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TOTAL DIFFERENTIAL EQUATIONS

1. Total Differential Equation.

The equation

$$P\,dx + Q\,dy + R\,dz = 0$$

where P, Q and R are functions of x, y and z, is called a **total differential equation.**

If u(x, y, z) = c satisfies the equation, it is said to be the solution or integral of the given differential equation. If it is integrable, we shall find the condition which P, Q, R must satisfy so that the total differential equation is integrable. As such, the condition is called **condition of integrability.**

2. Condition for the integrability of the Equation

$$P\,dx + Q\,dy + R\,dy = 0 \qquad \text{..... (1)}$$

Theorem : A necessary and sufficient condition for the integrability of the equation

$$P\,dx + Q\,dy + R\,dz = 0$$

is that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \text{ .....(2)}$$

Proof : Necessary Condition : Let the equation (1) be integrable. Then a functional relation f(x, y, z) = c must be an integral of the equation (1). Taking the total differential of f, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad \text{.....(3)}$$

Comparing this with $P\,dx + Q\,dy + R\,dz = 0$,
we have

$$\frac{\frac{\partial f}{\partial x}}{P} = \frac{\frac{\partial f}{\partial y}}{Q} = \frac{\frac{\partial f}{\partial z}}{R} = \lambda \text{ (say)}$$

where λ is a function of x, y, z. Therefore

$$\frac{\partial f}{\partial x} = \lambda P,\ \frac{\partial f}{\partial y} = \lambda Q,\ \frac{\partial f}{\partial z} = \lambda R$$

But $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore, $$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial}{\partial x}(\lambda Q)$$

or, $$\lambda \frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

or, $$\lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y}$$

Similarly,

$$\lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial y} - R\frac{\partial \lambda}{\partial x}$$

and

$$\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z}$$

Multiplying these three equations with R, Q and P respectively and adding, we get

$$\lambda\left[P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] = 0$$

Since $\lambda \neq 0$, the condition (2), the integrability of equation (1) follows.

Sufficient Condition : To prove this part, take z to be constant and also suppose that Pdx + Qdy is a perfect differential is x, y say, dV(x, y, z). If it is not, we shall find a suitable integrating factor $\lambda(x, y, z)$ such that $\lambda P\, dx + \lambda Q\, dy$ is a perfect differential of x, y. To establish the sufficiency part, let us consider the equation

$$\lambda P\, dx + \lambda Q\, dy + \lambda R\, dz = 0$$

It is easy to see that if (2) holds for P, Q, R, it also holds for λP, λQ, λR.

For the function V(x, y, z), where z is constant,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

If $\quad dV = P\,dx + Q\,dy,$

We have $P = \dfrac{\partial V}{\partial x},\ Q = \dfrac{\partial V}{\partial y}$.

Substituting these values in (2), we have

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right) + R\left(\frac{\partial^2 V}{\partial y \partial x} - \frac{\partial^2 V}{\partial x \partial y}\right) = 0$$

i.e.
$$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

i.e. Jacobian $\quad \dfrac{\partial\left(V, \dfrac{\partial V}{\partial z} - R\right)}{\partial(x, y)} = 0$

which shows that $\dfrac{\partial V}{\partial z} - R$ and V are dependent and that there is a relation between them which is indendent of x, y

As such,

$$\frac{\partial V}{\partial z} - R = f(V, z) \text{ only.}$$

Then $\quad R = \dfrac{\partial V}{\partial z} - f(V, z).$

With this value of R, equation (1) takes the form

$$\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \left(\frac{\partial V}{\partial z} - f(V, z)\right) dz = 0$$

i.e. $\quad \left(\dfrac{\partial V}{\partial x}dx + \dfrac{\partial V}{\partial y}dy + \dfrac{\partial V}{\partial z}dz\right) - f(V, z)dz = 0$

or, $\quad dV - f(V, z)dz = 0$

or, $\quad \dfrac{dV}{dz} = f(V, z)$

This is a first degree and first order differential equation is V, z. Let $\phi(V, z) = C$ be its solution, where C is arb. constant. Substituting for V, we have a relation in x, y, z

$$\psi(x, y, z) = C,$$

This is the integral of the equation (1). Thus we have established that the condition (2) is sufficient for integrability of total differential equation (1).

3. Different Methods to solve theTotal Differential Equations.

We shall discuss different methods to solve the differential equation (1).

Method (a). Inspection Method : Sometimes by rearranging the terms of the given equation or by dividing by a suitable function of x, y, z the equation contains several parts which are exact differentials. In such case, we need not apply the condition of integrability and the integration can easily be obtained. Following list of exact differentials may be helpful :

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(y/x\right)$

(iii) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log \dfrac{y}{x}\right)$

(iv) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{y}{x}\right)$

(v) $\dfrac{2xy\,dy - y^2\,dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

(vi) $\dfrac{2x^2 y\,dy - 2xy^2\,dx}{x^4} = d\left(\dfrac{y^2}{x^2}\right)$

(vii) $2xy\,dy + y^2\,dx = d(y^2x)$

(viii) $xy\,dz + xz\,dy + yz\,dx = d(xyz)$.

Expample 1. Solve $zx\,dx + zy\,dy + (x^2 + y^2)dz = 0$

Solution. The equation may be written as

$$\frac{x\,dx + y\,dy}{x^2 + y^2} + \frac{dz}{z} = 0$$

Integrating, $1/2 \log(x^2 + y^2) + \log z = \log C'$.

or, $(x^2 + y^2)z^2 = C$.

Example. 2. Solve $dx + dy + (x + y)dz = 0$

Solution. The equation may be written as

$$\frac{dx + dy}{x + y} + dz = 0.$$

Integrating, $\log(x + y) + z = \log c$

or, $\log\{(x+y)/c\} = -z$

$\therefore$ $x + y = ce^{-z}$.

Example 3. Solve $(y^2 + z^2 - x^2)dx - 2xy\,dy - 2xz\,dz = 0$.

Solution. Adding and subtracting $x^2\,dx$, the equation takes the form

$$(x^2 + y^2 + z^2)dx - 2x(x\,dx + y\,dy + z\,dz) = 0$$

or, $$\frac{dx}{x} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$$

Integrating, $\log x + \log c = \log(x^2 + y^2 + z^2)$

or, $cx = x^2 + y^2 + z^2$.

Example 4. Solve $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$

Solution. Dividing the equation by x^2y^2, we have

$$\left(\frac{1}{y} - \frac{y}{x^2} - \frac{z}{x^2}\right)dx + \left(\frac{1}{x} - \frac{z}{y^2} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} + \frac{1}{y}\right)dz = 0$$

Rearranging, we have

$$\frac{y\,dx - x\,dy}{y^2} + \frac{x\,dy - y\,dx}{x^2} + \frac{x\,dz - z\,dx}{x^2} + \frac{y\,dz - z\,dy}{y^2} = 0$$

or, $$d\left(\frac{x}{y}\right) + d\left(\frac{y}{x}\right) + d\left(\frac{z}{x}\right) + d\left(\frac{z}{y}\right) = 0$$

Integrating, $$\frac{x}{y} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} = C$$

or, $x^2 + y^2 + z(x + y) = Cxy$.

Example 5. Solve $(x^2 - y^2 - z^2 + 2xy + 2xz)dx + (y^2 - z^2 - x^2 + 2yz + 2yx)dy + (z^2 - x^2 - y^2 + 2zx + 2zy)dz = 0$.

Solution. Adding and subtracting x^2dx, y^2dy, z^2dz in first, second and third term respectively and simplifying, we have

$[-(x^2 + y^2 + z^2) + 2x(x + y + z)]dx + [-(x^2 + y^2 + z^2) + 2y(x + y + z)] + dy[-(x^2 + y^2 + z^2) + 2z(x + y + z)]dz = 0$

or, $-(x^2 + y^2 + z^2)(dx + dy + dz) + (x + y + z)(2x\,dx + 2y\,dy + 2z\,dz) = 0$

or, $$\frac{dx + dy + dz}{x + y + z} = \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2}$$

Integrating, $\log(x + y + z) = \log(x^2 + y^2 + z^2) + \log C$

or, $x + y + z = C(x^2 + y^2 + z^2)$, c being arb. constant.

Example 6. Find $f(z)$ such that $(y^2 + z^2 - x^2)dx - 2xy\,dy + 2x\,f(z)dz = 0$ is integrable. Hence Solve it.

Solution. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we have

$$P = y^2 + z^2 - x^2, \quad Q = -2xy, \quad R = 2xf(z)$$

Substituting, these in the equation

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

and writing f for f(z), we have

$$(y^2 + z^2 - x^2)(0 - 0) - 2xy(2f - 2z) + 2xf[2y - (-2y)] = 0$$

or, $$-4xy(f - z) + 8xyf = 0$$

or, $$f = -z$$

Now the given equation becomes

$$(y^2 + z^2 - x^2)dx - 2xy\,dx - 2xz\,dz = 0$$

Adding and subtracting x^2dx, this equation reduces to

$$(x^2 + y^2 + z^2)dx - 2x(x\,dx + y\,dy + z\,dz) = 0$$

or, $$\frac{dx}{x} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$$

Integrating, $$\log x + \log C = \log(x^2 + y^2 + z^2)$$

or, $$Cx = x^2 + y^2 + z^2.$$

Method (b). One Variable is taken as Constant.

When the equation $P\,dx + Q\,dy + R\,dz = 0$ is integrable, take z as constant, so $dz = 0$. Then

$$P\,dx + Q\,dy = 0 \text{ is solved.}$$

If $\phi(x, y) = C$ be a solution of $P\,dx + Q\,dy = 0$, the arbitrary constan. C be replaced by f(z), so that

$$\phi(x, y) - f(z) = 0$$

The total differential of $\phi(x, y) - f(z) = 0$ given by

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy - f'(z)dz = 0 \quad \text{is compared with}$$

$$P\,dx + Q\,dy + R\,dz = 0$$

to obtain an ordinary differential equation is f and z.

This equation is solved to find f as a function of z say $f_1(z)$ and arb. constant C. Then $\phi(x, y) = f_1(z) + C$ is required solution.

Example 7. Solve $(xz^2 - ayz)dx + (yz^2 - azx)dy + axy\,dz = 0$.

Solution. Here $P = xz^2 - ayz, \quad Q = yz^2 - azx, \quad R = axy$

$$\therefore \quad \frac{\partial P}{\partial x} = z^2 \qquad \frac{\partial Q}{\partial x} = -az \qquad \frac{\partial R}{\partial x} = ay$$

$$\frac{\partial P}{\partial y} = -az \qquad \frac{\partial Q}{\partial y} = z^2 \qquad \frac{\partial R}{\partial y} = ax$$

$$\frac{\partial P}{\partial z} = 2xz - ay \qquad \frac{\partial Q}{\partial z} = 2zy - ax \qquad \frac{\partial R}{\partial z} = 0$$

Then

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$= (xz^2 - ayz)(2yz - ax - ax) + (yz^2 - azx)(ay - 2xz + ay) + axy(-az + az)$

$= (xz^2 - ayz)(2yz - 2ax) + (yz^2 - azx)(2ay - 2xz)$

$= 2z(xz - ay)(yz - ax) + 2z(yz - ax)(ay - xz) = 0$

Hence the equation is integrable.

Let us take z as constant. Then dz = 0. The given equation reduces to

$$(xz^2 - ayz)dx + (yz^2 - azx)dy = 0$$

or, $$z^2(x\,dx + y\,dy) - az(y\,dx + x\,dy) = 0$$

or, $$\frac{1}{2}z^2 d(x^2 + y^2) - az\,d(xy) = 0$$

Inegrating, $$\frac{1}{2}z^2(x^2 + y^2) - az\,xy = C = f(z),$$

where f(z) is an arbitrary function of z.

Differentiating this, equation, we get

$(x^2 + y^2)z\,dz + z^2(x\,dx + y\,dy) - axy\,dz - azy\,dx - azx\,dy = f'(z)dz$

or, $(xz^2 - ayz)dx + (y^2z - axz)dy + [z(x^2 + y^2) - axy - f'(z)]dz = 0$

Comparing this equation with the given equation, we have

$$1 = 1 = \frac{z(x^2 + y^2) - axy - f'(z)}{axy}$$

or, $$f'(z) = z(x^2 + y^2) - 2axy$$

$$\therefore \frac{df}{dz} = \frac{2}{z}\left[\frac{1}{2}z^2(x^2 + y^2) - axyz\right].$$

or, $$\frac{df}{dz} = \frac{2}{z}f \quad \text{, since } f = \frac{1}{2}z^2(x^2 + y^2) - axyz$$

or, $$\frac{df}{f} = 2\frac{dz}{z}$$

Integrating, $\log f = \log z^2 + \log C$, where C is arbitrary constant

or, $f = cz^2$

or, $\frac{1}{2} z^2(x^2 + y^2) - axyz = cz^2$

or, $z(x^2 + y^2) - 2\,axy = 2\,Cz$

or, $z(x^2 + y^2) = 2\,axy + 2\,Cz.$

Aliter : The given equation may be written as

$$z^2(x\,dx + y\,dy) - az\,(y\,dx + x\,dy) + axy\,dz = 0$$

or, $\frac{1}{2} z^2 d(x^2 + y^2) - azd(xy) + axy\,dz = 0$

or, $\frac{1}{2} d(x^2 + y^2) - a\frac{\{zd(xy) - xy\,dz\}}{z^2} = 0$

or, $\frac{1}{2} d(x^2 + y^2) - ad\left(\frac{xy}{z}\right) = 0$

Integrating $\frac{1}{2}(x^2 + y^2) - \frac{axy}{z} = C$

or, $z(x^2 + y^2) - 2axy = 2Cz.$

Example 8. Solve

$$(2yz + zx - z^2)dx - zx\,dy - (x^2 + xy - xz)\,dz = 0$$

Solution. Here

$$P = 2yz + zx - z^2, \quad Q = -zx, \quad R = -(x^2 + xy - xz)$$

$\frac{\partial P}{\partial x} = z$ $\qquad \frac{\partial Q}{\partial x} = -z$ $\qquad \frac{\partial R}{\partial x} = -(2x + y - z)$

$\frac{\partial P}{\partial y} = 2z$ $\qquad \frac{\partial Q}{\partial y} = 0$ $\qquad \frac{\partial R}{\partial y} = -x$

$\frac{\partial P}{\partial z} = 2y + x - 2z$ $\qquad \frac{\partial Q}{\partial z} = -x$ $\qquad \frac{\partial R}{\partial z} = z$

Then $\quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (2yz + zx - z^2)(-x + x) - zx(-2x - y + z - 2y - x + 2z) - (x^2 + xy - xz)(2z + z)$

$= 3(zx^2 + xyz - xz^2 - zx^2 - xyz + xz^2) = 0$

Therefore, the euation is integrable. Taking x as constant, dx = 0, so that the given equation reduces to

$$-zx\,dy - (x^2 + xy - xz)dz = 0$$

i.e. $$zx\frac{dy}{dz} + x^2 + xy - xz = 0$$

or, $$\frac{dy}{dz} + \frac{x}{z} + \frac{y}{z} - 1 = 0$$

or, $$\frac{dy}{dz} + \frac{1}{z}y = \left(1 - \frac{x}{z}\right)$$

This equation is linear in y. The integrating factor is z. Then its solution is

$$yz = \int\left(1 - \frac{x}{z}\right)z\,dz + C$$

or, $$yz = \frac{z^2}{2} - xz + C$$

Writing C as an arbitrary function of x, we have

$$yz = \frac{z^2}{2} - xz + f(x) \qquad .....(1)$$

or, $$f(x) - xz - yz + \frac{1}{2}z^2 = 0$$

Differentiating it, we get

$$f'(x)dx - (x\,dz + z\,dx) - (y\,dz + z\,dy) + z\,dz = 0$$

or, $$[f'(x) - z]dx - zdy - (x + y - z)\,dz = 0$$

Comparing this with the given equation, we get

$$\frac{f'(x) - z}{2yz + zx - z^2} = \frac{-z}{-zx} = \frac{1}{x}$$

Hence $$f'(x) = z + \frac{1}{x}(2yz + zx - z^2)$$

$$= \frac{2}{x}(yx + zx - \frac{1}{2}z^2) = \frac{2}{x}f(x) \qquad \text{form (1)}$$

$\therefore$ $$\frac{f'(x)}{f(x)} = \frac{2}{x}$$

Integrating w.r.t. x, we have

$$\log f(x) = 2\log x + \log C$$

or, $$f(x) = Cx^2$$

Hence the solution of the equation is

$$yz = \frac{z^2}{2} - xz + cx^2$$

or, $$2yz + 2xz - z^2 = 2cx^2.$$

Example 9. Solve

$$(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0$$

Solution. Taking x as constant, dx = 0. The equation reduces to

$$dy + 2z\,dz = 0$$

Integrating it, we get

$$y + z^2 = C$$

Writing C = f(x), an arbitrary function of x, we have

$$y + z^2 = f(x)$$

Differentiating it, we get

$$dy + 2z\,dz - f'(x)dx = 0$$

Comparing this equation with the given differential equation, we get

$$\begin{aligned} -f'(x) &= 2x^2 + 2xy + 2xz^2 + 1 \\ &= 2x^2 + 1 + 2x(y + z^2) \\ &= 2x^2 + 1 + 2xf \qquad \text{, since } f = y + z^2. \end{aligned}$$

$$\therefore \qquad \frac{df}{dx} + 2xf = -(2x^2 + 1)$$

This is a linear equation in f. Its integrating factor is $e^{2\int x\,dx} = e^{x^2}$ and its solution is.

$$\begin{aligned} fe^{x^2} &= -\int (2x^2 + 1)e^{x^2}\,dx + C \\ &= -\int [x(2xe^{x^2})]dx - \int e^{x^2}\,dx + C \\ &= -\left[x.\left(e^{x^2}\right) - \int 1.e^{x^2}\,dx\right] - \int e^{x^2}\,dx + C \\ &= -x\,e^{x^2} + \int e^{x^2}\,dx - \int e^{x^2}\,dx + C \\ &= -x\,e^{x^2} + C \end{aligned}$$

or, $$(y + z^2)e^{x^2} = -x\,e^{x^2} + C$$

or, $$x + y + z^2 = Ce^{-x^2}$$

Example 10. Find the value of f(z) and then solve

$$yz \log z\, dx - f(z)\, dy + xy\, dz = 0 \qquad \text{......(i)}$$

Solution. To find the value of f(z), let us consider the integrability of the differential equation. Then P, Q, R satisfy

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

i.e. $yz \log z[-f'(z) - x] - f(z)[y - y(1 + \log z)] + xy[z \log z - 0] = 0$

or, $-yz \log z[f'(z) + x] + y \log z\, f(z) + xyz \log z = 0$

or, $-yzf'(z) - xyz + y\, f(z) + xyz = 0$

$$z\, f'(z) = f(z)$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z}$$

Integrating, $\log f(z) = \log z + \log a$

$f(z) = az,$ where a is constant.

The given equation takes the form

$$yz \log z\, dx - az\, dy + xy\, dz = 0 \qquad \text{.....(ii)}$$

To solve it, take y as a constant, then dy = 0.

Then the equation (ii) becomes

$$yz \log z\, dx + xy\, dz = 0$$

or, $$\frac{dx}{x} + \frac{dz}{z \log z} = 0$$

Integrating, $\log x + \log(\log z) = \log C$

or, $x \log z = C = u(y),$ .....(iii)

where u is arbitrary function of y.

Differentiating this equation, we get

$$\log z\, dx + \frac{x}{z}dz - u'(y)dy = 0$$

or, $z \log z\, dx - zu'(y)dy + x\, dz = 0$ .....(iv)

Comparing equation (ii) and (iv), we have

$$\frac{u'(y)}{a} = \frac{1}{y}$$

or, $$\frac{du}{dy} = \frac{a}{y}$$

Integrating, we get

$$u(y) = a \log y + \log C$$

or, $x \log z = a \log y + \log C$

$\therefore$ $z^x = cy^a$

Method (c) When $P\,dx + Q\,dy + R\,dz = 0$ is homogeneous equation i.e. P, Q, R are homogeneous function of the same degree.

There are two ways of solving such equations.

First, let us put

$$x = zu, \qquad y = zv$$

so that $$dx = u\,dz + z\,du, \qquad dy = v\,dz + z\,dv$$

Substituting these in the given equation and simplifying, we get

$$f(u, v)du + h(u, v)dv + \frac{dz}{z} = 0$$

Integrating it, we get the result.

Next, If the equation $Pdx + Qdy + Rdz = 0$ is homogeneous, then it can be transformed to an integrable one,

if $\quad Pdx + Qdy + Rdz \neq 0.$

We find an integrating factor (I.F.) as $\dfrac{1}{(Px + Qy + Rz)}$. Multiplying by it and integrating, we find the solution.

In case $Px + Qy + Rz = 0$, the equation is exact i.e. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ etc. and it is integrable immediately.

Example 11. Solve

$$(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$$

Solution. It is homogeneous equation. Let us put

$$x = uz \qquad \text{and } y = vz$$

so that $dx = z\,du + u\,dz, \qquad dy = z\,dv + v\,dz.$

Substituting these values in the equation, we get

$$(z^2v^2 + z^2u)(z\,du + u\,dz) + (z^2 + z^2u)(z\,dv + v\,dz) + (v^2z^2 - uvz^2)dz = 0$$

or, $$(v^2 + v)(z\,du + u\,dz) + (1 + u)(z\,dv + v\,dz) + (v^2 - uv)dz = 0$$

or, $$z[(v^2 + v)du + (1 + u)dv] + [(v^2 + v)u + (1 + u)v + v^2 - uv]dz = 0$$

or, $$z[(v^2 + v)du + (1 + u)dv] + [uv^2 + uv + v + v^2]dz = 0$$

or, $$\frac{(v^2 + v)du}{uv^2 + uv + v + v^2} + \frac{(1+u)dv}{uv^2 + uv + v + v^2} + \frac{dz}{z} = 0$$

or, $$\frac{v(v+1)du}{(u+1)(v^2 + v)} + \frac{(1+u)dv}{(u+1)(v^2 + v)} + \frac{dz}{z} = 0$$

or, $$\frac{du}{u+1} + \frac{dv}{v(v+1)} + \frac{dz}{z} = 0$$

or, $$\frac{du}{u+1} + \frac{dv}{v} - \frac{dv}{v+1} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log(u + 1) + \log v - \log(v + 1) + \log z = \log C$$

or, $$\frac{zv(u+1)}{v+1} = C$$

or, $$\frac{y\left(\frac{x}{z}+1\right)}{\left(\frac{y}{z}\right)+1} = C$$

or, $$y(x + z) = C(y + z).$$

Example 12. Solve

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$$

Solution. Let us put

$$x = uz, \qquad y = vz$$

$\therefore$ $$dx = z\,du + u\,dz, \qquad dy = z\,dv + v\,dz$$

The given equation reduces to

$$(v^2z^2 + vz^2 + z^2)(z\,du + u\,dz) + (z^2 + uz^2 + u^2z^2)(z\,dv + v\,dz) + (u^2z^2 + uvz^2 + v^2z^2)dz = 0$$

or, $$(v^2 + v + 1)(z\,du + u\,dz) + (1 + u + u^2)(z\,dv + v\,dz) + (u^2 + uv + v^2)dz = 0$$

or, $$z[(v^2 + v + 1)du + (1 + u + u^2)dv] + [u(1 + v + v^2) + v(1 + u + u^2) + (u^2 + uv + v^2)]dz = 0$$

or, $$\frac{(v^2 + v + 1)du + (u^2 + u + 1)dv}{uv^2 + 3uv + u^2v + u + v + u^2 + v^2} + \frac{dz}{z} = 0$$

or, $$\frac{(v^2 + v + 1)du + (u^2 + u + 1)dv}{uv(u + v + 1) + u(v + u + 1) + v(u + v + 1)} + \frac{dz}{z} = 0$$

or, $$\frac{(v^2 + v + 1)du + (u^2 + u + 1)dv}{(u + v + 1)(uv + u + v)} + \frac{dz}{z} = 0$$

or, $$\frac{(u + v + 1)\{(u + 1)dv + (v + 1)du\} - (uv + u + v)(du + dv)}{(u + v + 1)(uv + u + v)} + \frac{dz}{z} = 0$$

or, $$\frac{(u + 1)dv + (v + 1)du}{uv + u + v} - \frac{(du + dv)}{u + v + 1} + \frac{dz}{z} = 0$$

Integrating, $\log(uv + u + v) - \log(u + v + 1) + \log z = \log C$

$$\log\left[\frac{uv + u + v}{u + v + 1}\,z\right] = \log C$$

or, $$\frac{uv+u+v}{u+v+1}.z = C$$

or, $$\frac{xy+zx+zy}{x+y+z} = C$$

or, $xy + zx + zy = C(x + y + z)$

Aliter. Inspection method may be easier for this problem. For

$$y^2 + yz + z^2 = (y+z)^2 - yz = (y+z)^2 - (xy+yz+zx) + x(y+z)$$
$$= (y+z)(x+y+z) - (xy+yz+zx),$$
$$z^2 + zx + x^2 = (z+x)(x+y+z) - (xy+yz+zx)$$

and $x^2 + xy + y^2 = (x+y)(x+y+z) - (xy+yz+zx)$

Hence the given equation takes the form

$$(x+y+z)[(y+z)dx+(z+x)dy+(x+y)dz] - (xy+yz+zx)(dx+dy+dz) = 0$$

or, $(x+y+z)[(x\,dy+y\,dx)+(y\,dz+z\,dy)+(z\,dx+x\,dz)] - (xy+yz+zx)(dx+dy+dz) = 0$

or, $(x+y+z)\,d(xy+yz+zx) - (xy+yz+zx)(dx+dy+dz) = 0$

or, $$\frac{d(xy+yz+zx)}{xy+yz+zx} - \frac{dx+dy+dz}{x+y+z} = 0$$

Integrating, $\log(xy+yz+zx) - \log(x+y+z) = \log C$

i.e., $xy + yz + zx = C(x + y + z)$.

Example 13. Solve the equation

$$(2xy - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$$

Solution. We see that P, Q, R are homogeneous and

$$xP + yQ + zR = 2x^2z - xyz + 2y^2z - xyz - y^2z + xyz - x^2z$$
$$= x^2z - xyz + y^2z = z(x^2 - xy + y^2) \neq 0$$

Hence, $\frac{1}{z(x^2 - xy + y^2)}$ is I.F. Multiplying the equation by this factor, we have

$$\frac{(2xz - yz)dx}{z(x^2 - xy + y^2)} + \frac{(2yz - zx)dy}{z(x^2 - xy + y^2)} - \frac{(x^2 - xy + y^2)dz}{z(x^2 - xy + y^2)} = 0$$

i.e., $$\frac{(2x-y)dx + (2y-x)dy}{x^2 - xy + y^2} - \frac{dz}{z} = 0$$

or, $$\frac{d(x^2 - xy + y^2)}{(x^2 - xy + y^2)} - \frac{dz}{z} = 0$$

Integrating,

$$\log(x^2 - xy + y^2) - \log z = \log C$$

i.e. $x^2 - xy + y^2 = cz$.

4. To Find the solution of non-integrable total differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \quad .....(1)$$

which passes through $f(x, y, z) = C$ .....(2)

In such cases, verify the non-integrability. Differentiating (2) and eliminating z, dz from (1) and this equation, we have a differential equation in variables x and y, say

$$L\,dx + M\,dy = 0 \quad .....(3)$$

Integrating (3), we have $u(x, y) = K$, ...(4)

Then equation (2) and (4) will represent the solution of (1).

This procedure is also applicable for integrable equations (1).

Example 14. Find the curves satisfying the differential equation

$$y\,dx + (z - y)dy + x\,dz = 0 \quad .....(1)$$

and which lie on the plane $2x - y - z = 1$ .....(2)

Solution. Differentiating $2x - y - z = 1$, we get

$$2dx - dy - dz = 0 \quad .....(3)$$

Eliminating z and dz from (1) and (3), we get

$$(y + 2x)dx + (x - 2y - 1)dy = 0$$

or, $$(y\,dx + x\,dy) + 2x\,dx - 2y\,dy - dy = 0$$

Integrating, $$xy + x^2 - y^2 - y = C \quad .....(4)$$

The required curves are intersection of plane (2) and rectangular hyperbolic cylinder (4).

Example 15. Find the curves represented by the solution of

$$yz\,dx + z^2\,dy + y(z + x)dz = 0 \quad .....(i)$$

which lie on the surfaces.

$$zx = C \quad .....(ii)$$

Solution. Differentiating (ii), we have

$$z\,dx + x\,dz = 0 \quad .....(iii)$$

Eliminating z and dz from (i) and (iii), we get

$$\frac{cy}{x}dx + \frac{c^2}{x^2}dy + y\left(\frac{c}{x}+x\right)\left(-\frac{c}{x^2}dx\right) = 0$$

or, $$cx^2y\,dx + c^2x\,dy - c^2y\,dx - cx^2y\,dx = 0$$

or, $$x\,dy - y\,dx = 0$$

or, $$\frac{dy}{y} = \frac{dx}{x}$$

Integrating, $$y = Kx \quad .....(iv)$$

Hence, $y = Kx$, $zx = c$ give the required system of curves.

5. Geometrical Interpretation of $P\,dx + Q\,dy + R\,dz = 0$

Let us take $u(x, y, z) = C$ as the solution of this equation. Then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

represents that d.c.s. of the normal to the surface $u = c$ at the point (x, y, z) are proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$. It suggests that the d.c.s of the normal to the surface satisfying $P\,dx + Q\,dy + R\,dz = 0$ at the point (x, y, z) are proportional to P, Q, R.

Since $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ represent a system of curves such that the d.c.s of tangent to any curve of the system at any point (x, y, z) on it are proportional to P, Q, R at that point. Thus, geometrically, the curves represented by $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ are normal to the surface represented by $P\,dx + Q\,dy + R\,dz = 0$.

Problems

Solve.

1. $(y + z)dx + (z + x)dy + (x + y)dz = 0$ **[Ans.** $xy + xz + yz = C$]

2. $yz\,dx - xz\,dy - (x^2 + y^2)\tan^{-1}\left(\frac{y}{x}\right) dz = 0$ **[Ans.** $z\tan^{-1}\left(\frac{y}{x}\right) dz = C$]

3. $(x + xy + yz + zx)dx + (y + xy + yz + zx)dy + (z + xy + yz + zx)dz = 0$

[**Hint :** Write the equation as $(x\,dx + y\,dy + z\,dz) + (xy + yz + zx)$

$$(dx + dy + dz) = 0 \text{ or } \frac{1}{2}d(x^2 + y^2 + z^2) + (xy + yz + zx)\,d(x + y + z) = 0$$

If $x^2 + y^2 + z^2 = v$ and $(x + y + z) = u$, then $xy + yz + zx = \frac{u^2 - v}{2}$ etc.]

[Ans. $(x^2 + y^2 + z^2 - 1) = (x + y + z + 1) + Ce^{(x+y+z)}$]

4. $yz\,dx - zx\,dy - y^2dz = 0$ **[Ans.** $xyz = Ce^{-(x+y+z)}$]

5. $\sqrt{a^2 - y^2 - z^2}\,dx - y\,dy - z\,dz = 0$ **[Ans.** $y^2 + z^2 = a\sin(2x + c)$]

6. $dx + dy + (x + y + z)dz = 0$ **[Ans.** $x + y + z = ce^{-z}$]

7. $(z + z^2)dx - (z + z^2)dy + (1 - z^2)(y - \sin x)dz = 0$ **[Ans.** $(\sin x - y)(z^2 + 1) = cz$]

8. $zy\,dx + (x^2y - zx)dy + (x^2z - xy)dz = 0$ **[Ans.** $x^2(y^2 + z^2 - 2C) = 2xyz$]

9. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$

[**Ans.** $x^2 - xy + y^2 = cz$]

10. $(yz + z^2)dx - xz\,dy + xy\,dz = 0$ [**Ans.** $xz = C(y + z)$]

11. $yz(y + z)dx + zx(x + z)dy + xy(x + y)dz = 0$

[**Ans.** $xyz = c(x + y + z)$]

12. $z^2\,dx + (z^2 - 2yz)dy + (2y^2 - yz - xz)dz = 0$

[**Ans.** $(x + y)z - y^2 = cz^2$]

13. $3x^2\,dx + 3y^2\,dy - (x^3 + y^3 + e^{2z})dz = 0$

[**Hint :** $u = x^3 + y^3$ **Ans.** $x^3 + y^3 = e^{2z} + Ce^z$]

14. $yz(1 + x)dx + zx(1 + y)dy + xy(1 + z)dz = 0$

[**Hint :** Divide by xyz, **Ans.** $xyz = C\,e^{-(x+y+z)}$]

15. Show that the curves of

$$x\,dx + y\,dy + C\sqrt{\left(1 - x^2/a^2 - y^2/b^2\right)}\,dz = 0$$

that lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

lie also on the family of concentric sphere $x^2 + y^2 + z^2 = K^2$

16. Show that there is no single integral of $dz = 2y\,dx + x\,dy$. Prove that the curves of this equation that lie in the plane $z = x + y$ lie also on surfaces of the family $(x - 1)^2(2y - 1) = C$.