Putting $m + n = 1$ so that $m = 1 - n$ and using the known result

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \frac{\pi}{\sin n\,\pi}, \quad 0 < n < 1$$

we get $\dfrac{\pi}{\sin n\,\pi} = \dfrac{\lceil\overline{(1-n)}\,\lceil\overline{n}}{\lceil\overline{1}}$ which gives $\lceil\overline{n}\,\lceil\overline{(1-n)} = \dfrac{\pi}{\sin n\,\pi}$

§ 4·11. Value of the integral $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$, for positive values of m and n.

(Gorakhpur 84, 2016; Purvanchal 90

From definition

$$B\,(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m > 0 \text{ and } n > 0$$

Put $x = \sin^2\theta$; so that $dx = 2\sin\theta\cos\theta\,d\theta$.

$$\therefore \quad B\,(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta . 2\sin\theta\cos\theta\,d\theta$$

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

Therefore $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}.$

Remark. Replacing $2m - 1$ by m and $2n - 1$ by n we get the alternative form of above result as

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma\left(\dfrac{m+1}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right)}{2\Gamma\left(\dfrac{m+n+2}{2}\right)}$$

§ 4·12. Walli's formula.

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \dfrac{(n-1)\,(n-3)\,......\,4\cdot 2}{n\,(n-2)\,......\,3\cdot 1}, & \text{if } n \text{ is odd} \\[2ex] \dfrac{(n-1)\,(n-3)\,......\,3\cdot 1}{n\,(n-2)\,......\,4\cdot 2}\,\dfrac{\pi}{2}, & \text{if } n \text{ is even} \end{cases}$$

Proof. If n is odd, we put $n = 2m + 1$, where $m = 0, 1, 2, 3, ...$.

Then $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^{2m+1} x \cos^0 x\, dx$

$$= \frac{\Gamma(m+1)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2m+3}{2}\right)} \qquad \text{(by art 4·10)}$$

$$= \frac{m(m-1)(m-2)\,......\,3 \cdot 2 \cdot 1\sqrt{\pi}}{2 \cdot \Gamma\left(m+\frac{3}{2}\right)}$$

$$= \frac{m(m-1)(m-2)\,......\,3 \cdot 2 \cdot 1\sqrt{\pi}}{2 \cdot \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\,......\,\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}$$

$$= \frac{2m(2m-2)(2m-4)\,......\,6 \cdot 4 \cdot 2}{(2m+1)(2m-1)(2m-3)\,......\,5 \cdot 3 \cdot 1}$$

$$= \frac{(n-1)(n-3)(n-5)\,......\,6 \cdot 4 \cdot 2}{n(n-2)(n-3)\,......\,5 \cdot 3 \cdot 1}$$

If n is even, we put $n = 2m$ where m is positive integer. Then

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^{2m} x \cos^0 x\, dx = \frac{\Gamma\left(\frac{2m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2m+2}{2}\right)}$$

$$= \frac{\Gamma\left(m+\frac{1}{2}\right) \cdot \sqrt{\pi}}{2\Gamma(m+1)}$$

$$= \frac{\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\,......\,\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \cdot \sqrt{\pi}}{2 \cdot m \cdot (m-1)(m-2)\,......\,3 \cdot 2 \cdot 1}$$

$$= \frac{(2m-1)(2m-3)\,......\,3 \cdot 1 \cdot \pi}{2 \cdot 2m(2m-2)(2m-4)\,......\,6 \cdot 4 \cdot 2}$$

$$= \frac{(n-1)(n-3)\,......\,3 \cdot 1}{n(n-2)(n-4) \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

Example 1. *Evaluate* $\int_0^a x^2(a^2 - x^2)^{3/2}\, dx.$

Sol. Put $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$

$$\therefore \quad \int_0^a x^2(a^2-x^2)^{3/2}\, dx = \int_0^{\pi/2} a^2\sin^2\theta\, a^3\cos^3\theta\, a\cos\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$$

$$= \frac{a^6 \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= \frac{a^6 \frac{1}{2}\sqrt{\pi} \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2\cdot 3\cdot 2\cdot 1} = \frac{\pi a^6}{32}.$$

Example 2. *Show that* $\int_0^{2a} x^3 (2ax - x^2)^{1/2} \, dx = \frac{7}{8}\pi a^5.$

Sol. Let $I = \int_0^{2a} x^3 (2ax - x^2)^{1/2} \, dx$

$$= \int_0^{2a} x^{7/2} (2a - x)^{1/2} \, dx$$

Put $x = 2a \sin^2\theta$, so that $dx = 4a \sin\theta \cos\theta \, d\theta$

$$\therefore \; I = \int_0^{\pi/2} (2a \sin^2\theta)^{7/2} (2a - 2a \sin^2\theta)^{1/2} \, 4a \sin\theta \cos\theta \, d\theta$$

$$= 64a^5 \int_0^{\pi/2} \sin^8\theta \cos^2\theta \, d\theta$$

$$= 64a^5 \frac{\Gamma\left(\frac{9}{2}\right) \Gamma\left(\frac{3}{2}\right)}{2\Gamma(6)}$$

$$= 64a^5 \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{2 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{7\pi a^5}{8}$$

Example 3. *Evaluate* $\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx$

Sol. Let $I = \int_0^{\pi/4} \sin^4 x \cos^2 x \, dx$

$$= \int_0^{\pi/4} \sin^2 x \cos^2 x \sin^2 x \, dx$$

$$= \int_0^{\pi/4} \frac{\sin^2 2x}{4} \cdot \frac{1}{2} (2 \sin^2 x) \, dx$$

$$= \frac{1}{8} \int_0^{\pi/4} \sin^2 2x \, (1 - \cos 2x) \, dx$$

Put $2x = t$, so that $2dx = dt$. Also when $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}$, $t = \frac{\pi}{2}$

$$\therefore \qquad I = \frac{1}{8}\int_0^{\pi/2} \sin^2 t\,(1 - \cos t)\,\frac{1}{2}\,dt$$

$$= \frac{1}{16}\left[\int_0^{\pi/2} \sin^2 t\,dt - \int_0^{\pi/2} \sin^2 t\cos t\,dt\right]$$

$$= \frac{1}{16}\left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} - \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(1)}{2\Gamma\left(\frac{5}{2}\right)}\right]$$

$$= \frac{1}{16}\left[\frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2\,.\,1} - \frac{\frac{1}{2}\,.\,\sqrt{\pi}\,.\,1}{2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\sqrt{\pi}}\right]$$

$$= \frac{1}{16}\left[\frac{\pi}{4} - \frac{1}{3}\right]$$

Example 4. *If m, n are positive integers, then prove that*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \frac{1\,.\,2\,.\,3\,\ldots\,(m-1)}{n\,(n+1)\,\ldots\,(n+m-1)}$$

Sol. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B\,(m, n) = B\,(n, m)$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)\,!}$$

$$= \frac{(m-1)\,!\,(n-1)\,!}{(n+m-1)\,(n+m-2)\,\ldots\,n\,(n-1)\,(n-2)\,\ldots\,2\,.\,1}$$

$$= \frac{(m-1)\,!\,(n-1)\,!}{(n+m-1)\,(n+m-2)\,\ldots\,(n+1)\,n\,(n-1)\,!}$$

$$= \frac{(m-1)\,(m-2)\,\ldots\,3\,.\,2\,.\,1}{(n+m-1)\,(n+m-2)\,\ldots\,(n+1)\,n}$$

Example 5. *Obtain the value of the integral*

$$\int_0^{\pi} x\sin^6 x\cos^4 x\,dx$$

Sol. Let $I = \int_0^{\pi} x\sin^6 x\cos^4 x\,dx$ ...(i)

$$= \int_0^{\pi} (\pi - x) \sin^6 (\pi - x) \cos^4 (\pi - x)\, dx$$

(by property 4 of definite integral)

or $$I = \int_0^{\pi} (\pi - x) \sin^6 x \cos^4 x\, dx \quad \text{...(ii)}$$

Adding (i) and (ii) we get

$$2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x\, dx$$

$$= 2\pi \int_0^{\pi/2} \sin^6 x \cos^4 x\, dx \text{ (by property 6 of definite integral)}$$

$$= 2\pi \frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{5}{2}\right)}{2\Gamma(6)}$$

$$= \frac{2\pi \frac{5}{2} . \frac{3}{2} . \frac{1}{2} \sqrt{\pi} . \frac{3}{2} . \frac{1}{2} . \sqrt{\pi}}{2.5.4.3.2.1}$$

$$= \frac{3\pi^2}{256}$$

$$\therefore \quad I = \frac{3\pi^2}{512}$$

Example 6. *Show that* $\int_0^{\infty} x^6 e^{-2x} dx = \frac{45}{8}$

Solution. Putting $2x = t$, we get

$$\int_0^{\infty} x^6 e^{-2x} dx = \int_0^{\infty} \left(\frac{t}{2}\right)^6 e^{-t} . \frac{dt}{2}$$

$$= \frac{1}{2^7} \int_0^{\infty} e^{-t} t^6\, dt$$

$$= \frac{1}{2^7} \int_0^{\infty} e^{-t} t^{7-1}\, dt = \frac{\Gamma(7)}{2^7}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^7} = \frac{45}{8}.$$

Example 7. *Show that* $\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}$.

(Gorakhpur 85, 96, 2002, 09, Sidh. 2017)

Solution. Putting $x^2 = y$ or $x = y^{1/2}$

so that $dx = \frac{1}{2} y^{-1/2} dy$.

We get $\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y} \frac{1}{2} y^{-1/2} dy$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{\frac{1}{2}-1}\, dy = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$ **Proved.**

Example 8. *Show that* $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

(Purvanchal 89; Gorakhpur 95, 2005, 12)

Solution. Let $I = \int_0^\infty \frac{x^c}{c^x}\,dx$ Put $c^x = e^t$ *i.e.* $x \log c = t$

$$\therefore \quad x = \frac{t}{\log c}\,;\, dx = \frac{dt}{\log c}$$

$$\therefore \quad I = \int_0^\infty \left(\frac{t}{\log c}\right)^c \frac{1}{e^t}\frac{dt}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\, t^c\, dt$$

$$= \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\, t^{c+1-1}\, dt$$

$$= \frac{\Gamma(c+1)}{(\log c)^{n+1}}$$

Example 9. *Prove that* $\int_0^{\pi/4} (\cos 2\theta)^{3/2}\cos\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}$

Solution. $\int_0^{\pi/4} (\cos 2\theta)^{3/2}\cos\theta\, d\theta = \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2}\cos\theta\, d\theta$

(putting $\sqrt{2}\sin\theta = \sin\phi$ so that $\cos\theta\, d\theta = \frac{1}{\sqrt{2}}\cos\phi\, d\phi$)

$$= \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2}\frac{1}{\sqrt{2}}\cos\phi\, d\phi$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^4\phi\, d\phi$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sin^0\phi\cos^4\phi\, d\phi$$

$$= \frac{1}{\sqrt{2}}\frac{\Gamma(1/2)\,\Gamma(5/2)}{\Gamma(3)}$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{\pi}\cdot(3/2)\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 2\cdot 1} = \frac{3\pi}{16\sqrt{2}}$$

Example 10. *Evaluate* $\int_0^1 \sqrt{(1-x^4)}\,dx$

Solution. $\int_0^1 \sqrt{(1-x^4)}\,dx = \int_0^1 (1-x^4)^{1/2}\,dx$

putting $x^4 = t$ or $x = t^{1/4}$ so that $dx = \frac{1}{4} t^{-3/4}\, dt$

$$= \int_0^1 (1-t)^{1/2} . \frac{1}{4} t^{-3/4}\, dt$$

$$= \frac{1}{4}\int_0^1 t^{-3/4} (1-t)^{1/2}\, dt$$

$$= \frac{1}{4}\int_0^1 t^{1/4-1} (1-t)^{3/2-1}\, dt$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\Gamma(1/4)\,\Gamma(3/2)}{\Gamma\left(\frac{1}{4}+\frac{3}{2}\right)}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{4}\left\{\frac{\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{2}\sqrt{\pi}}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)}\right\} = \frac{1}{6}\left[\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right]$$

Example 11. *Show that* $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} \times \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \pi.$

Solution. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \int_0^{\pi/2} (\sin\theta)^{-1/2}\cos^0\theta\, d\theta$

$$= \left(\frac{\Gamma(1/4)\,\Gamma(1/2)}{2\Gamma\left(3/4\right)}\right)$$

and $\displaystyle\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2}\cos^0\theta\, d\theta$

$$= \frac{\Gamma(3/4)\,\Gamma(1/2)}{2\Gamma\left(5/4\right)}$$

Multiplying (i) and (ii), we get

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} \times \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \frac{\Gamma(1/4)\,\Gamma(1/2)}{2\Gamma\left(3/4\right)} \times \frac{\Gamma(3/4)\,\Gamma(1/2)}{2\Gamma\left(5/4\right)}$$

$$= \frac{\Gamma(1/4)\sqrt{\pi}\sqrt{\pi}}{4\Gamma\left(5/4\right)}$$

$$= \frac{\pi\,\Gamma(1/4)}{4\times\frac{1}{4}\Gamma\left(1/4\right)} = \pi.$$

Prov

Example 12. *Prove that* $\displaystyle\int_0^1 \frac{x^2\, dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}.$

Solution. Let $\displaystyle I_1 = \int_0^1 \frac{x^2\, dx}{\sqrt{(1-x^4)}}$

Put $x^2 = \sin\theta$ $\therefore$ $2x\,dx = \cos\theta\,d\theta$

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{(\sin\theta)}\cdot\cos\theta\,d\theta}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{1/2}\,d\theta$$

$$\therefore \quad I_1 = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)}$$

$$\therefore \quad I_1 = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \qquad \text{...(i)}$$

$$\because \left(\Gamma\left(\frac{5}{4}\right) = \left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)\right)$$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{(1+x^4)}}$ Put $x^2 = \tan\phi$

so that $2x\,dx = \sec^2\phi\,d\phi$

$\therefore \; dx = \frac{\sec^2\phi\,d\phi}{2\sqrt{\tan\phi}}$

$$\because \quad I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\phi\,d\phi}{\sqrt{\tan\phi}\cdot\sec\phi}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{(\sin\phi\cos\phi)}} = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\phi}{\sqrt{(\sin 2\phi)}}$$

Put $2\phi = t$; $2d\phi = dt$

$$\therefore \quad I_2 = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} (\sin t)^{-1/2}\,dt$$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \quad I_2 = \frac{1}{4\sqrt{2}}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \text{...(ii)}$$

Multiplying (i) and (ii) we get

$$\text{L.H.S.} = I_1 \times I_2 = -\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \times \frac{1}{4\sqrt{2}}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\pi}{4\sqrt{2}} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

Example 13. *Prove that* $2^{2n-1}\,\Gamma(n)\,\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2n)$. ***(Purvanchal 9***

Solution. We know that $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$$= \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \qquad \text{...(i)}$$

Putting $m = \frac{1}{2}$ in it, we get

$$\int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(1/2)\,\Gamma(n)}{2\Gamma(n+\frac{1}{2})} = \frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n+\frac{1}{2})} \qquad \text{...(ii)}$$

Again if we put $m = n$ in (i), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(n)\,\Gamma(n)}{2\Gamma(2n)}$$

or $$\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1}\, d\theta = \frac{2^{2n-1}\{\Gamma(n)\}^2}{2\,\Gamma(2n)}$$

(on multiplying both sides by 2^{2n-1})

or $$\int_0^{\pi/2} (\sin 2\theta)^{2n-1}\, d\theta = \frac{2^{2n-1}\{\Gamma(n)\}^2}{2\,\Gamma(2n)}$$

or $$\frac{2^{2n-1}\{\Gamma(n)\}^2}{2\,\Gamma(2n)} = \frac{1}{2}\int_0^{\pi} (\sin t)^{2n-1}\, dt$$

(by the substitution $2\theta = t$)

$$= \int_0^{\pi/2} (\sin t)^{2n-1} dt \qquad \text{(by property 6 of definite integrals)}$$

$$= \int_0^{\pi/2} \left\{\sin\left(\frac{\pi}{2} - t\right)\right\}^{2n-1} dt \quad \text{by property 4}$$

$$= \int_0^{\pi/2} (\cos t)^{2n-1} dt = \int_0^{\pi/2} (\cos\theta)^{2n-1} d\theta$$

or $$\int_0^{\pi/2} (\cos\theta)^{2n-1} d\theta = \frac{2^{2n-1}\{\Gamma(n)\}^2}{2\,\Gamma(2n)} \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$\frac{2^{2n-1}\{\Gamma(n)\}^2}{2\,\Gamma(2n)} = \frac{\sqrt{\pi}\,\Gamma(n)}{2\,\Gamma(n+\frac{1}{2})}$$

or $$2^{2n-1}\,\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2n).$$ **Hence Proved.**

EXERCISE 4·1

1. $\displaystyle\int_0^\infty e^{-x} x^{2/3}\, dx$ 2. $\displaystyle\int_0^\infty e^{-3x} x^4\, dx$

3. $\displaystyle\int_0^\infty \frac{e^{-1/x}}{x^{3/2}}\, dx$ 4. $\displaystyle\int_0^1 \frac{dx}{(1-x^6)^{1/6}}$

Show that

5. $\displaystyle\int_0^\infty x^n e^{-k^2x^2}\, dx = \Gamma\left(\frac{n+1}{2}\right) / 2k^{n+1}$ and hence solve $\displaystyle\int_0^\infty e^{-k^2x^2}\, dx$

(Gorakhpur 2004, 06, 08)

6. $\displaystyle\int_0^\infty x^{2n-1} e^{-ax^2}\, dx = \frac{\Gamma(n)}{2a^n}$

7. $\displaystyle\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$

8. $\displaystyle\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

9. $\displaystyle\int_0^1 x^{3/2}(1-x)^{3/2}\, dx = \frac{3\pi}{128}$

10. $\displaystyle\int_0^1 x^2(1-x^2)^{3/2}\, dx = \frac{\pi}{32}$

11. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\Gamma\left(\frac{1}{n}\right)\sqrt{\pi}}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

(Gorakhpur 88; 96)

12. $\displaystyle\int_0^\infty \frac{x^2\, dx}{1+x^4} = \Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)$

(Gorakhpur 2000, 11)

$$\left[\textbf{Hint : Put } x = \sqrt{\tan\theta} \text{ then the integral} = 2\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta\right]$$

13. Show that $\displaystyle\int_0^1 (1-x^3)^{-1/2}\, dx = \frac{1}{3}B\left(\frac{1}{3},\frac{1}{2}\right)$

14. Express $\displaystyle\int_0^1 x^m(1-x^n)^p\, dx$ in terms of Beta functions and hence

evaluate $\displaystyle\int_0^1 x^5(1-x^3)^{10}\, dx.$ *(Gorakhpur 2003, 2007, 2008)*

15. Show that $B(m,n) = B(m+1,n) + B(m,n+1).$ *(Gorakhpur 99, 2005)*

16. Show that $\displaystyle\int_0^\infty \frac{x^8(1-x^6)\, dx}{(1+x)^{24}} = 0.$

[**Hint :** Given integral $= B\,(9\;15) - B\,(15, 9) = 0$].

17. Prove that $B\,(n, n) = \dfrac{\sqrt{\pi}\;\Gamma(n)}{2^{2n-1}\,\Gamma(n+\frac{1}{2})} = 2\displaystyle\int_0^{1/2} (x - x^2)^{n-1}\,dx$

(Purvanchal 89, 2003; Gorakhpur 99, 2002)

18. Show that

$$\int_0^{\pi/2} \sqrt{\tan\theta}\;d\theta = 2\int_0^{\infty} \frac{x^2\,dx}{1+x^4} = \frac{1}{2}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

(Sidh. 2017)

19. Prove that $\displaystyle\int_0^{2a} x^3\sqrt{2ax - x^2}\;dx = \frac{7\pi\,a^5}{8}$

(Gorakhpur 99)

20. Prove that $\left|\overline{\frac{3}{2} - x}\right.\;\left|\overline{\frac{3}{2} + x}\right. = \left(\frac{1}{4} - x^2\right)\pi\sec\pi x$

21. Evaluate $\displaystyle\int_0^{\pi/6} \cos^4 3\phi\,\sin^3 6\phi\;d\phi$

22. Evaluate $\displaystyle\int_0^{a} x^4\,(a^2 - x^2)^{1/2}\,dx$

23. $\displaystyle\int_0^{\infty} \frac{x^4\,dx}{(a^2 + x^2)^4}$

24. $\displaystyle\int_0^{a} x^3\,(2ax - x^2)^{3/2}\,dx$

25. If $b > a$, $m > 0$, $n > 0$ prove that

$$\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\,B\,(m, n)$$

26. Prove that $\displaystyle\int_{-1}^{1} (1-x)^{p-1}\,(1+x)^{q-1}dx = 2^{p+q-1}\,B\,(p, q)$

27. Prove that $\displaystyle\int_{-\pi/4}^{\pi/4} (\cos x + \sin x)^{m-1}\,dx = \frac{2^{m-\frac{3}{2}}\,\Gamma\left(\frac{m}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{m+1}{2}\right)}$

□□