Chapter 8

MULTIPLE INTEGRALS

§ 8·1. Double Integrals.

Let A be a finite region of the xy-plane and let $f(x, y)$ be a function of the independent variables x and y defined at every point in A. Divide the region A into n parts of area $\delta A_1, \ldots\ldots \delta A_n$. Let (x_r, y_r) be any point inside the rth elementary area δA_r. Then the limit of the sum if it exists of $\sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r$ when the number of sub-divisions tends to infinity and thereby making area of each sub-division tend to zero, is called the double integral of $f(x, y)$ over the region A and is denoted by

$$\iint_A f(x, y)\, dA. \qquad \text{...(1)}$$

Thus $$\iint_A f(x, y)\, dA = \lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r.$$

The region "A" is called the region of integration. The term double integral refers to the dimensionality of the region A. It can be noted here that this definition corresponds to the definition

$$\int_a^b f(x)\, dx = \lim_{\substack{n \to \infty \\ \delta x \to 0}} \sum_{r=1}^{n} f(x_r)\, \delta x_r \qquad \text{...(2)}$$

for the definite integral of a single variable.

§ 8·2. Tripple Integrals.

The results obtained above for two dimensions can be extended to finite regions in three dimensions. Suppose $f(x, y, z)$ is a function defined in a closed region R. Divide the region into n sub-regions $\delta r_1, \delta r_2, \ldots\ldots \delta r_n$. Let δv_j be the volume of the jth region δr_j. If (x_j, y_j, z_j) is any point in this region then

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j, y_j, z_j)\, \delta v_j.$$

if it exists is denoted by $\iiint_R f(x, y, z)\, dv$

and is called the tripple integral of $f(x,y,z)$ over R. In this chapter we shall however confine ourselves mainly to double integrals.

§ 8·3. Evaluation of double integrals.

Theorem. *If the region A is the area bounded by the curves* $y=f_1(x)$, $y=f_2(x)$ *and the ordinates* $x=a$ and $x=b$ *then*

$$\iint_A f(x,y)\,dA = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right) dx$$

where the integration is carried with respect to y first and x is treated as constant.

Proof. Let us divide the whole region A into elementary rectangles of dimensions δx and δy by drawing lines parallel the coordinate axes. One such reccangle is shown in the figure 9.1 by the shaded region.

Then from definition

$$\iint_A f(x,y)\,dA$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma f(x_r,y_r)\,\delta x\,\delta y \qquad \text{...(i)}$$

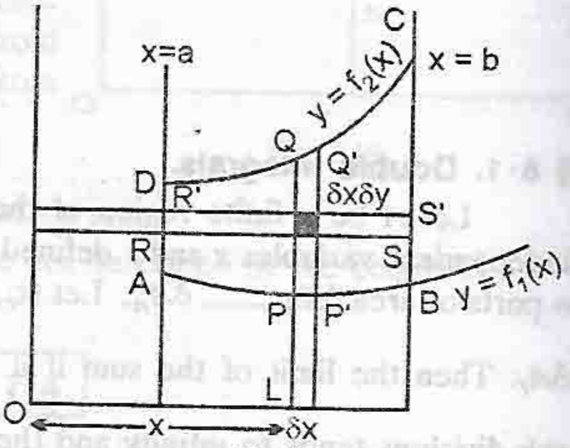

Fig. 8.1

where (x_r,y_r) is a point inside the rth rectangle and the summation extends to all such rectangles into which the region is divided.

Let us consider a vertical strip $PQQ'P'$. We first sum up all the elementary rectangles into which this strip may be supposed to be divided. Thus we get the sum of $f(x_r,y_r)\,\delta x\,\delta y$ over the strip $PQQ'P'$. Let n be the total number of such strips. Then sum of all such strips gives the sum (i).

Therefore $\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma f(x_r,y_r)\,\delta x\,\delta y$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum_{r=1}^{n} \left[\sum_{s=1}^{m} f(x_r,y_s)\,\delta y \right] \delta x \qquad \text{...(ii)}$$

Where (x_r,y_s) is a point inside the sth rectangle in the rth strip and m the number of rectangles in the rth strip. Summation inside the bracket is to be performed first keeping x_r constant.

Now we can write

$$\lim_{\delta y \to 0} \sum_{r=1}^{m} f(x_r,y_s)\,\delta y = \int_{y_1}^{y_2} f(x_r,y)\,dy \qquad \text{....(iii)}$$

where y_1 and y_2 are the extreme values of y in the rth strip. Since the region A is bounded below and above by the curves.

$y=f_1(x)$ and $y=f_2(x)$ we can take

$y_1=f_1(x_r)$ and $y_2=f_2(x_r)$

Therefore from (iii) we have

$$\lim_{\delta y \to 0} \sum_{s=1}^{m} f(x_r, y_s)\,\delta y = \int_{f_1(x_r)}^{f_2(x_r)} f(x_r, y)\,dy$$

$$= F(x_r) \text{ say.} \qquad \text{....(iv)}$$

Introducing this result in (ii) we have

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma f(x_r, y_r)\,\delta x\,\delta y = \lim_{\delta x \to 0} \sum_{r=1}^{n} F(x_r)\,\delta x$$

$$= \int_a^b F(x)\,dx = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y)\,dy\right] dx \qquad \text{...(v)}$$

Hence from (i) and (v) we get

$$\iint_A f(x, y)\,dA = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y)\,dy\right] dx \qquad \text{...(vi)}$$

General practice is to omit the brackets on the right and write it simply as

$$\iint_A f(x, y)\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\,dx\,dy \qquad \text{....(vii)}$$

Remark 1. It can be shown in the same way that if the region is bounded by the curves $x = g_1(y), x = g_2(y)$ and $y = c$ and $y = d$, then

$$\iint_A f(x, y)\,dA = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} f(x, y)\,dx\right] dy \qquad \text{...(viii)}$$

$$= \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\,dy\,dx$$

if the brackets are omitted.

Here the integration is first carried with respect to x treating y as a constant *i.e.* we sum along a horizontal strip first.

Remark 2. If the region of integration is bounded by two curves as in firuge 9.2 even then we can consider it bounded by the four lines

$$y = f_1(x),\ y = f_2(x)$$

$$x = a \text{ and } x = b.$$

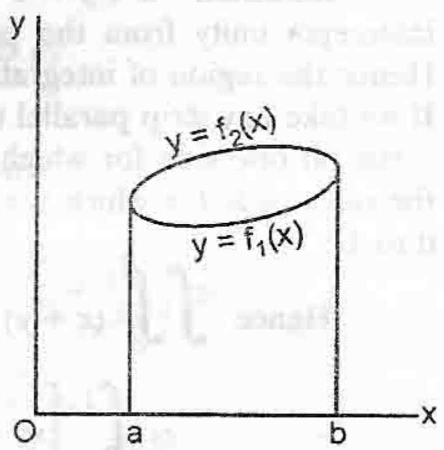

Fig. 8.2

§ 8·4. Limits of Integration for ∫∫ f (x, y) dx dy.

We have seen above that the integral is evaluated by integrating with respect to y first treating x as constant. In other words we first integrate in a vertical strip (strip parallel to y-axis). Limits of this integration therefore are the values of y for the lowest and highest points of the strip. These values are in general function of x and are obtained from the equations of the curves bounding the strip.

Second integration is made with respect to x which gives strip-wise summation from the first to the last strip. Hence the extreme left value of x for the region gives the lower limit and extreme right one as the upper limit. Examples given below make clear the task of deciding the limits and evaluating such integrals.

Example 1. *Evaluate* $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} dx\,dy$

Solution. $$\int_0^2 \int_0^{\sqrt{(2x-x^2)}} dx\,dy = \int_0^2 \left[y\right]_0^{\sqrt{(2x-x^2)}} dx$$

(on integrating with respect to y first, treating x as cnstant)

$$= \int_0^2 [\sqrt{(2x-x^2)}]\,dx = \int_0^2 \sqrt{\{x\,(2-x)\}}\,dx$$

putting $x = 2\sin^2\theta$, $dx = 4\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sqrt{(2\sin^2\theta . 2\cos^2\theta)} . 4\sin\theta\cos\theta\,d\theta$$

$$= 8\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{8\,\Gamma(3/2)\,\Gamma(3/2)}{2\,\Gamma(3)} = \frac{8 \times \frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2.2.1}$$

$$= \frac{\pi}{2}.$$

Example 2. *Evaluate*
$\iint (x+y)\,dx\,dy$ *over the region in the positive quadrant for which* $x + y \le 1$.

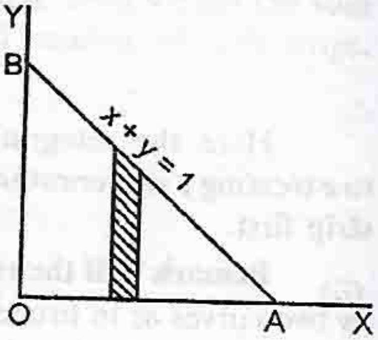

Fig. 8.3

Solution. $x + y = 1$ is a straight line making intercepts unity from the positive direction of the axes. Hence the region of integration is the $\Delta\,OAB$ in this case. If we take any strip parallel to y-axis then it is bounded by x-axis on one side for which $y = 0$ and by the line AB on the other side for which $y = 1 - x$. Limits of x are clrealy 0 to 1.

Hence $$\iint_A (x+y)\,dx\,dy = \int_0^1 \int_0^{1-x} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left\{x\,(1-x) + \frac{(1-x)^2}{2}\right\} dx$$

$$= \int_0^1 \left\{\frac{2\,(x-x^2) + 1 + x^2 - 2x}{2}\right\} dx$$

$$= \int_0^1 \left(\frac{(1-x^2)}{2} \right) dx$$

$$= \frac{1}{2}\left[x - \frac{x^3}{3} \right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}.$$

Example 3. *Evaluate* $\iint xy(x+y)\,dx\,dy$ *over the area between* $y^2 = x$ and $y = x$. *(Gorakhpur 2006, 2005)*

Solution. Let us first mark the region of integration which is bounded by

$$y = x \qquad \text{...(i)}$$

which is a st. line

and $$y^2 = x \qquad \text{...(ii)}$$

which is a parabola.

Obviously the region of integration is the region

OPAQO.

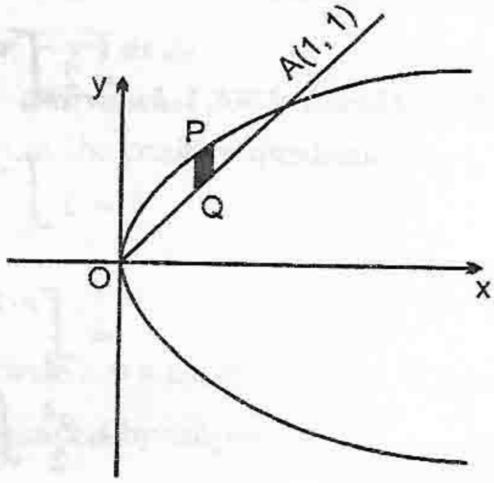

Fig. 8.4

Solving (i) and (ii) we get

$$x^2 = x.$$

$\therefore \quad x = 0 \quad \text{or} \quad x = 1,$

which gives $y = 0$ and $y = 1$ respectively. Thus the point A is (1, 1).

Now divide the region into strips parallel to y-axis and consider one such strip PQ. This strip is bounded by the line $y = x$ on one side and $y^2 = x$ or $y = \sqrt{x}$ on the other side

Thus the limits for y will be $y = x$ as lower limit and $y = \sqrt{x}$ as upper limit. Then the limits for x will be 0 to 1.

Hence the given integral $= \int_0^1 \int_x^{\sqrt{x}} xy(x+y)\,dx\,dy$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_x^{\sqrt{x}} dy$$

$$= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{x^{5/2}}{3} \right) - \left(\frac{x^4}{2} + \frac{x^4}{3} \right) \right] dx$$

$$= \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{5}{6}x^4 \right) dx$$

$$= \left[\frac{x^4}{8} + \frac{2x^{7/2}}{21} - \frac{x^5}{6} \right]_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{6} = \frac{3}{56}$$

Example 4. *Evaluate* $\iint r\,d\theta\,dr$ over the area of the circle

$$r = a \sin\theta$$

Solution. $r = a\sin\theta$ is a circle with its centre C on the line $\theta = \frac{\pi}{2}$. Initial line is tangent at the pole O.

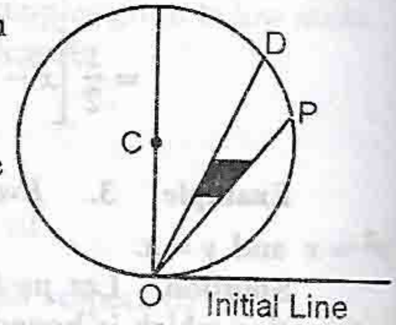

Fig. 8.5

Circle is symmertical about the line $\theta = \frac{\pi}{2}$. Hence the value of the integal

= 2 value of the integral of the half circle

$$= 2\int_0^{\pi/2}\int_0^{a\sin\theta} r\,d\theta\,dr$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/2} a^2\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi a^2}{4}.$$

EXERCISE 8·1

Evaluate the following integrals

1. $\int_1^a\int_1^b \frac{dx\,dy}{xy}$

2. $\int_1^2\int_0^x \frac{dx\,dy}{x^2+y^2}$

3. $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx$

4. $\int_0^1\int_0^{\sqrt{(1-y^2)}} 4y\,dy\,dx$

5. Prove that $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy = \frac{123}{4}$

6. Prove that $\int_0^a\int_0^{\sqrt{(a^2-x^2)}} x^2y\,dx\,dy = \frac{a^5}{15}$

7. Find the value of the integral $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

8. Find the value of $\iint \frac{xy}{\sqrt{(1-y^2)}}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$. *(Gorakhpur 1983, 2004, 2008, 2010)*

9. Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2+y^2 \le 1$.

10. Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$

11. Find the value of $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} \sqrt{(a^2-x^2-y^2)}\,dx\,dy$. *(Purvanchal 2003; Gorakhpur 87)*

12. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.

13. Evaluate $\int_0^{\pi} \int_0^{a \sin\theta} r\,d\theta\,dr$.

14. Evaluate $\iint r^2\,d\theta\,dr$ over the area of the circle $r = a\cos\theta$.

15. Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
(Gorakhpur 92, 99)

§ 8·5. Change of order of integration.

Let us consider the integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$

This integral is a summation of strips parallel to y-axis. If we change the order of integration, then the summation will be that of strips parallel to x-axis, and so before adding strips we must add up all elements in a strips parallel to the x-axis.

Following procedure is adopted for the change of order of integration.

(i) Mark the region of integration by drawing the curves $y=f_1(x)\, y=f_2(x)$, $x=a$ and $x=b$.

(ii) Now divide the region of integration into different parts (if necessary) by drawing lines parallel to x-axis through the points where strips parallel to x-axis change their character. These points may be points of intersection of any two of $y=f_1(x)$, $y=f_2(x)$, $x=a$ and $x=b$ falling in the region of integration.

(iii) Draw elementary strips parallel to x-axis in any one part of the region. The value of x in terms of y at the extremities of the elementary strips give limits of x for that part of the region.

(iv) For other parts of the region we proceed similarly and add all such integrals. Following examples will clarify the method.

Example 5. *Change the order of integration in*
$$\int_0^a \int_0^x f(x,y)\,dx\,dy.$$

Solution. The region of integration is bounded by $y = 0, y = x, x = 0$ and $x = a$. Thus OAB is the region of integration (figure 9.6)

Consider an elementary strip parallel to the axis of x. Values of x at the extremities of this strip are $x = y$ and $x = a$. These will be the lower and upper limits of x.

Also y varies from O to B, hence its limits are 0 to a.

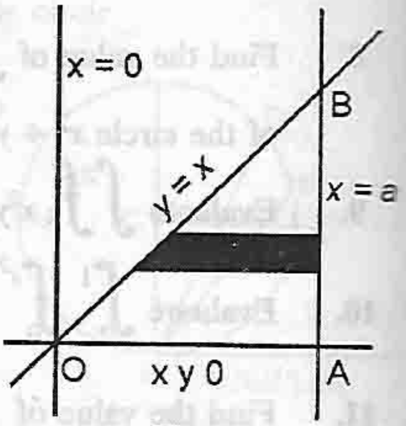

Fig. 8.6

Therefore $\displaystyle\int_0^a \int_0^x f(x,y)\, dx\, dy$

$$= \int_0^a \int_y^a f(x,y)\, dy\, dx$$

Example 6. *Change the order of integration in*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$$

and hence find its value. **(Gorakhpur 2004)**

Solution. The region of integration is bounded by the lines $y = x, x = 0$ (y-axis) and an infinite boundary. Thus it is the upper half of the first quadrant, as shown in figure 9.7.

Let us divide the region into strips parallel to x-axis and consider one such strip. The extremities of this strip lie on $x = 0$ on one side and on $y = x$ or $x = y$ on the other side. Hence the limits of x are from 0 to y.

Limits of y are clearly 0 to ∞.

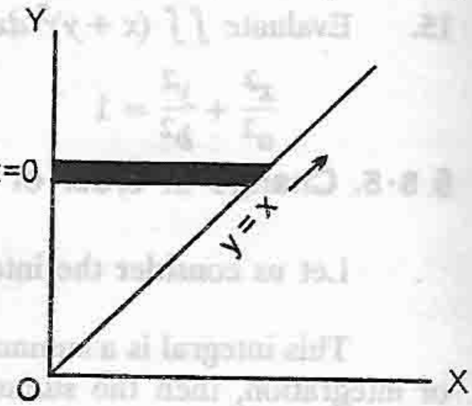

Fig. 8.7

Therefore

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dy\, dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} \Big[x\Big]_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy = \int_0^\infty e^{-y}\, dy$$

$$= \Big[-e^{-y}\Big]_0^\infty = 1.$$

Example 7. *Change the order of integration in the double integral*

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2 - x^2)}} f(x,y)\, dx\, dy.$$ **(Gorakhpur 88; Purvanchal 90)**

Solution. Here the region of integration is bounded by the line $y = x\tan\alpha$ the circle $y^2 = a^2 - x^2$ or $x^2 + y^2 = a^2$ the line $x = 0$ which is y-axis and the line $x = a\cos\alpha$ which is parallel to y-axis.

Thus the region of integration is clearly OAB as shown in figure 9.8

The circle $x^2 + y^2 = a^2$ and the line $y = x \tan \alpha$ intersect at A. Solving them we find the coordinates of A as $(a \cos \alpha, a \sin \alpha)$

Now we can see from the figure that the strips parallel to x-axis change their character at A. Thus the region of integration is divided into two parts namely OLA and LAB.

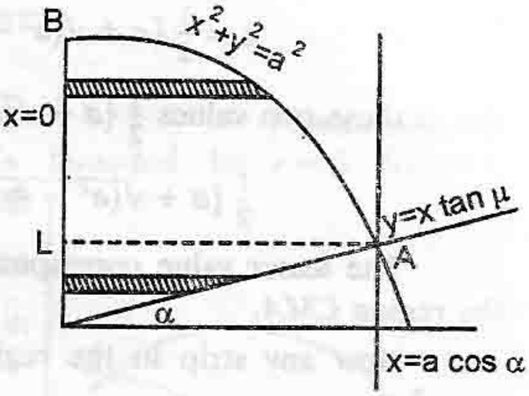

Fig. 8.8

Now in the region LAB any strip parallel to x-axis is bounded by y-axis on one side and the circle

$$x^2 + y^2 = a^2$$

or
$$x = \sqrt{(a^2 - y^2)}$$

on the other side. Hence the limits of x are 0 to $\sqrt{(a^2 - y^2)}$ and the limits of y are OL to OB i.e. $a \sin \alpha$ to a.

Again in the region OAL any strip parallel to x-axis lies on y-axis on one side and the line $y = x \tan \alpha$ or $x = y \cot \alpha$ on the oher side. Therefore the limits of x will be 0 to $y \cot \alpha$ and the limits of y are clearly 0 to $a \sin \alpha$.

Hence we have

$$\int_0^{a \cos \alpha} \int_{x \tan \alpha}^{\sqrt{(a^2 - x^2)}} f(x, y)\, dx\, dy = \int_0^{a \sin \alpha} \int_0^{y \cot \alpha} f(x, y)\, dy\, dx$$

$$+ \int_{a \sin \alpha}^{a} \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dy\, dx.$$

Example 8. *Change the order of* integration in the integral

$$\int_0^a \int_{\sqrt{ax - x^2}}^{\sqrt{ax}} V\, dx\, dy$$

(Gorakhpur 99)

Solution. The region of integration is bounded by $y = \sqrt{(ax - x^2)}$ or $x^2 + y^2 - ax = 0$ (a circle), $y = \sqrt{(ax)}$ or $y^2 = ax$ (a parabola) and the lines $x = 0$ and $x = a$. Thus the region of integration is $OPACO$, fig. 9.9.

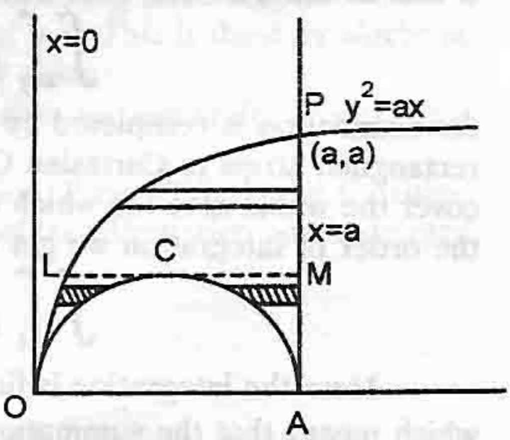

Fig. 8.9

The strips parallel to x-axis change their character at the highest point C of the circle. Let LM be the tangent at C to the circle. Then the region of integration is obviously divided into three parts namely OLC, CMA and LPM.

From the equation of the circle, we have

$$x^2 - ax + y^2 = 0$$

$\therefore \qquad x = \frac{1}{2}\{a \pm \sqrt{(a^2 - 4y^2)}\}$

out of these two values $\frac{1}{2}\{a - \sqrt{(a^2 - 4y^2)}\}$ is lesser than

$$\frac{1}{2}\{a + \sqrt{(a^2 - 4y^2)}\}$$

The lesser value corresponds to the region *OLC* whereas the greater one to the region *CMA*.

Now any strip in the region *OLC* is bounded by the parabola $y^2 = ax$ or $x = \frac{y^2}{a}$ on one side and the circle on the other side for which we shall take the lesser value $x = \frac{1}{2}\{a - \sqrt{(a^2 - 4y^2)}\}$. Since the point *C* is $(a/2, a/2)$ hence the limits for y are from 0 to $a/2$.

In the region *CMA,* the strip lies on the circle on one side and the line $x = a$ on the other. So the limits of x are from $\frac{1}{2}\{a + \sqrt{(a^2 - 4y^2)}\}$ to a and the limits of y are obviously 0 to $\frac{a}{2}$.

Again in the region *LPM* the strip lies on the parabola on one side and the line $x = a$ on the other. So the limits for x are $\frac{y^2}{a}$ to a whereas the limits for y will be $\frac{a}{2}$ to a as the point *P* is (a, a). Hence the given integral

$$= \int_0^{a/2} \int_{y^2/a}^{\frac{1}{2}\{a - \sqrt{(a^2 - 4y^2)}\}} V\,dy\,dx.$$

$$+ \int_0^{a/2} \int_{\frac{1}{2}\{a + \sqrt{(a^2 - 4y^2)}\}}^{a} V\,dy\,dx + \int_{a/2}^{a} \int_{y^2/a}^{a} V\,dy\,dx$$

§ 8·6. Change of order in polar coordinate.

Process is similar to that adopted in Cartesian Coordinates. The only difference is that in order to evaluate an integral of the type

$$\iint_A f(r, \theta)\,d\theta\,dr$$

the summation is completed by dividing the region into triangular strips (unlike the rectangular strips in Cartesian Coordinates). Then we make strip-wise summation to cover the whole area for which the value θ will be suitably chosen. When we change the order of integration we get an integral of the type

$$\iint_A f(r, \theta)\,dr\,d\theta$$

Here the integration is first performed with respect to θ, r being kept constant which means that the summation is first made inside a circular strip. Now the value of r is chosen so as to cover the whole region. Following example is given as an illustration of the method.

Example 9. *Change the order of integration in*

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, d\theta\, dr.$$

(Gorakhpur 83)

Solution. The region of integration is bounded by $r = 0$ (origin), $r = 2a\cos\theta$ (a circle with centre $(a, 0)$, $\theta = 0$ which is the initial line and $\theta = \frac{\pi}{2}$ (y-axis). Thus the region of inte- gration is the semi-circle $OQAO$ as shown in the figure 9.10.

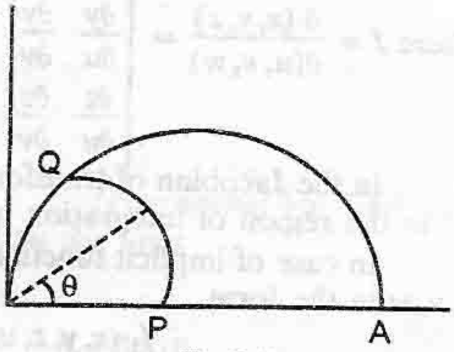

Fig. 8.10

To change the order of integration consider the elementary circular arc PQ (θ varying and r remaining constant). Thus r is bounded by the initial line one side for which $\theta = 0$ and the circle on the other side for which

$$\theta = \cos^{-1}\left(\frac{r}{2a}\right)$$

which is obtained from the equation of the circle. Since the diameter of the circle is $2a$ hence the limits for r will be 0 to $2a$. Hence the given integral

$$= \int_0^{2a}\int_0^{\cos^{-1}(r/a)} f(r,\theta)\, dr\, d\theta$$

§ 8·7. Change of variables.

(*Transformation*) In the evaluation of multiple integral sometimes it is convenient to change the variables. The process of changing the variables in a multiple integral is called the transformation of multiple integrals. Suppose we want to transform the multiple integral

$$\iint_A f(x,y)\, dx\, dy$$

to another system of variables, u, v, where

$$x = \phi(u,v) \text{ and } y = \psi(u,v), \text{ say}$$

then the transformation would be completed in the following three steps.

(i) To determine the function $f(x,y)$ in terms of u, v. This is done by algebraic substitutions and eliminations. Let $F(u,v)$ be its new value

(ii) The assignment of new limits. This is also an algebraic process and geometrical considerations.

(iii) To determine the new element of integration. From Differential Calculus, we know that $dx\, dy = J\, du\, dv$, where J is the Jacobian of transformation and is defined as

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Thus we have

$$\iint f(x,y)\, dx\, dy = \iint F(u,v)\, J\, du\, dv.$$

Now, for the third part, we may write

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} \{x(u,v,w), y(u,v,w), z(u,v\,w)\}$$

$$\text{where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

in the Jacobian of transformation from the coordinates (x,y,z) to (u,v,w) and V′ in the region of integration in the new coordinates system.

In case of implicit function *i.e.*, if we are given the relations between x,y,z and u,v,w in the form

$$f_1(x,y,z,u,v,w) = 0$$

$$f_2(x,y,z,u,v,w) = 0$$

$$f_3(x,y,z,u,v,w) = 0$$

then the Jacobian J is given by

$$J = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} \neq 0$$

Example 10. *Evaluate the double integral*

$$\int_0^a \int_0^{\sqrt{(a^2-x^2)}} (x^2+y^2)\,dx\,dy,$$

by changing to polar coordinates.

Solution. Laws of transformation from cartesian to polar is

$$x = r\cos\theta,\ \ y = r\sin\theta.$$

$$\therefore \quad x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial v}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \quad dx\,dy = r\,d\theta\,dr.$$

Now the region of integration is bounded by $y = 0$ (x-axis), $y = \sqrt{(a^2-x^2)}$ (a circle with centre at oigin), $x = 0$ and $x = a$, This is clearly the positive quadrant of the circle of radius a. In order to cover this area by means of polar coordinates we find that the limits of r should be from 0 to a and those of θ from 0 to $\frac{\pi}{2}$. Hence we have

$$\int_0^a \int_0^{\sqrt{(a^2-x^2)}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^a r^2 . r\,d\theta\,dr$$

$$= \int_0^{\pi/2} {}_0^a\left[\frac{r^4}{4}\right] d\theta = \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}\,[\theta]_0^{\pi/2} = \frac{\pi a^4}{8}.$$

Example 11. *Evaluate the integral*

$$\int_0^1 \int_0^x \frac{x^3\,dx\,dy}{\sqrt{(x^2+y^2)}}$$

by changing to polar coordinates. ***(Purvanchal 90; GKP 2013)***

Solution. Region of integration is bounded by the lines

$y = 0,\ y = x,\ x = 0$

and $x = 1$.

Thus the region of integration is the region OAB (fig. 9.11) where B is the point (1, 1).

Equation of the line AB is

$$x = 1$$

or $$r\cos\theta = 1$$

$$\therefore \quad r = \sec\theta.$$

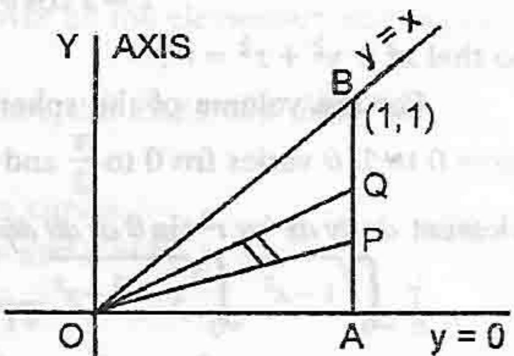

Fig. 8.11

Therefore in any triangular strip OPQ into which the area OAB is divided r varies from 0 to $\sec\theta$ and then for all such strips θ varies from 0 to $\frac{\pi}{4}$.

Thus the integral when transformed to polars

$$= \int_0^{\pi/4}\int_0^{\sec\theta} \frac{(r\cos\theta)^3\, r\,d\theta\,dr}{r}$$

$$= \int_0^{\pi/4} \left[\frac{r^4}{4}\right]_0^{\sec\theta} \cos^3\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta}{4}\,d\theta$$

$$= \frac{1}{4}\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}$$

$$= \frac{1}{4}\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)$$

$$= \frac{1}{4}\log(\sqrt{2}+1).$$

Example 12. Evaluate the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$ by

changing it to spherical polar coordinates.

Solution. Here, we can see that the region of integration is bounded by $z = 0$.

$$z = \sqrt{1 - x^2 - y^2} \quad (i.e., x^2 + y^2 + z^2 = 1); \quad y = 0$$

$$y = \sqrt{1 - x^2} \quad (i.e., x^2 + y^2 = 1), \quad x = 0, x = 1$$

which is the volume of the sphere.

$x^2 + y^2 + z^2 = 1$ in the positive octant.

Changing the Cartesian coordinates (x, y, z) to spherical polar coordinates by (r, θ, ϕ)

using $\quad x = r \sin\theta \cos\phi,$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta$$

so that $x^2 + y^2 + z^2 = r^2$.

For the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant r varies from 0 to 1, θ varies fro 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$. Now, replacing the volume element $dz\, dy\, dx$ by $r^2 \sin\theta\, dr\, d\theta\, d\phi$, we have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1 - x^2 - y^2 - z^2}} dy$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1 - r^2}} dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1 - (1 - r^2)}{\sqrt{1 - r^2}} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left(\frac{1}{\sqrt{1 - r^2}} - \sqrt{1 - r^2} \right) \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\sin r - \left(\frac{\partial \sqrt{1 - r^2}}{2} + \frac{1}{2} \sin^{-1} r \right) \right]_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\pi}{2} - \frac{\pi}{\phi} \right) \sin\theta\, d\theta\, d\phi$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \left[-\cos\theta \right]_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4} \int_0^{\pi/2} d\phi$$

$$= \frac{\pi}{2} \left[\phi \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$$

§ 8·8. Area by double integral.

Element of area in Cartesian Coordinates is $dx\, dy$ and $r\, d\theta\, dr$ in polar coordinates. Hence the area is obtained by evaluating the double integral $\int\int dx\, dy$ in cartesian coordinates or $\int\int r\, d\theta\, dr$ in case of polar coordinates, integration extending over the area under consideration. Thus the area bounded by the curves $y = f_1(x)$, $y = f_2(x)$, $x = a$ and $x = b$ is given by the double integral

$$= \int_a^b \int_{f_1(x)}^{f_2(x)} dx\, dy$$

§ 8·9. Volume under a Surface.

Let A be the region in the xy-plane and $z = f(x, y)$ be a surface. We have to find the volume between this surface and the region A.

In order to get this volume let us consider a small rectangle of area $dx\, dy$ in the region A. Now construct a vertical prism with $dx\, dy$ as base bounded at the top by the given surface. The volume of this prism is therefore $z\, dx\, dy$. Now the required volume is composed of similar prisms constructed over all the elementary rectangles in the given region

Hence the required volume $= \int\int z\, dx\, dy$ where the limits of x and y are yet to be assigned.

If the region of integration is bounded by the curves

$$y = f_1(x),\ y = f_2(x),\ x = a \text{ and } x = b,$$

then the required volume

$$= \int_a^b \int_{f_1(x)}^{f_2(x)} z\, dx\, dy$$

Now substituting the value of z in terms of x and y from the equation of the surface. We can evaluate the integral to get the volume.

Note. If we consider the area $dy\, dz$ on yz plane and construct prism as above by drawing lines parallel to x axis the required volume $= \int\int x\, dy\, dz$ under due limits of integration.

Similarly by considering the area $dz\, dx$ on xz plane and erecting the prism on this base by drawing lines parallel to y-axis the required volume $= \int\int y\, dx\, dz$ under due limits of integration.

Example 12. *Evaluate the triple integral* $\int_0^1 \int_0^{\sqrt{1-x^2-y^2}} \dfrac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ *by changing it to spherical polar coordinates.*

Solution. Here, we can see that the region of integration is bounded by $z = 0$ $z = \sqrt{1-x^2-y^2}$ (*i.e.*, $x^2 + y^2 + z^2 = 1$), $y = 0$ $y = \sqrt{1-x^2}$ (*i.e.*, $x^2 + y^2 = 1$), $x = 0, x = 1$ which is the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant. Changing the carterian coordinates (x, y, z) to spherical polar coordinates by (r, θ, ϕ) using $x = r \sin\theta \cos\theta$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

For the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant r varies from 0 to 1, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$. Now, replacing the volume element $dz\, dy\, dx$ by $r^2 \sin\theta\, dr\, d\theta\, d\phi$, we have

$$\int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right) \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left[\sin^r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r\right)\right]_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left(\frac{\pi}{2} - \frac{\pi}{\phi}\right) \sin\theta\, d\theta\, d\phi$$

$$= \frac{\pi}{2}\int_0^{\pi/2} [-\cos\theta]_0^{\pi/2}\, d\phi = \frac{\pi}{4}\int_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4}[\phi]_0^{\pi/2} = \frac{\pi^2}{8}$$

Example 13. *Find by double integration the area inside the cardiod $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. $r = a$ is a circle with centre as pole and radius a where as

$$r = a(1 + \cos\theta)$$

is the cardiod *ABOCA* thus the required area is as shows by the shaded region in fig 9.12.

In order to evaluate this area let us integrate $r\, d\theta\, dr$ on the shaded area. This area can be divided into triangular strips. Consider one such strip *OPQ*. In this strip θ remains constant and r varies from $r = a$ (on the circle) to $r = a(1 + \cos\theta)$ on the cardiod. For all the strips into which the area can be divided θ varies from $-\pi/2$ to $\pi/2$.

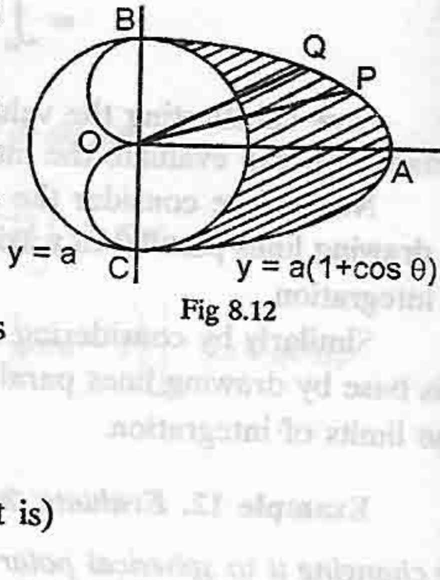

Fig 8.12

$$\text{Thus area} = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\, d\theta\, dr$$

(because area above the initial line is same as below it is)

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2} \{(1+\cos\theta)^2 - 1\}\, d\theta$$

$$= a^2 \left[\int_0^{\pi/2} \left(\cos^2\theta + 2\cos\theta\right) d\theta\right]$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta + 2a^2 \left(\sin\theta\right)_0^{\pi/2}$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta + 2a^2$$

$$= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + 2a^2$$

$$= \frac{a^2}{2} \left[\frac{\pi}{2}\right] + 2a^2 = a^2 \left[\frac{\pi}{4} + 2\right].$$

Example 14. *Find the mass of the plate in the form of the cardiod $r = a(1 + \cos\theta)$ whose density varies as square of radius vector.*

Solution. Element of area $= r\, d\theta\, dr$

density $\rho \propto r^2$ or $\rho = kr^2$,

where k is a constant.

$\therefore$ Element of mass $= kr^2 \cdot r\, d\theta\, dr$

$$= kr^3\, d\theta\, dr.$$

Again the curve is symmetrical about the initial line, hence mass of the plate

$$= 2 \int_0^{\pi} \int_0^{a(1+\cos\theta)} kr^3\, d\theta\, dr$$

$$= 2k \int_0^{\pi} \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2ka^4}{4} \int_0^{\pi} (1 + \cos\theta)^4\, d\theta$$

$$= \frac{ka^4}{2} \int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta = 8ka^4 \int_0^{\pi} \cos^8\frac{\theta}{2}\, d\theta$$

$$= 16ka^4 \int_0^{\pi/2} (\cos t)^8\, dt \text{ by substitution } \frac{\theta}{2} = t$$

$$= 16ka^4 \frac{\Gamma(9/2)\,\Gamma(1/2)}{2\Gamma(5)}$$

$$= \frac{16ka^4 \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}\sqrt{\pi}}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{35ka^4\pi}{16}$$

Example 15. *Evaluate by double integral the volume of the region enclosed by the plane*

$$x = 0,\ y = 0,\ z = 0, \text{ and } x + y + z = a$$

Solution. Here a vertical column is bounded by the planes $z = 0$ and $z = a - x - y$. The latter plane cuts the xy-plane in the line $a - x - y = 0$. So the area A above which the volume stands is the region in xy-plany bounded by the lines

$$y = 0, y = a - x, x = 0, x = a$$

Hence the volume $= \int_0^a \int_0^{a-x} z\,dx\,dy$

$$= \int_0^a \int_0^{a-x} (a - x - y)\,dx\,dy$$

$$= \int_0^a \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

$$= \int_0^a \frac{(a-x)}{2}(2a - 2x - a + x)\,dx$$

$$= \int_0^a \frac{(a-x)^2}{2}\,dx = \frac{1}{2}\left[-\frac{(a-x)^3}{3}\right]_0^a = \frac{a^3}{6}$$

EXERCISE 8·2

Change the order of integration in the following integrals.

1. $\int_0^4 \int_x^{2\sqrt{x}} f(x, y)\,dx\,dy$

2. $\int_0^a \int_x^{a^2/x} V\,dx\,dy$

   (GKP 2014, Sid. 2017)

3. $\int_0^{a/2} \int_{x^2/a}^{x - x^2/a} V\,dx\,dy$

   (Gorakhpur 86, 95, 2003)

4. Give a sketch of the region of integration and change the order of integration $\int_0^a \int_0^{x^2} V\,dx\,dy$

5. Show that $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx\,dy = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{(ay)}} dy\,dx$

6. Change the order of integration in $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} V\,dx\,dy$

7. Change the order of integration in $\int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} V\,dx\,dy$

   (Purvanchal 2003; Gorakhpur 87, 2017)

8. Change the order of integration $\int_0^a \int_{mx}^{lx} V\,dx\,dv$

9. Evaluate $\int_0^\infty \int_0^x xe^{-x^2/y}\,dx\,dy$ by changing the order of integration

   (Gorakhpur 92, 94, 99, 2006, 2007; Purvanchal 94)

10. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\, dx\, dy$ *(Purvanchal 89, GKP 2006, 09)*

[Hint : Change the order of integration]

Evaluate the following integrals by changing to polar coordinates.

11. $\int_0^a \int_0^x \frac{x\, dx\, dy}{x^2 + y^2}$

12. $\int_0^1 \int_0^x \frac{x^2\, dx\, dy}{\sqrt{(x^2 + y^2)}}$

13. Transform to polar coordinates and integrate

$$\iint \sqrt{\left[\frac{(1 - x^2 - y^2)}{1 + x^2 + y^2}\right]}\, dx\, dy$$

the integral being extended over all positive values of x and y subject to $x^2 + y^2 \leq 1$.

14. Find by double integration the area of a circle of radius a.

15. Find by double integration the area of one loop of the curve $r^2 = a^2 \cos 2\theta$.

16. Find the mass of a circular plate of diameter a, whose density at any point is k times the distance from a fixed point on the circumference.

17. Find the mass of area between $y^2 = x$ and $x^2 = y$. If $\rho = k\,(x^2 + y^2)$ *(Purvanchal 89)*

$\left[\text{Hint : Mass } \int_0^1 \int_0^{\sqrt{x}} k\,(x^2 + y^2)\, dx\, dy\right]$

18. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

19. Find the volume in the positive octant of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

20. Find the volume of the cylinder $x^2 + y^2 - ax = 0$ bounded by the planes $z = 0$ and $z = x$.

21. Transform the integral $\int_0^a \int_0^{\sqrt{a^2 - x^2}} y^2 \sqrt{x^2 + y^2}\, dx\, dy$ by changing to polar coordinated and hence evaluate it. *(Gorakhpur 2008)*

22. Evaluate $\int_0^2 \int_0^{\sqrt{2x - x^2}} \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$. *(Gorakhpur 2002)*

23. (a) Evaluate by changing to polar form

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

(Gorakhpur 2005)

23. (b) Evaluate by changing the order of integration :

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

(GKP 2016)

24. Prove that the area in the positive quadrant bounded by the curves $y^2 = 4ax$, $y^2 = 4bx, xy = c^2$ and $xy = d^2$ is $\frac{1}{3}(d^2 - c^2)\log\frac{b}{a}$.

25. Evaluate $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx$ by, changing it to polar coordinates.

§ 8·10. Triple Integrals

Let the function $f(x, y, z)$ of the point $P(x, y, z)$ be continuous for all points within a finite region V and on its boundary. Divide the region V into n parts, let $\delta V_1, \delta V_2 \ldots\ldots \delta V_n$ be their volumes. Take a point in each part and form the sum

$$S_n = f(x_1, y_1, z_1)\,\delta V_1 + f(x_2, y_2, z_2)\,\delta V_2 + \ldots. + f(x_n, y_n, \{z_n)\,\delta V_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r \qquad \ldots(1)$$

Then the limit to which the sum (1) tends when n tends to infinity and the dimensions of each subdivision tends to zero, is called the triple integral of the function $f(x, y, z)$ over the region V. This is denoted by

$$\iiint_V f(x, y, z)\,dV$$

or

$$\iiint_V f(x, y, z)\,dx\,dy\,dz$$

Evaluation of Triple integrals :

(a) If the region V be specified by the inequalities,

$$a \le x \le b, \qquad c \le y \le d, \qquad e \le z \le f$$

then the triple integral

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x, y, z)\,dx\,dy\,dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\,dz$$

Here the order of integration in immaterial and the integration with respect to any of x, y and z can be performed first.

(b) If the limits of z are given as functions of x and y, the limits of y as functions of x while x takes the constant values say from $x = a$ to $x = b$ then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz$$

The integration with respect to z is performed first regarding x and y as constants, then the integration w.r. to y is performed regarding x as a constant and in the last we perform the integration w.r. to x.

Example : Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz.$

Solution. Here $x - z$ to $x+z$ are the limits of integration of y, 0 to z are there of x and -1 to 1 are those of z. The given triple integral is

$$= \int_{-1}^{1}\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)-x(x-3)\right.$$

$$\left.-\frac{(x-z)^2}{2}-z(x-z)\right]dx\,dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}(4xz+2z^2)\,dx\right]dz$$

$$= \int_{-1}^{1}[2zx^2+2z^2x]_0^z\,dz$$

$$= \int_{-1}^{1}(2z-z^2+2z^2\cdot z)\,dz = 4\int_{-1}^{1}z^3\,dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 1[1-1] = 0.$$

Example. Evaluate $\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,dx\,dy\,dz.$

Solution. We have

$$\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$$

$$= \int_{0}^{\log 2}\int_{0}^{x}\left[e^{x+y+z}\right]_{0}^{x+\log y}dx\,dy$$

integrating w.r. to z regarding x and y as constants.

$$= \int_{0}^{\log 2}\int_{0}^{x}\left[e^{x+y+z+\log y}-e^{x+y}\right]dx\,dy$$

$$= \int_{0}^{\log 2}\int_{0}^{x}[e^{2x}e^{y}e^{\log y}-e^{x}e^{y}]\,dx\,dy$$

$$= \int_{0}^{\log 2}\left[\int_{0}^{x}2e^{2x}y\,e^{y}\,dy-\int_{0}^{x}e^{x}e^{y}\,dy\right]dx$$

$$= \int_0^{\log 2} \left[e^{2x} \{y e^y\}_0^x - e^{2x} \int_0^x e^y \, dy - e^x \{e^y\}_0^x \right]_0^x dx$$

Integrating w.r. to y regarding x as a constant to integrate ye^y we have applied integration by parts :

$$= \int_0^{\log 2} \left[e^{2x} \cdot xe^x - e^{2x} (e^y)_0^x - e^x [e^x - 1] \right] dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{2x} (e^x - 1) - e^{2x} + e^x] \, dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{3x} + e^x] \, dx$$

$$= \int_0^{\log 2} xe^{3x} \, dx - \int_0^{\log 2} e^{3x} \, dx + \int_0^{\log 2} e^x \, dx$$

$$= \frac{1}{3} \left[xe^{3x} \right]_0^{\log 2} - \frac{1}{3} \int_0^{\log 2} e^{3x} \, dx - \int_0^{\log 2} e^{3x} \, dx + \int_0^{\log 2} e^x \, dx$$

$$= \frac{1}{3} (\log 2) \, e^{3 \, Log \, 2} - \frac{4}{3} \left[\frac{e^{3x}}{3} \right]_0^{\log 2} + \left[e^x \right]_0^{\log 2}$$

$$= \frac{1}{3} (\log 2) \, e^{\log 8} - \frac{4}{9} (e^{3 \log 2} - 1) + (e^{\log 2} - 1)$$

$$= \frac{8}{3} \log 2 - \frac{4}{9} (8 - 1) + (2 - 1) = \frac{8}{3} \log 2 - \frac{28}{9} + 1$$

$$= \frac{8}{3} \log 2 - \frac{19}{9}.$$

EXERCISE 8·3

1. Evaluate $\int_{z=-c}^{c} \int_{y=-b}^{b} \int_{x=-a}^{a} (x^2 + y^2 + z^2) \, dx \, dy \, dz$

2. Evaluate $\int_{-1}^{1} \int_0^3 \int_{x-z}^{x+z} (x + y + z) \, dx \, dy \, dz$

3. Evaluate $\iiint_V (x^2 + y^2 + z^2) \, dz \, dy \, dx$, where V is the region bounded between the xy-plane and the sphere $x^2 + y^2 + z^2 = 1$.

4. Evaluate $\iiint (x + y + z) \, dx \, dy \, dz$ over the tetra hedron bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

□□