1 Free-Oscillations

Q. 1 : A U-tube of uniform cross-section 'a' containes amercury column of lenght $2l$. In one arm of the U tube, the mercury lelvel in depresent and released. Find the period of oscillations of the merury column.

(2001)

Sol^n : When the mercury column in one arm of the tube in depressed by amount ψ the mercury level in the other arm is raised by amount 2ψ . The weight of the mercury column of length provide the return force.

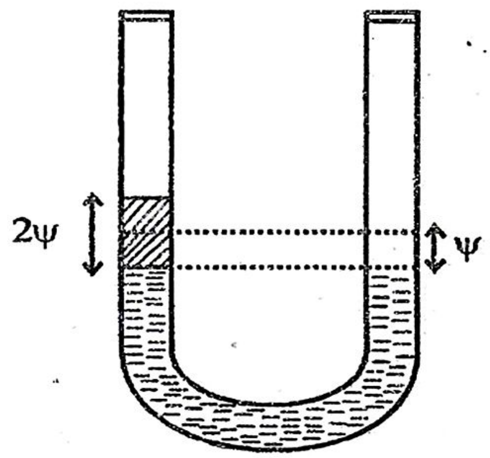
Return force = (mass)g $= a.2\psi.\rho.g$

जब किसी ट्यूब की एक भुजा मे पारे का स्तम्भ. ψ मात्रा में घटता है तो दूसरी भुजा में पारे का स्तर 2ψ मात्रा बढ़ता है। 2ψ लम्बाई के पारे के स्ताम्भ का भार रिटर्न बल के बराबर होता है।

$$\text{प्रत्यानन बल} = (\text{द्रव्यमान}).\ g = a.2\psi.\rho.g$$

Where ρ is the density of mercury entire mass of the mercury column is $a.2l.\rho$.

जहा ρ पारे का घनत्व है। जबकि पारे के स्तम्भ का सम्पूर्ण द्रव्यमान $a.2l.\rho$ है।



Hence the equation of motion is

$$a.2l.\rho\frac{d^2\psi}{dt^2} = -2a\rho g\psi$$

or, $$\frac{d^2\psi}{dt^2} = -\frac{g}{l}\psi$$

or,Frequency of oscillation = $\omega_0 = \sqrt{g/l}$

orPeriod of osillation = $T = 2\pi\sqrt{\frac{l}{g}}$

Q.2 : Two mass $m_1 = 1.0kg$ and $m_2 = 2.0kg$ kg are tied together with spring of force constant 250 N/M Calculate the frequency of oscillation ?

(2001)

Sol^n : Given $m_1 = 1.0kg$, $m_2 = 2.0kg$ and k = 250 N/M

Required mass of the system is

$$\mu = \frac{m_1.m_2}{m_1+m_2} = \frac{1\times 2}{1+2} = \frac{2}{3}kg$$

Therefore the frequency of oscillation =

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{250\times 3}{2}} = 19.36$$

or, $\omega = 19.36\ Hz$ Ans.

Q.3 : Define simple harmonic oscillation and establish differential equation for It ? **(1999,2013)**

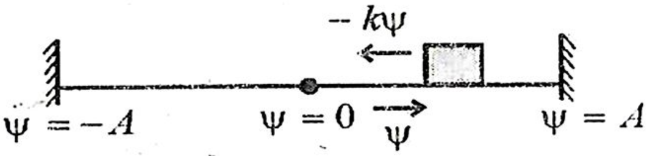
Sol^n: Simple harmonic motion is the simplest possible periodic motion of a single particle of one dimensional system. It is defined as the motion of an oscillating particle moving back about an equlibrium position under the influence of a (restoring) force which is directly proportional to the displacement but opposite to it in direction. The particle is called a simple harmonic oscillator.

सरल आवर्त गति किसी कण अथवा एक आयामी निकाय की सबसे सरलतम सम्भावित आवर्त गति होती है। यह इस प्रकार परिभाषित किया जाता है:– किसी दोलनी कण की गति जो कि एक प्रत्यानन बल के प्रभाव के कारण किसी निश्चित माध्य स्थिति के दोनों ओर इस प्रकार गति करे कि प्रत्यानयन बल विस्थापन के समानुपाती हो तथा बल की दिशा विस्थापन के विपरीत हो। कण को सरल आवर्त लोलक कहा जाता है।

Let the position of the oscillating particte at any instant from the mean position be ψ. According to the characteristics of the simple harmonic oscillations, the return force is proportional to displacement ψ and directed towards the equlibrium position.

Therefore return force $= k\,\psi$

Where k is constant called force constant.



माना किसी क्षण दोलनी कण की माध्य स्थिति से विस्थापन ψ है। सरल आवर्त गति के प्रगुण के अनुसार प्रत्यानयन बल विस्थापन ψ के समानुपाती होगा तथा इसकी दिशा माध्य स्थिति की तरफ होगी।

अतः प्रत्यानयन बल $= k\,\psi$

जहां k को बल नियतांक कहते हैं।

According to newton's law of motion, the equation of the motion of the particle of mass 'm' is

न्यूटन के गति नियम के अनुसार किसी द्रव्यमान के कण की गति का समीकरण

$$m\frac{d^2\psi}{dt^2} = -k\,\psi \quad \text{where } \frac{d^2\psi}{dt^2} \text{ is acceleration}$$

or, $$\frac{d^2\psi}{dt^2} = \frac{-k}{m}\,\psi \qquad \text{........(1)}$$

Let $$\frac{-k}{m} = \omega_0^2$$

so that $$\frac{d^2\psi}{dt^2} = -\omega_0^2\,\psi$$

or $$\frac{d^2\psi}{dt^2} + \omega_0^2\,\psi = 0 \qquad \text{....... (2)}$$

It is the differential equation of simple harmonic oscillation.

यह सरल आवर्त गति का अवकलन समीकरण है।

Q.4 : Derive an exprersion for the energy of a body executing simple harmonic motion ? **(2003)(SU-2016)**

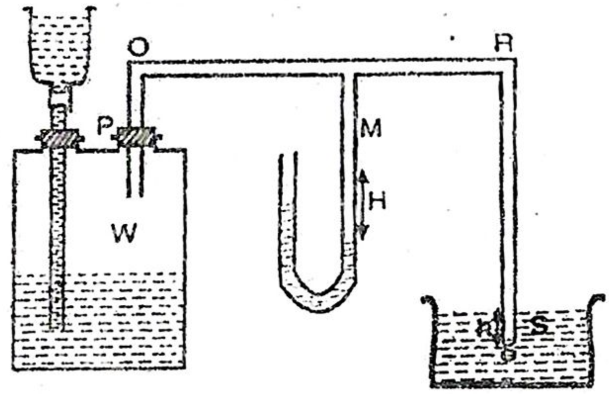
Or

tension of a liquid. it can also be used to study the variation of surface tension with temperature.

किसी द्रव में बुलबुले निर्माण कर के तथा अधिक्य दाब p (=2T/R) को आकलित करके जैगर ने द्रव के पृष्ठ तनाव की गणना किया। उन्होंने ताप के साथ पृष्ठ तनाव के बदलाव को भी अध्ययन किया।

APPARATUS :-

The apparatus, used for this purpose is shown in the following figures. It consist of woult's bottle W. In one mouth of the bottle, a dropping funnel, contaning water, is fitted and in the other a glass tube PQRS. To this tube a manometer M is connected which contains a liquid of low density, sothat the difference in the levels of the liquid in its



limbs may be large for a given pressure difference. The end S of the tube is drawn into a fine capillary and its top is cut perpendicular tothe axis of the tube and quite smooth so that there appears no roughness even when seen under a microscope. The opening S of the tube lies at a depth of 4 to 5 cm inside the liquid, of which the surface tension is to be determined.

इस उद्देश्य के लिए प्रयुक्त उपकरण चित्र (1) में चित्रित है। इसमे एक बुल्ट की बोतल W है। बोतल के एक मुँह में एक बूँद कीप ड्रापिंग फनेल जो कि पानी को रखता है, जोड़ा जाता है, जबकि दूसरे में एक ग्लास ट्यूब PQRS इस ट्यूब के साथ एक मैनोमीटर, जो कि एक कम घनत्व के द्रव को रखता है, जुड़ा रहता है। जिससे किसी दिये हुए दाबान्तर पर द्रव के लेयर मे अंतर ज्यादा हो सकता है। ट्यूब का सिरा S एक अच्छे केशनली में अंदर जाता है और इसका उपरी भाग ट्यूब के अक्ष के लम्बवत कट किया जाता है तथा यह इतना चिकना होता है कि सूक्ष्मदर्शी से देखने पर भी खुरदरापन दिखाई नहीं देती है। ट्युब की सिरा S द्रव 4 से 5 सेमी गहराई में होती है। जिस पर पृष्ठ तनाव की गणना की जाती है।

THEORY :-

Let P be the atmospheric pressure. Now consider the equilbrium of the bubble just before breaking.

The pressure inside the bubble = $P + H\rho g$

Where H is the maximum difference of the levels of the liquid in the manometer and ρ its density.

The presure outside the bubble = $P + hdg$

Where h is the depth of orifice S below the free surface of the experimental liquid and d its density.

Excess of pressure inside the bubble = $H\rho g - hdg$

But at the excess of pressure inside the bubble = $2T/r$

Therefore,

$$2T/r = (H\rho - hd)g$$

or

$$T = (H\rho - hd)rg/2$$

Thus by knowning the values of various quantities, we can calculate the value of T for the given liquid.

बल के विपरीत कण को उस स्थिति तक विस्थापित करने में लगती है।
यह दिखाता है कि स्थितिज उर्जा E_p समय के साथ बदलती है।

$$E_p = \int_0^{\psi} k\,\psi\, d\psi$$

or, $$E_p = \frac{1}{2} k\,\psi^2$$

or, $$E_p = \frac{1}{2} ka^2 \sin^2(\omega t + \delta) \qquad \text{.......... (2)}$$

This shows that the potential energy varies with time t

When, $\sin^2(\omega t + \delta) = 1$ i.e. $\psi = \pm a$, E_p is maximum $\frac{1}{2}ka^2$

Total Energy :- The total energy E, of the particle is the sum of the kinetic energy and potential energy.

किसी कण की कुल उर्जा E स्थितिज उर्जा तथा गतिज उर्जा के बराबर होती है।

$$E = E_k + E_p$$

$$E = \frac{1}{2}ka^2 \cos^2(\omega t + \delta) + \frac{1}{2}ka^2 \sin^2(\omega t + \delta)$$

or $$E = \frac{1}{2}ka^2 \qquad \text{...... (3)}$$

We thus, see that the total energy is constant and has the value $\frac{1}{2}ka^2$. It is also the maximum value of potential or kinetice energy.

Also $k = \omega^2 m$ and $\omega = \frac{2\pi}{T} = 2\pi n$, where 'n' is the frequency of oscillation.
Therefore

$$E = \frac{1}{2}ka^2 = \frac{m\omega^2 a^2}{2} = \frac{2\pi^2 ma^2}{T^2} = 2\pi^2 mn^2 a^2$$

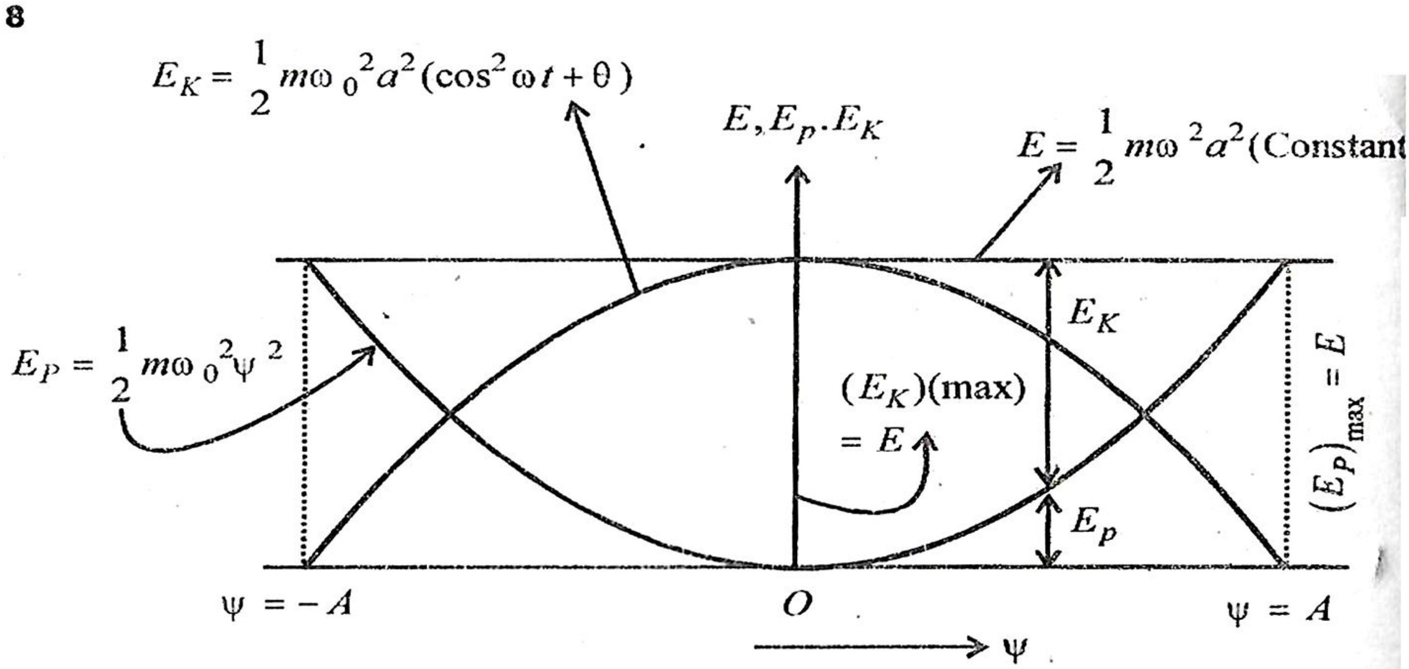
That is, the energy of the particle is directly propartional to the square of the amplitude and to the square of the frequency.

The variation of kinetic energy and potential energy with position is shown in the figure.

अतः हम देखते हैं कि कुल उर्जा नियत है तथा इसका मान $\frac{1}{2}ka^2$ है जो कि स्थितिज उर्जा तथा गतिज उर्जा का अलग–अलग अधिकतम मान है।

अतः कण की उर्जा अयाम के वर्ग के समानुपाती तथा आवृति के वर्ग के भी समानुपाती होती है। गतिज उर्जा तथा स्थितिज उर्जा की विस्थापन के साथ का परिवर्तन चित्र में प्रदर्शित है।

It is clear from the figure that at equilibrium position ($\psi = 0$), kinetic energy is max (E) and potential energy in minimum (= zero) while at the position of max. displacement ($\psi \pm a$), kinetic energy is min. (= zero) and potential energy



is max. (= E). At points in between the equilibrium and max. displacement position the kinetic and potential energies are such that their sum is equal to total energy.

चित्र से स्पष्ट है कि साम्यावस्था में $\psi = 0$ गतिज उर्जा अधिकतम है तथा स्थितिज उर्जा निम्नतम है जबकि अधिकतम स्थिति पर गतिज उर्जा निम्नतम है तथा स्थितिज उर्जा अधिकतम है। उन बिन्दुओ पर जो कि साम्यावस्था तथा अधिकतम अवस्था के बीच है गतिज उर्जा तथा स्थितिज उर्जा इस तरह है कि योग कुल उर्जा के बंराबर है।

Q.5 : Show that for a harmonic oscillation time average of kinetic energy and potential energy over one period are equal. (GKP-2000,2016)

Sol^n. The time average kinetic energy in one complete period is

$$\overline{E}_k = \frac{1}{T}\int_0^T E_K \, dt$$

or $$\overline{E}_k = \frac{1}{T}\int_0^T \frac{1}{2} m\dot{\psi}^2 \, dt$$

or, $$\overline{E}_k = \frac{1}{T}\int_0^T \frac{1}{2} m\,\omega^2 a^2 \sin^2(\omega t + \delta)\, dt$$

or, $$\overline{E}_k = \frac{1}{4} m\omega^2 a^2$$

or, $$\overline{E}_k = \frac{1}{4} k\, a^2 \qquad \text{....... (1)}$$

The time average potential energy in one complete period is

$$\overline{E}_p = \frac{1}{T}\int_0^T E_P \, dt \qquad \text{or,} \qquad \overline{E}_p = \frac{1}{T}\int_0^T \frac{1}{2} k\,\psi^2 \, dt$$

or, $$\overline{E}_p = \frac{1}{T}\int_0^T \frac{1}{2} k\,\omega^2 a^2 \cos^2(\omega t + \delta)\, dt$$

or $$\overline{E}_p = \frac{1}{4} m\,\omega^2 a^2$$

or $$\overline{E}_p = \frac{1}{4} k\,a^2 \qquad \text{...... (2)}$$

Q 6 : Write short notes on the Lissajous figures? (2001,2009) OR

(a) What are Lissajous figures? Obtain the expression for the resultant of two mutually perpendicular simple harmonic oscillation having the same frequency but differing in amplitude and phase, what will be the type of motion for phase difference. (2013)(SU-2016)

(b) What are Lissajous figures? Obtain the expression for the resultant of two mutually perpendicular simple harmonic oscillation of same amplitude where frequencies in the ratio 1:1 Discuss the Lissajous figure for phase difference $\phi = 0$ and $\phi = \pi/2$. (2002,2011,2015)

OR

A point moves in a plane performing two orthogonal simple harmonic motion given by $x = a_1 \cos\omega t$ and $y = a_2 \cos(\omega t + \phi)$. Show that in general the point moves in an elliptical path. Write the condition for the path to be a circle. (GKP-2008,2016)

Sol^n : (a) **LISSAJOUS FIGURES** :- When a particle has superimposed upon it by two mutually perpendicular simple harmonic motion simultaneously, the resultant path of the particle is known as a Lissagous Figures' . The form of the figure depends upon

(i) The ratio of the frequencies of two vibration

(ii) The amplitudes of the component vibration and

(iii) The relative phase of the two component motion.

लीसॉजूस फीगर:--जब एक कण इस प्रकार अध्यारोपित किया जाय कि वह दो लम्बवत दिशाओ में सरल आवर्त गति करे तो कण के परिणामी पथ को लीसाजुस फीगर कहा जाता है।

'**लीसाजूस फीगर**' का प्रकृति निर्भर करता है--

1. दोनों कम्पन्न या दोलन की आवृत्तियों के अनुपात के
2. घटक दोलन के आयाम पर तथा
3. दोनो घटक गति के आपेक्षिक कला पर

Consider two S.H.M. of equal periods superimpsed on a particle is mutually perpendcular directions one along the x-axis and the other along y - axis

माना एक कण दो लम्बवत दिशाओं मे सरल आवर्त गति कर रहा है जिनका आवर्तकाल समान है। माना एक दिशा x- अक्ष तथा दूसरी दिशा y अक्ष है।

तब $$x = a\cos\omega t \qquad \text{...... (1)}$$

$$y = b\cos(\omega t + \phi) \qquad \text{...... (2)}$$

Where ϕ is the phase difference between them and 'a' and 'b' are their amplitude. The equation of the resultant path, traced out by the particle can be obtained by eliminating 't' from eqn. (1) and (2).

जहां ϕ दोनो आवर्त गति की कलाओं का अन्तर है। तथा 'a' और 'b' उनके आयाम

है। कण के द्वारा परिणामी पथ का समीकरण समीकरण 1 तथा समीकरण 2 से 't' को विलुप्त करने पर प्राप्त होगा।

Equation may be written as -

$$\cos\omega t = x / a$$

$$\sin\omega t = \sqrt{1-\frac{x^2}{a^2}}$$

or, $$\sin\omega t = \frac{\sqrt{a^2 - x^2}}{a}$$

Now from equation (2)

$$\cos(\omega t + \phi) = \frac{y}{b}$$

or, $$\cos\omega t \cos\phi - \sin\omega t \sin\phi = \frac{y}{b}$$

or, $$\cos\phi - \frac{\sqrt{a^2 - x^2}}{a}\sin\phi = \frac{y}{b}$$

or $$\left(\frac{x}{a}\cos\phi - \frac{y}{b}\right) = \frac{\sqrt{a^2 - x^2}}{a}\sin\phi$$

squaring on both sides, we get

$$\frac{x^2}{a^2}\cos^2\phi + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \left(1-\frac{x^2}{a^2}\right)\sin^2\phi$$

or, $$\frac{x^2}{a^2}\left(\cos^2\phi + \sin^2\phi\right) + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi \qquad \text{...... (3)}$$

It is equation of the resultant curve. The shape of the curve depend on the phase difference ϕ.

Special Case :

Case (i) : Let $\phi = 0$ then $\sin\phi = 0$ and $\cos\phi = 1$, Putting this value in equation (3), we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$ or, $\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$. This represent a pair of straight line and the equation of each line is $y = \frac{b}{a}x$

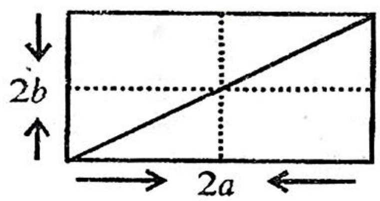


Figure - 1(a)

Case (ii) : $\phi = \pi / 2$ then $\sin\phi = 1$ and $\cos\phi = 0$

Putting these value in equation (3), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Which is an ellipse whose axis concide with the coordinate axis. Figure 1(b)

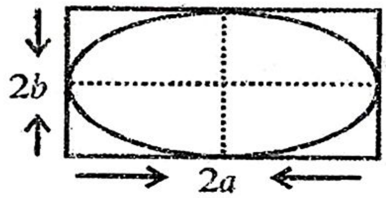


Figure - 1(b)

If $a=b$, the path of the particle becomes a circle $x^2+y^2=a^2$. Figure 1(c).

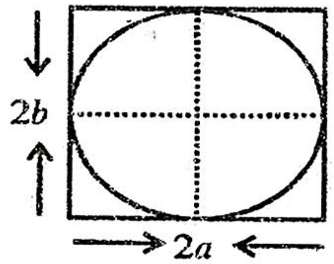


Figure - 1(c)

Case (iii) : Let $\phi=3\pi/4$, then $\sin\phi=\frac{1}{\sqrt{2}}$ and $\cos\phi=-\frac{1}{\sqrt{2}}$

putting these valves in equation 3, we gets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\sqrt{2}\,\frac{xy}{ab}=\frac{1}{2}$$

this again represents an oblique ellipse, fig. 1(d)

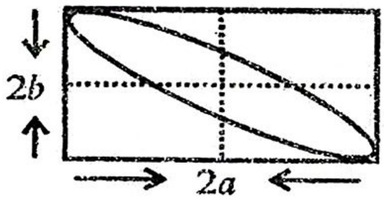


Figure - 1(d)

(b) Sol^n : See question no. (6), (putting a = b)

Q. 7 : A solid cylinder floating in a liquid in equlibirium is slightly pressed and released show that the vertical up-down of the cylinder is simple hormonic motion. (2003)

Sol^n : Consider a cylinder of material of denisity ρ, cross-sectional area 'A' and length 'l', which floats upright with a length immersed in a liquid of density σ.

माना एक सिलिंडर जिसका घनत्त्व ρ परिच्छेद क्षेत्रफल 'A' तथा लम्बाई 'l' है जो कि एक तरल पदार्थ जिसका घनत्व σ है, में अपराइट तैरता है।

If the cylinder is slightly depresed or raised from its normal position, there is restoring force equal to the increase or decrease in the weight of liquid displaced. The mass 'm' of the cylender is $m=lA\rho$.

जब सिलिंडर अपनी साम्यावस्था से अंशतः घटाया या बढ़ाया जाता है तब एक प्रत्यानयन बल जो कि घटाये या बढ़ाये गये तरल पदार्थ के द्रव्यमान के बराबर है।

सिलिंडर का द्रव्यमान 'm' है $m=lA\rho$

Let the cylinder be at a distance 'x' from its normal floating level. The weight of the liquid displaced is $= (A\sigma x)g$

जब सिलिंडर अपनी सामान्य अवस्था से 'x' दूरी पर स्थित हो तो विस्थापित द्रव का भार $(A\sigma x)g$

The equation of motion is given by

$$m\frac{d^2x}{dt^2} = -(\sigma A\, x)g$$

$$\frac{d^2x}{dt^2} + \frac{\sigma A\, x\, g}{Al\rho} = 0$$

$$\frac{d^2x}{dt^2} + \left(\frac{\sigma g}{l\rho}\right)x = 0$$

or,
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Which is similar to general equation of simple hormonic motion. Thus the vertical up-down of the cylinder is simple hormonic motion.

अतः सिलिंडर का उर्ध्वाधर (गति जो कि उपर से नीचे व नीचे से उपर हो रहा है) एक सरल आवर्त गति है।

Q. 8 : The S.H.M. of a body is expressed by $x = 5\cos\left(\frac{100\pi t}{3}\right)$ c.m. Evaluate its amplitide and time period?

Sol^n : Given that the S.H.M. of a body is

$$x = 5\cos\left(\frac{100\pi t}{3}\right) \quad \text{....... (1)}$$

the general equation of S.H.M. is

$$x = a\cos\omega_0 t \quad \text{........ (2)}$$

from equation (1) and (2), we get amplitude a = 5 unit

and frequency $\omega_0 = \frac{100\pi}{3}$

$$\Rightarrow \quad 2\,\pi\, n = \frac{100\pi}{3}$$

$$\Rightarrow \quad n = \frac{50\pi}{3} \ \sec^{-1}$$

Thus time period $T = 1/n$

or. $T = \frac{3}{50}$ sec Ans.

Q.9 : Set up the diffrential equation for the change of an oscillatory L.C. circuit ? (2000)

OR

Show that the charging and discharging of a condensor through an inductance is simple harmonic. (2008)

Sol^n : When a charged capacitor is allowed to discharge through inductance.

The flow of charge or the current in the circuit is oscillatory. Let at any instant 't', the charge on the plate of the capacitor be 'Q' and current in the inductance be 'I', the equation of electromotive force from kirchoffs law, is given by

जब एक आवेशित संधारित्र को प्रेरकत्व के द्वारा अनावेशित किया जाता है तो परिपथ में आवेश या धारा का प्रवाह दोलनी होता है। माना किसी क्षण 't' पर संधारित्र पर आवेश 'Q' है तथा प्रेरकत्व में धारा 'I' है तो किरचॉफ के नियम से विद्युत वाहक बल का समीकरण होगा –

$$L\frac{dI}{dt}+\frac{Q}{C}=0$$

Since $$I=\frac{dQ}{dt}$$

We have $$L\frac{d^2Q}{dt^2}=-\frac{Q}{C}$$

or, $$\frac{d^2Q}{dt^2}=-\frac{1}{LC}Q=-\omega_0^2Q \qquad \text{...... (1)}$$

Where $$\omega_0^2=\frac{1}{LC}$$

The angular frequency of oscillation is

$$\omega_0=\frac{1}{\sqrt{LC}}$$

or, $$2\pi n_0=\frac{1}{\sqrt{LC}}$$

or, $$n_0=\frac{1}{2\pi}\frac{1}{\sqrt{LC}} \qquad \text{......(2)}$$

Equation (1) represent the differential equation for the change of an oscillatory LC circuit. It is similar to the equatin of motion of simple hormonic motion therefore the charging and discharging of a condenser through an inductance is simple harmonic.

समीकरण 1 किसी दोलनी LC परिपथ के आवेश का अवकलन समीकरण है। यह सरल आवर्त गति के समीकरण के समान हैं अतः आवेशित तथा प्रेरकत्व के द्वारा अनावेशित सधारित्र की गति एक सरल आवर्त है।

अवमंदित दोलन में आयाम का क्षय निम्न समीकरण द्वारा होता है –
किसी दोलक के अवमंदित दोलन की गति का समीकरण है।

Q.10 : Derive expression for kinetic, potential and total energies of a harmonic oscillation and show their dependence on displacements drawing a neat graph. (2005)
or
Show that the average values of kinetic and potential energies in one complete period are equal.
or
Show that the charging and discharging of a condensor through an inductance is simple harmonic ?
or
Establish the differential equation for the oscillation of a LC circuit. (2005)

Sol^n : For the expression of kintetic, potential and total energy see question no. (4) and for the expression of average values of kinetic and potential energy see

question no. (5)

Q.11 : What are Lissajous figures ? discuss with necessary theory. The superposition on two rectangular simple harmonic vibrations of same frequencies but different amplitudes. Discribe what happens if one of the vibrations changes its phase from 0^0 to 180^0 relative to other. (2004)

Sol^n. for the solution of this equation see question no. (6)

Case - 4: Let , $\phi = \pi$, then $\sin\phi = 0$, $\cos\phi = -1$

putting those value in equation (3), one get,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+2\frac{xy}{ab}=0$$

or ,
$$\left(\frac{x}{a}+\frac{y}{b}\right)^2=0$$

which again represents a pair of straight line and the equation of each being

$$y=-\frac{b}{a}x$$

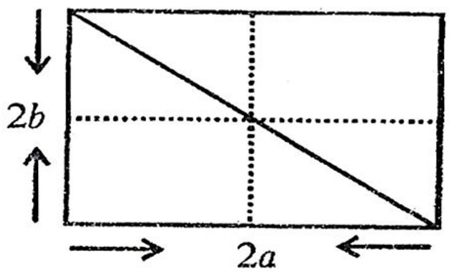


Figure - 1(e)

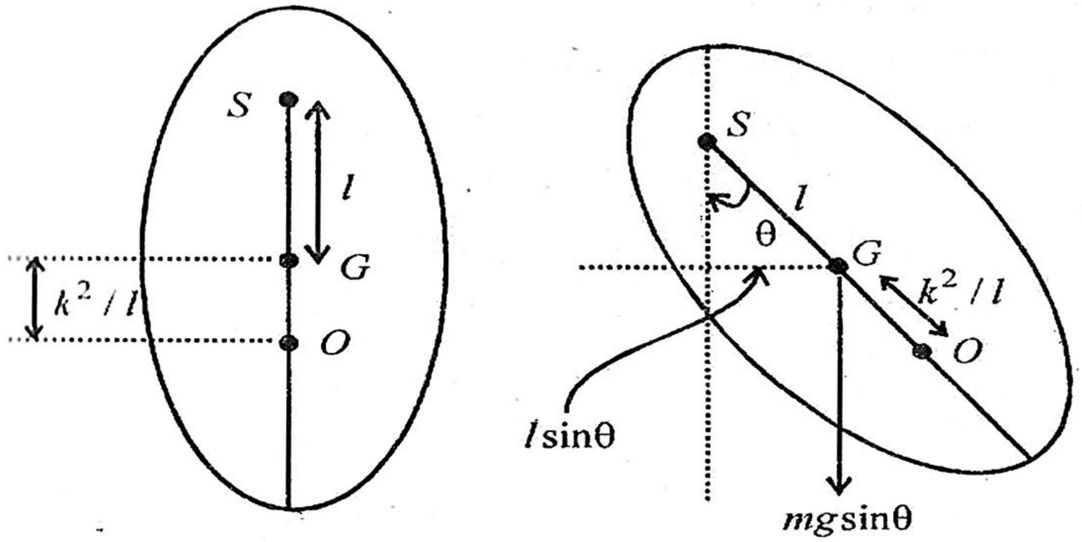
Q. 12 : Write short notes on the Compound pendulum ? (2011)

Sol^n. A rigid body capable of oscillating in a vertical plane about a horizontal axis is called a compound pendulum. When the pendulum is displaced from its equilibrium position through a small angle and released it executes angular simple harmonic motion. The vertical plane containing the motion of centre of gravity cuts the axis of suspension at a point called the centre of suspension. (C.S.)

संयुक्त दोलक :— एक दृढ़ पिण्ड, जो एक उर्ध्वाधर तल में किसी क्षैतिज अक्ष के परितः दोलन करता हो उसे संयुक्त दोलक कहते हैं। जब दोलक को इसकी साम्याएस्था से अत्यन्त छोटे कोण पर विस्थापित कर छोड़ दिया जाता है, तो यह एक कोणीय सरल आर्त गति प्रदर्शित करता है। उर्ध्वाधर तल में स्थिति गुरूत्व केन्द्र, गति करते समय, निलंबन अक्ष को एक बिन्दु पर काटता है, इस बिन्दु को निलंबन बिन्दु कहते हैं।

let the distance of C.S. from the C.G. be 'l'. When the pendulum is displaced by an angle θ , the moment of the restoring force about the C.S.is $mgl\sin\theta$ .If I is the moment of inertia of the pendulum about the C.S. then the equation of motion of pendulum is

माना निलंबन बिन्दु और गुरूत्व केन्द्र के बीच की दूरी 'l'है। जब दोलक को एक छोटे कोण θ पर विस्थापित किया जाता है तो निलंबन बिन्दु के परितः प्रत्यानयन बल का आघूर्ण $mgl\sin\theta$ होता है। यदि I निलंबन बिन्दु के परितः दोलक को जड़त्व आघूर्ण हो तो दोलक के गति का समीकरण



$$I.\frac{d^2\theta}{dt^2} = -mgl\sin\theta \qquad ....(1)$$

for small angular displacement,

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta \qquad ......\ (2)$$

If I_g is the moment of inertia of the pendulum about an axis passing through C.G. and parallel to the axis of suspension then,

$$I = I_g + ml^2$$

$$\text{or } I = mk^2 + ml^2$$

Wher K is radius of gyration about an axis passing through C.G. Thus

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{m(k^2+l^2)}\theta = \frac{g}{(l+k^2/l\ )}\theta$$

Evidently the angular frequency of oscillation is

$$\omega = \sqrt{\frac{g}{l+\left(k^2/l\right)}}$$

or time period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l+\left(k^2/l\right)}{g}}$$

The length is $\left(l+\frac{k^2}{l}\right)$ called the length of equivalent simple pendulum. A point lying on th line SG at a distance $\frac{k^2}{l}$ from the C.G. is called the centre of oscillation.

यदि I_g गुरूत्व केन्द्र से गुजरने वाली अक्ष के परितः जो कि निलंबन अक्ष के समानान्तर हो, के परितः दोलक का जड़त्व आघूर्ण हो।

Q. 13 : A particle of mass 'm' moves in one dimensional potential field in which the potential energy of the particle at point 'x' is given by $$U(x)=\frac{a}{x^2}-\frac{b}{x}$$

where 'a' and 'b' are constants. Find the force constant of the system.

Sol^n :
$$U(x)=\frac{a}{x^2}-\frac{b}{x} \qquad \text{.......... (1)}$$

At the equilbrium position,
$$\frac{dU}{dx}=0$$

or,
$$-\frac{2a}{x^3}+\frac{b}{x^2}=0$$

or,
$$x=\frac{2a}{b} \qquad \text{....... (2)}$$

Force constant k of the force field is
$$k=\left(\frac{d^2U}{dx^2}\right)_{equilibrium} \Rightarrow k=\left[\frac{6a}{x^4}-\frac{2b}{x^3}\right]_{x=\frac{2a}{b}}$$

or,
$$k=\frac{b^4}{8a^3}$$
Ans.

Ques. 14. The displacement of a simple harmonic oscillation in given by
$$x=a\sin(\omega t+\phi)$$
If the oscillation is started at time $t=0$, from the position 'x_0' with velocity $x=V_0$, show that
$$\tan\phi=\frac{x_0\,\omega}{V_0} \quad \text{and} \quad a=\left(V_0^2+\omega^2 x_0^2\right)^{1/2}/\omega$$

Sol^n : the equation of simple harmonic oscillation is
$$x=a\sin(\omega t+\phi) \qquad \text{........ (1)}$$
at $t=0$, we have from equation (1)
$$x_0=a\sin\phi \qquad \text{........ (2)}$$
or,
$$\omega x_0=a\omega\sin\phi \qquad \text{....... (3)}$$
Now Differentiating eqn (1) w.r.t 't', we get
$$\frac{dx}{dt}=a\omega\cos(\omega t+\phi)$$
at $t=0$, we have
$$\frac{dx}{dt}=x=V_0=a\omega\cos\phi$$
or,
$$V_0=a\omega\cos\phi \qquad \text{....... (4)}$$

Now from equation (2) and (4), we get

$$\tan\phi = \frac{x_0\,\omega}{V_0}$$

Now squaring and adding equation (3) and (4) we get

$$\omega^2 x^2 + V_0^2 = \omega^2 a^2 \sin^2\phi + a^2\omega^2\cos^2\phi$$

or, $$\omega^2 x^2 + V_0^2 = a^2\omega^2(\sin^2\phi + \cos^2\phi)$$

or, $$a^2\omega^2 = x_0^2\omega^2 + V_0^2$$

or, $$a\,\omega = (x_0^2\omega^2 + V_0^2)^{1/2} \qquad a = \frac{(x_0^2\omega^2 + V_0^2)^{1/2}}{\omega}$$

Q.15 : If two simple harmonic motion **(2006)**

$$y_1 = a_1\sin(\omega_1 t + \phi_1) \quad \text{and} \quad y_2 = a_2\sin(\omega_2 t + \phi_2)$$

superpose under the following conditions, explain what will be the resulting motion.

(a) $a_1 \neq a_2,\ \phi_1 \neq \phi_2,\ \omega_1 \neq \omega_2$ **and the superposition in linear.**

(b) $a_1 \approx a_2,\ \phi_1 \approx \phi_2,\ \omega_1 \approx \omega_2$ **and the superposition is linear**

(c) $a_1 \neq a_2,\ \phi_1 \neq \phi_2,\ \omega_1 = \omega_2$ **and the superposition is perpendicular what will happen if in condition (c),** ω_1 **is slightly different from** ω_2**?**

or

Two simple harmonic motion of same frequency ω **but having displacement in two perpendicular direction act simultaneously on a particle**

$$x = a\sin(\omega t + \alpha_1) \qquad y = b\sin(\omega t + \alpha_2)$$

Find the resultant motion for various values of the phase difference $\delta = \alpha_1 - \alpha_2$. **(2007)**

Sol :- The two simple harmonic motion is represent as

$$y_1 = a_1\sin(\omega_1 t + \phi_1) \quad \text{....... (1)}$$

$$y_2 = a_2\sin(\omega_2 t + \phi_2) \quad \text{........ (2)}$$

(a) If $a_1 \neq a_2, \phi_1 \neq \phi_2$ and $\omega_1 = \omega_2 = \omega$ then the resultant motion is given by

$$y = y_1 + y_2$$

or $$y = a_1\sin(\omega t + \phi_1) + a_2\sin(\omega t + \phi_2)$$

or $$y = a_1\{\sin\omega t\cos\phi + \cos\omega t\sin\phi_1\} + a_2\{\sin\omega t\cos\phi_2 + \cos\omega t\sin\phi_2\}$$

or $$y = \sin\omega t(a_1\cos\phi_1 + a_2\cos\phi_2) + \cos\omega t(a_2\sin\phi_2 + a_2\sin\phi_2)$$

Let - $$a_1\cos\phi_1 + a_2\cos\phi_2 = r\cos\theta \quad \text{...........(3)}$$

and $$a_1\sin\phi_1 + a_2\sin\phi_2 = r\sin\theta \quad \text{............(4)}$$

where 'r' and 'θ' are new constants.

putting these valve in the above equation, then we get -

$$y = \sin\omega t(r\cos\theta) + \cos\omega t(r\sin\theta)$$

or $$y = r\sin(\omega t + \theta) \qquad \text{............(5)}$$

This equation is of the same type as equation (1) and (2) which represent simple harmonic motion. Hence the resultant motion of the particle is also simple harmonic with same frequency as the individual motions, but with same frequency as the individual motions, but with a different amplitude 'r' and a different initial phase 'θ' .

Squaring and adding equation (3) and (4), then

we get - $$a_1^2 + a_2^2 + 2a_1a_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) = r^2$$

or $$a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2) = r^2$$

or $$r = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)}$$

Now Dividing equation (4) by equation (3), we get-

$$\tan\theta = \frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2} \qquad \text{............(6)}$$

(b) If $a_1 \simeq a_2$, $\phi_1 \simeq \phi_2$, $\omega_1 \simeq \omega_2$ then the two simple harmonic motion will be represented as

$$y_1 = a\sin\omega_1 t \qquad \text{..............(7)}$$

$$y_2 = a\sin\omega_2 t \qquad \text{..............(8)}$$

where $a_1 + a_2 = a$

The resultant motion is given by $y = y_1 + y_2$

or, $$y = a\sin\omega_1 t + a\sin\omega_1 t$$

or $$y = 2a\sin\left(\frac{\omega_1 + \omega_1}{2}\right)t\cos\left(\frac{\omega_1 - \omega_1}{2}\right)t$$

or $$y = \left\{2a\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t\right\}\sin\left(\frac{\omega_1 + \omega_2}{2}\right)t$$

or $$y = A\sin\left(\frac{\omega_1 + \omega_2}{2}\right)t \qquad \text{............(9)}$$

where $A = 2a\cos\left(\frac{\omega_1 - \omega_1}{2}\right)t$ is the amplitude of resultant motion,

Thus the resultant amplitude of motion varies periodically between '2a' and zero under the inflerence of cosine term-

(c) If $a_1 \neq a_2$, $\phi_1 \neq \phi_2$, $\omega_1 = \omega_2$ then the above equation can written as

$$y_1 = a\sin(\omega t + \delta) \qquad \text{................(10)}$$

and $$y_2 = b\sin(\omega t + \delta) \qquad \text{................(11)}$$

where a_1 = a, a_2 = b and $\omega_1 = \omega_2 = \omega$

The equation of the resultant path, traced out by the particle can be obtained by eliminating t from equation (10) and (11).

Now equation (10) may be written as $\frac{y_1}{a} = \sin(\omega t + \delta)$

or $$\frac{y_1}{a} = (\sin\omega t\cos\delta + \cos\omega t\sin\delta) \qquad \text{..........(12)}$$

and from equation (11)

$$\sin\omega t = \frac{y_2}{b} \qquad \text{and} \qquad \cos\omega t = \sqrt{1 - \frac{y_2^2}{b^2}}$$

Now from equation (12), we have

$$\frac{y_1}{a} = \frac{y_2}{b}\cos\delta + \sqrt{\left(1 - \frac{y_2^2}{b^2}\right)}\ \sin\delta$$

or, $$\frac{y_1}{a} - \frac{y_2}{b}\cos\delta = \sqrt{\left(1 - \frac{y_2^2}{b^2}\right)}\ \sin\delta$$

Squaring on both side, then we get

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2}\cos^2\delta - \frac{2y_1y_2}{ab}\cos = \left(1 - \frac{y_2^2}{b^2}\right)\ \sin^2\delta$$

or $$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} - \frac{2y_1y_2}{ab}\cos\delta = \sin^2\delta \qquad \text{................(13)}$$

This represent an ellipse, contained within a rectangle of sides '2a' and '2b', but of an eccentricity and at an inclination depending on the phase difference 'δ' and on the individual amplitudes.

Q.16 : A point is executing S.H.M. with a period of π sec. When it is passing through the centre of its path, its velocity is 0.1 m/sec. What is its velocity when it is at a distance 0.03 m from the mean position.

Solⁿ. Given $T = \pi$ sec

therefore from the relation $T = \frac{2\pi}{\omega}$, we have

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2\ \text{sec}^{-1}$$

When it is passes through the centre i.e. $x = 0$, its velocity is 0.1 m/sec.

Therefore from the relation $v = \omega\sqrt{A^2 - x^2}$

we have

$$0.1 = \omega\sqrt{A^2 - 0^2}$$

or, $$0.1 = A\,\omega$$

or, $$A = \frac{0.1}{\omega} = \frac{0.1}{2} = 0.05\ m.$$

Now the velocity of the point, when it is at a distance 0.003 m from the mean position is $v = \omega\sqrt{A^2 - x^2}$

or, $v = 2\sqrt{(0.05)^2 - (0.03)^2}$

or, $v = 2\sqrt{(0.08) \times (0.02)}$

or, $v = 2\sqrt{0.0016}$

or, $v = 2 \times 0.04$

or, $v = 2 \times 0.04$ m/sec

Q.17 : Solve the diferential equation $\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2 = E$ and show that x(t) represent simple harmonic motion. What is the frequency and amplitude of vibration of the motion.

Sol^n : The given differential equation is

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2 = E$$

Differentiating this equation, then we get

$$\frac{1}{2} m \,.\, 2\left(\frac{dx}{dt}\right) . \frac{d^2x}{dt^2} + \frac{1}{2} k . 2x \frac{dx}{dt} = 0 \quad \text{[Since } k = \text{constant]}$$

$$\left(\frac{dx}{dt}\right)\left[m \,.\, \frac{d^2x}{dt^2} + kx\right] = 0$$

or, $m \frac{d^2x}{dt^2} + kx = 0$

or $\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$ ....(1)

Let, $\frac{k}{m} = \omega^2$

then we have $\frac{d^2x}{dt^2} + \omega^2 x = 0$ ....(2)

It is the differential equation of a S.H.M. The solution of this equation can be written as $x = A\, e^{\alpha t}$ ....(3)

Where A and α are arbitrary constant. Differentiating this equation twice with respect to t, then we get,

$$\frac{dx}{dt} = \alpha A e^{\alpha t} \quad \text{and} \quad \frac{d^2x}{dt^2} = \alpha^2 A e^{\alpha t}$$

putting this value in equation (2), then we get

$$A\alpha^2 e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

or, $A e^{\alpha t} (\alpha^2 + \omega^2) = 0$

or, $\alpha^2 + \omega^2 = 0$

or, $\alpha = \pm \sqrt{-\omega^2} = \pm j\omega$ ....(4)

Thus there are two possible solution of equation (2). These are

$$x = A_1 e^{j\omega t} \quad \text{and} \quad x = A_2 e^{-j\omega t}$$

The general solution of equation (2) will be sum of these two solution i.e.

$$x = A_1 e^{j\omega t} + A_2 e^{-j\omega t} \qquad ....(5)$$

Where A_1 and A_2 are arbitrary constant. This solution can also be written as

$$x = A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t)$$

$$x = (A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t \qquad ....(6)$$

We can make a change in arbitrary constant by putting

$A_1 + A_2 = B\cos\phi$ and $j(A_1 - A_2) = B\sin\phi$

Where B and ϕ are new constant such that .

$$B^2 = 4A_1A_2$$

Now we have from equation (6) $x = B\cos\phi\cos\omega t + B\sin\phi\sin\omega t$

$$x = B\cos(\omega t - \phi)$$

It is the solution of the above given equation.

Q. 18 : What are the uses of Lissazous figures. (2008)

Sol[n] : The Lissajous figures have number of practical uses-

लिसजस चित्रों के अनेक प्रायोगिक प्रयोग हैं–

(i) If the frequencies of two vibrating systems are commensurable, i.e. in a whole number ratio as 1 : 1, 2 : 1, 3 : 1 the ratio may be read off at once from the steady figure they produce.

यदि दो कंपनिक निकायों की आवृत्ति अनुपातिक हैं अर्थात् पूर्ण संख्याओं के अनुपात में हैं जैसे– 1 : 1, 2 : 1, 3 : 1 तो अनुपात को उनके द्वारा उत्पन्न स्थिर चित्रों द्वारा तुरन्त पढ़ा जा सकता है।

(ii) The frequencies of two systems, if nearly commensurate, may be compared with the help of these figures.

दो निकायों की आवृत्ति यदि लगभग अनुपातिक हैं तो उनके चित्रों द्वारा उनके तुलना की जा सकती है।

(iii) The figure afford a good method for adjusting the frequencies of two forks to a given ratio.

यह चित्र दो स्वरित्रों की आवृत्तियों को एक दिये अनुपात में व्यवस्थित (संयोजित) करने का एक अच्छा तरीका प्रदान करता है।

(iv) They are specially useful in testing, the accuracy of tuning of some simple intervals between two forks. Even a slight mistuning causes the figure to vary in form.

यह दो द्विभुजों के बीच कुछ निश्चित अंतराल की ट्यूनिंग की शुद्धता जांचने में विशेष रूप से उपयोगी है। यहां तक कि एक छोटी सी गलत ट्यूनिंग से चित्र का रूप बदल जाता है।

(v) The principle of Lissajous figures may be employed to investigate how the period of a rod, fixed at one end, varies with the length of the rod.

लिसजस चित्र का सिद्धान्त यह जांच हेतु प्रयोग किया जा सकता है कि, कैसे एक छड़ जिसका एक सिरा फिक्स्ड है उसका समय अंतराल छड़ की लम्बाई से कैसे परिवर्तित होता है।

(vi) Helmholtz used the principle to investigate the vibration of a violin string.

हेमहोल्ट्ज ने इस सिद्धांत का प्रयोग वायलिन के तारों के कंपनों की जांच के लिये प्रयोग किया।

Q.19. Show that when a particle is moving in simple harmonic motion its velocity at a distance $\sqrt{\frac{3}{2}}$ of its amplitude from the central position is half its velocity in the central positions. (2009)

Sol^n : The displacement of a particle in S.H.M. is

$$x = a\sin(\omega t + \phi)$$

The velocity is

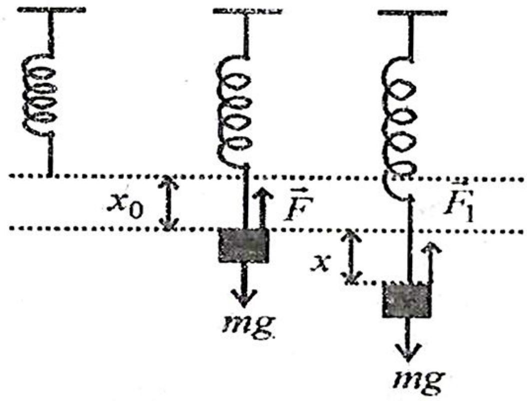
$$v = \frac{dx}{dt} = \omega a\cos(\omega t + \phi) \qquad v = \omega a\sqrt{\left(1 - \frac{x^2}{a^2}\right)} = \omega\sqrt{\left(a^2 - x^2\right)}$$

At $x = \frac{\sqrt{3}}{2}a$ the velocity is $v = \omega\sqrt{a^2 - \frac{3a^2}{4}} = \frac{1}{2}\omega a$

Which is half the velocity (ω, a) in the central position $x = 0$

Q.20. Write the differential equation for a spring mass system in which one end of the spring is fixed and other is attached to mass and has zero damping. Find the general solution and the frequency of oscillation if mass M = 4kg and force constant of spring K = 100 N/m. Mass of the spring is negligible. (2009)

Sol^n : Consider a massless spring of length 'l', hanging vertically as shown in the following figure.



Now a mass 'm' is attached to its lower and so that its length extected by x_0. Due to elasticity, spring exerets a upward force 'F' an the mass 'm'. According to Hook's law this force is $F = -k\,x_0$. where k is the force constant. The other force acting on the mass is its weight $+mg$ which is taken to be positive. Thus the total force acting on mass 'm' is $F + mg$.

अब एक द्रव्यमान 'm' इसके निचले छोर से जोड़ देते हैं जिससे इसकी लम्बाई x_0 बढ़ जाती है। प्रत्यास्थता के कारण द्रव्यमान 'm' पर, स्प्रिंग ऊपर की ओर एक बल लगाता है। हुक के नियम के अनुसार यह बल $F = -k\,x_0$ है। जहाँ k बल नियतांक है। दूसरा बल जो द्रव्यमान पर कार्य करता है, इसका भार $+mg$ है जो धनात्मक लेते हैं। इस प्रकार कुल बल जो द्रव्यमान 'm' पर $F + mg$ कार्य करता है।

As the mass has no acceleration, therefore the total force acting on it must be zero. That is

चूँकि द्रव्यमान में कोई त्वरण नहीं होता इसलिए इस पर कार्यकारी कुल बल शून्य होना चाहिए। इस प्रकार

$$F + mg = 0$$

or $$-kx_0 + mg = 0$$

or $$x_0 = mg / k \qquad \text{..........(1)}$$

This is the static equilbrium position of the block 'm'.

यह ब्लाक 'm' के लिए स्थैतिक संतुलन की दशा है।

During the oscillation, let the total extension in the spring is $x_0 + x$, then the force exerted on mass m by the spring is

$$F' = -k\,(x_0 + x)$$

$$F' = -k\left(\frac{mg}{k} + x\right)$$

$$F' = -mg - kx \qquad \text{..........(2)}$$

The other force acting on the mass 'm' is its weight $+mg$. Therefore, the net force acting on the block is

$$F'' = F' + mg$$

$$F'' = (-mg - kx) + mg$$

$$F'' = -kx$$

According to Newton's second law we have

$$F'' = -k\,x = m\frac{d^2x}{dt^2}$$

or, $$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \qquad \text{where} \qquad \omega^2 = \frac{k}{m}$$

This is the equation of S.H.M. Thus the block 'm' executes S.H.M. about the position $x_0 = mg / k$. Time period of the S.H.M.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency of oscillation is $n = \frac{1}{T}$

Given $m = 4$ kg and $k = 100\ N/m$

Therefore, $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow n = \frac{1}{2\times 3.14}\sqrt{\frac{100}{4}} \Rightarrow n = \frac{1}{6.28}\sqrt{25}$

$$n = \frac{5}{6.28} = \frac{500}{628} = 0.79 \text{ per second}$$

Q.21 : What will be the shape of Lissazous figure for the following mations.

$x = \cos 2\omega t \qquad y = \sin 2\omega t$ **(2010)**

Sol^n. The equation of motion for two S.H.O. is

$$x = \cos 2\omega t \qquad \text{...............(1)}$$

and $$y = \sin 2\omega t \qquad \text{...............(2)}$$

The equation of the resultant path, traced out by the particle can be obtained by eliminating 't' from equation (1) and (2). Now from equation (1)

$$\cos 2\omega t = x$$

$$1 - 2\sin^2 \omega t = x$$

or $$2\sin^2 \omega t = 1 - x$$

or $$\sin \omega t = \sqrt{\frac{1-x}{2}} \qquad \text{..............(3)}$$

Similarly from equation (2)

$$\sin 2\omega t = y$$

or, $$2\sin \omega t \cos \omega t = y$$

or, $$2\sqrt{\frac{1-x}{2}} \cos \omega t = y$$

or, $$\cos \omega t = \frac{y}{\sqrt{2(1-x)}} \qquad \text{................(4)}$$

Squaring and adding equation (3) and (4) we get,

$$\sin^2 \omega t + \cos^2 \omega t = \left(\frac{1-x}{2}\right) + \frac{y^2}{2(1-x)}$$

or, $$1 = \frac{(1-x)^2 + y^2}{2(1-x)}$$

or, $$(1-x)^2 + y^2 = 2(1-x)$$

or, $$1 + x^2 - 2x + y^2 = 2 - 2x$$

or, $$x^2 + y^2 = 2 - 1$$

or, $$x^2 + y^2 = 1$$

It is the euqtion of circle, Therefore the shape of the Lissazous figure will be Circle.

Q. 22 : Discuss the resultant motion when two Simple harmonic motions having a frequency ratio 1:2 and acting at right angles to each other superimpose. **(2010)**

Sol^n: Let the two simple harmonic oscillations which are to be companded, be

$$x = a\cos \omega t \qquad \text{.............(1)}$$

$$y = b\cos(2\omega t + \phi) \qquad \text{.............(2)}$$

The equation of the resultant curve can be obtained by eliminating 't' from equation (1) and (2). Now from equation (1)

$$\cos \omega t = \frac{x}{a} \Rightarrow \sin \omega t = \frac{\sqrt{a^2 - x^2}}{a}$$

Now from equation (2), we get $\cos(2\omega t+\phi)=\frac{y}{b}$

or, $$\cos 2\omega t\cos\phi-\sin 2\omega t\sin\phi=\frac{y}{b}$$

or, $$\left(2\cos^2\omega t-1\right)\cos\phi-2\sin\omega t\cos\omega t\sin\phi=\frac{y}{b}$$

or, $$\left(\frac{2x^2}{a^2}-1\right)\cos\phi-2\frac{\sqrt{a^2-x^2}}{a}.\frac{x}{a}\sin\phi=\frac{y}{b}$$

or, $$\left(\frac{2x^2}{a^2}-1\right)\cos\phi-\frac{y}{b}=\frac{2x}{a}\frac{\sqrt{a^2-x^2}}{a}\sin\phi$$

Squaring on both side, then we get,

$$\left(\frac{4x^4}{a^4}-\frac{4x^2}{a^2}+1\right)\cos^2\phi-\frac{2y}{b}\left(\frac{2x^2}{a^2}-1\right)\cos\phi+\frac{y^2}{b^2}=\frac{4x^2}{a^2}\left(\frac{a^2-x^2}{a^2}\right)\sin^2\phi$$

$$\frac{4x^4}{a^4}\left(\cos^2\phi+\sin^2\phi\right)-\frac{4x^2}{a^2}\left(\cos^2\phi+\sin^2\phi\right)-\frac{4x^2y}{a^2b}\cos\phi+\frac{y^2}{b^2}+\frac{2y}{b}\cos\phi+\cos^2\phi=0$$

or $$\frac{4x^4}{a^4}-\frac{4x^2}{a^2}-\frac{4x^2}{a^2}\frac{y}{b}\cos\phi+\left(\frac{y^2}{b^2}+\frac{2y}{b}\cos\phi+\cos^2\phi\right)=0$$

or $$\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}-\frac{y}{b}\cos\phi-1\right)+\left(\frac{y}{b}+\cos\phi\right)^2=0 \qquad \text{...........(3)}$$

It is the equation of the resultant path traced by a particle moving under the simultaneous action of two simple harmonic oscillations. For any value of ϕ, 'a' and 'b' this curve consists of two loops.

Particular Case : –

Case (i) – For $\phi=0$, the general equation (3) becomes

$$4.\frac{x^2}{a^2}\left(\frac{x^2}{a^2}-\frac{y}{b}-1\right)+\left(\frac{y}{b}+1\right)^2=0$$

$$4\frac{x^2}{a^4}-\frac{4x^2}{a^2}\left(\frac{y}{b}+1\right)+\left(\frac{y}{b}+1\right)^2=0$$

or, $$\left[\frac{2x^2}{a^2}-\left(\frac{y}{b}+1\right)\right]^2=0$$

This represent two coincident parabola's equation of each being

$$2\frac{x^2}{a^2}-\left(\frac{y}{b}+1\right)=0 \qquad \frac{2x^2}{a^2}=\left(\frac{y}{b}+1\right)$$

as $$x^2=\frac{a^2}{2b}(y+b) \qquad \text{..................(4)}$$

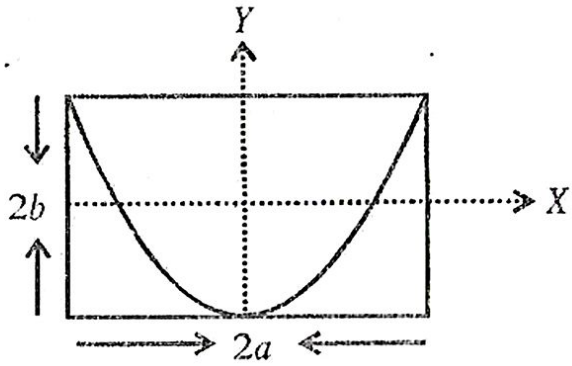
Case(ii) – When $\phi=\pi/2$ eugation (3) becomes

$$4\frac{x^2}{a^2}\left(\frac{x^2}{a^2}-1\right)+\frac{y^2}{b^2}=0 \qquad \text{...............(5)}$$

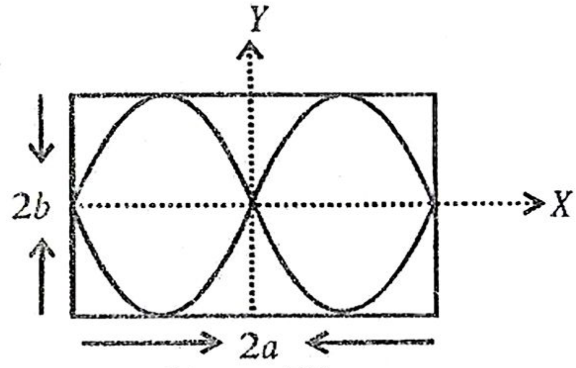
This represents a curve whose shape is ∞.

[Figure of eight (8) rotated by 90^0]

For other value of f the curve are complicated.



($\phi = 0$)



($\phi = \pi / 2$)

Q23 : Write short notes on Oscillations in a potential well.

Sol^n : When a particle moves in a potential field in such a way that its motion is confined to a certain bounded region, then the particle is said to be in a potential well or potential energy well.

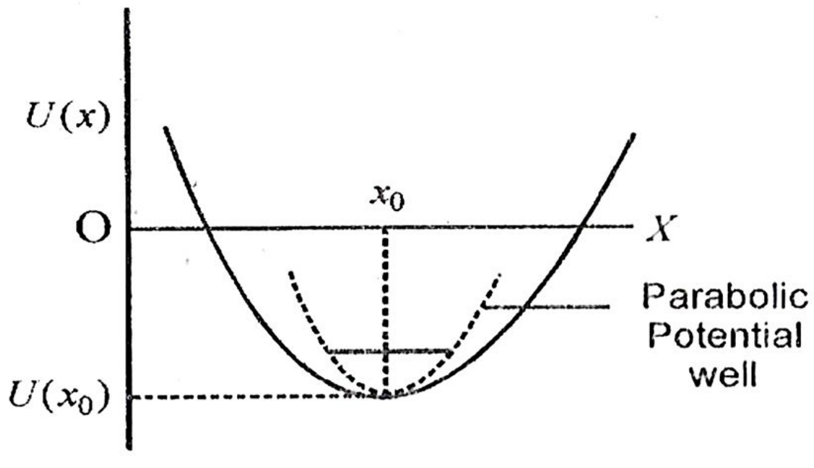
जब एक कण विभव क्षेत्र में इस प्रकार घूमता है कि इसका घूमना एक निश्चित क्षेत्र में हो तो कण को विभाव कप या विभव ऊर्जा कूप कहा जाता है।

Consider the motion of a prticle in one dimensional potential field in which the potential energy of the particle is $U(x)$. It is known that the potential energy is minimum for some value of 'x' say 'x_0'. The potential energy of the particle at any point 'x' may be expressed in the form of Taylor's series.

माना कण का धूमना एक विमीय क्षेत्र में इस प्रकार है कि इसकी विभव ऊर्जा $U(x)$ है। यह ज्ञात है कि विभव ऊर्जा $x = x_0$ (say) के लिए न्यूनतम है। कण की विभव ऊर्जा किसी बिन्दु x पर टेलर सीरिज के रूप में निम्न प्रकार लिखी जा सकती है।

$$U(x)=U(x_0)+(x-x_0)\left(\frac{dU}{dx}\right)_{x_0}+\frac{(x-x_0)^2}{2!}\left(\frac{d^2U}{dx^2}\right)_{x_0}+$$

$$\frac{(x-x_0)^3}{3!}\left(\frac{d^2U}{dx^3}\right)_{x_0}+\text{.........} \qquad \text{................(1)}$$



The point $x = x_0$, where the potential energy is minimum, the force acting on the particle is zero, is called the equilibrium point.

बिन्दु $x = x_0$, जहाँ विभव ऊर्जा न्यूनतम है, कण पर लगने वाला बल शून्य होगा , एक बिन्दु साम्यावस्था का बिन्दु कहलाता है।

$$F(x_0) = -\left(\frac{du}{dx}\right)_{x_0} = 0 \qquad \text{..................(2)}$$

It is convenient to take the potential energy at the equilibrium point equal to zero and measure the potential energies at other points with respect to this value. Thus

यह सुविधाजनक होता है कि साम्यावस्था के बिन्दु पर विभव ऊर्जा को शून्य माना जाय तथा अन्य बिन्दुओं पर इस बिन्दु के सापेक्ष विभव ऊर्जा का मान ज्ञात किया जाय।

$$U(x_0) = 0 \qquad \text{....................(3)}$$

Shifting the origin to the point $x = x_0$, we can write the Taylor's expansion for the potential energy as

यदि हम ओरिजिन को $x = x_0$ बिन्दु पर शिफ्ट करें तो हम टेलर सीरिज को निम्न प्रकार से लिख सकते हैं।

$$U(x) = \frac{x^2}{2!}\left(\frac{d^2u}{dx^2}\right)_{x_0} + \frac{x^3}{3!}\left(\frac{d^3u}{dx^3}\right)_{x_0} + \ldots\ldots$$

If the motion of the particle is confined to the region very near the equilibrium position the terms containing higher powers of x may be neglected in the expansion of $U(x)$. Under this approximation we can write.

यदि कण की गति साम्यावस्था के बिन्दु के नजदीक हो तो x के अधिक घात वाले पद को $U(x)$ के फैलाव में निकाल सकते हैं तथा इस समय लिख सकते हैं।

$$U(x) = \frac{1}{2}x^2\left(\frac{d^2u}{dx^2}\right)_{x_0} \qquad \text{....................(5)}$$

or $$U(x) = \frac{1}{2}k\,x^2 \qquad \text{......................(6)}$$

where $$k = \left(\frac{d^2u}{dx^2}\right)_{x_0} \qquad \text{....................(7)}$$

k is the force constant of the force field. Thus, for small values of x, the potential energy curve has the shape of a parabolic potential well. The force acting on the particle at point x is

जहाँ बल क्षेत्र का बल नियतांक है। अतः के छोटे मान के लिए विभव ऊर्जा वक्र का आकार परवलयाकार विभाव कूप के समान होगा । बिन्दु पर कण पर लगाने वाला बल होगा।

$$F = -\frac{du}{dx} = -k\,x \qquad \text{..................(8)}$$

and hence the equation of motion of the particle is $m\ddot{x} = -kx$

और कण के गति का समीकरण $m\ddot{x} = -kx$ होगा।

which represent the simple harmonic oscillation. Thus the small amplitude motion of the particle about the equilbrium position is simple harmonic.

जो सरल आर्वत गति को प्रदर्शित करता है।

Que.24 The displacement of a moving particle at any time is given by the equation. (SU-2016)

$$x = a\cos\omega t + +b\sin\omega t$$ **(GKP-2016)**

Show that the motion is simple harmonic.

Sol^n. The displacement of particle at any time is given by the equation

$$x = a\cos\omega t + b\sin\omega t \quad \text{....... (1)}$$

Differentiating equation (1) w.r.t. 't', we get

$$\frac{dx}{dt} = -a\omega\sin\omega t + b\omega\cos t$$

or, $$\frac{dx}{dt} = \omega(-a\sin \omega t + b\cos \omega t)$$

Differentiating this equation once again, we get

$$\frac{d^2x}{dt^2} = \omega(-a\omega\cos\omega t + b\omega\sin\omega t)$$

or, $$\frac{d^2x}{dt^2} = \omega^2(-a\cos\omega t + b\sin\omega t)$$

or, $$\frac{d^2x}{dt^2} = -\omega^2 x$$

or, $$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{....... (2)}$$

It is differential equation of S.H.M. Thus the motion of particle is simple harmonic motion.

Q.25 : A particle of mass 10 g.m excuting simple harmonic motion has amplitute of 4 c.m. If it makes 16 vibration per second, find its maximum velocity and energy at mean position. (2011)

Sol^n : The displacement of a particle in S.H.M. is -

$$x = a\sin(\omega t + \phi)$$

and its velocity -

$$v = \frac{dx}{dt} = \omega a\cos(\omega t + \phi)$$

It is maximum when $\cos(\omega t + \phi) = 1$

i.e. $$v_{\max} = \omega a$$

or, $$v_{\max} = 2\pi n a$$

or $$v_{\max} = 2 \times 3.14 \times 16 \times 4$$

or $$v_{\max} = 401.92 \text{ cm/sec.}$$

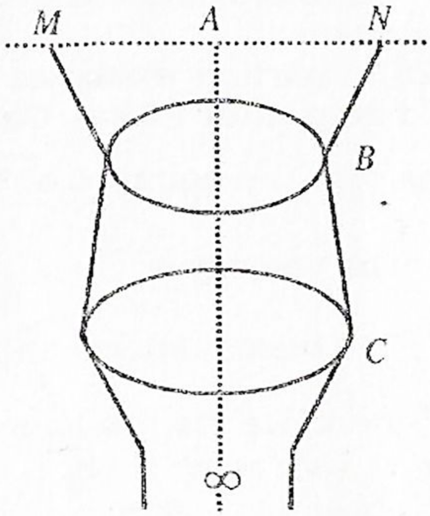
The energy at mean position is entirely kinetic and is given by

$$E = K_{\max} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2} \times 10 \times (401.92)^2 = 5 \times 161539.68$$

or, $$E = 807698.4 = 8.07 \times 10^5 \text{ erg}$$

Q.26 : What are Lissajou's figures ? How can they be demonstrated ?

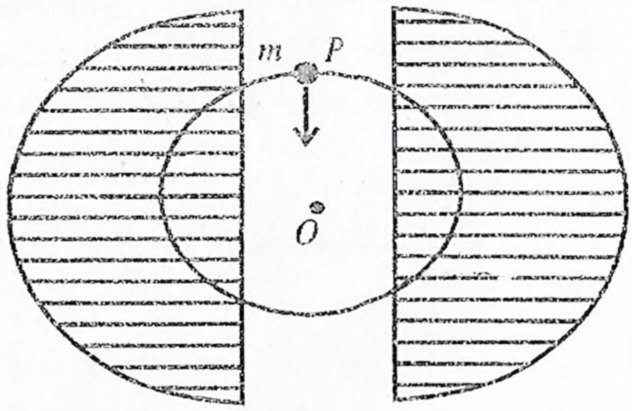
Sol^n^ Demonstration of Lissajous Figure : The Lissajous figure can be demonstrated by compounding two simple harmonic motion perpendicular to each other.



Black Burn's Pendilum : A mechanical device to demonstrate the formation of Lissajous figure is the Blackburn's pendulum. It consist of a string with its two ends are attached to a rigid beam at point M and N The string is cut at its centre and the ends so formed are attached to a heavy funnel C. The exit from the funnel is narrow so that when it is filled with sand a fine stream fall on the ground. A clip B is placed so as to bring the two strings together. It can be slipped up and down on the strings. If the funnel is pulled out stant wise and released it moves under the action of two S.H.M. at right angles. Hence the stream of stand escaping from the funnel traces a lissajous figure on the ground.

Q.27 : A straight tunnel is bored through the centre of earth. A body of mass m dropped into it. Prove that its motion is simple harmonic.

Ans. Consider a particle of mass m at r. O is the centre of the earth. The gravitational attraction of the earth for the particle P arises entirely from that portion of the earth of radius r. The external shell exerts no force on particle.



If 'ρ' is the density of earth, then the mass of the sphere of radius 'r' is

$$M' = \rho V' = \frac{4}{3}\pi r^3 \rho$$

so the force of attraction between the earth and the particle of mass m is

$$F = -G\frac{m\,M'}{r^2}$$

minus sign indicate that the force is attractive and directed towards the centre of earth. Therefore.

$$F = -G\frac{\left(\frac{4}{3}\pi r^3 \rho\right)m}{r^2} = -\left(\frac{4}{3}G\rho\pi m\right)r$$

or

$$F = -k\,r$$

where $k = \frac{4}{3} G \rho \pi m$ is a constant.

Therefore force is directly proportional to displacement r by oppositely directed.

Q28. A particle executes simple harmonic motion with time period 31.4 sec and amplitude 25 cm. Calculate its maximum velocity.

Ans. The equation of S.H.M. is $x = a\sin(\omega t + \phi)$

The velocity is $v = \frac{dx}{dt} = \omega\, a\cos(\omega t + \phi$

The maximum value of $\cos(\omega t + \phi)$ is 1,

therefore the maximum velocity $v_{\max} = \omega\, a$

Here $\omega = \frac{2\pi}{T} = 0.2$ radian/sec

and a =25 cm, therefore

$$v_{\max} = 0.2 \times 25 = 5.0 \text{ cm/sec.}$$

Q.29 : A particle executes simple harmonic motion along a straight line. Its velocity when passing through the points 3 cm and 4 cm away from its equilibrium position is 16 cm/sec and 12 cm/sec respectively. Find the amplitude of oscillation of the particle. **(2014)**

Sol^n : The displacement of a particle in S.H.M. is

$x = a\sin(\omega t + \phi)$ Where 'a is amplitude .

Its velocity is

$$\frac{dx}{dt} = \omega a\cos(\omega t + \phi) \Rightarrow \omega a\sqrt{1-\sin^2} \Rightarrow \omega a\sqrt{1-\left(\frac{x^2}{a^2}\right)}$$

or, $$\frac{dx}{dt} = \omega\sqrt{a^2 - x^2} \Rightarrow \left(\frac{dx}{dt}\right)^2 = \omega^2\left(a^2 - x^2\right) \qquad \text{........ (1)}$$

Now it is given that,

when $x = 3$ cm, $\frac{dx}{dt} = 16$ cm/sec

and when $x = 4$ cm, $\frac{dx}{dt} = 12$ cm/sec

Substituting in equation (1), we get

$$256 = \omega^2\left(a^2 - 9\right) \qquad \text{........ (2)}$$

$$144 = \omega^2\left(a^2 - 16\right) \qquad \text{........ (3)}$$

Dividing, We get,

$$\frac{16}{9} = \frac{a^2 - 9}{a^2 - 16} \Rightarrow 16\left(a^2 - 16\right) = 9\left(a^2 - 9\right)$$

or, $$7a^2 = 256 - 81 = 175$$

$$a^2 = \frac{175}{7} = 25 \Rightarrow a = 5 \text{ cm.}$$ **Ans.**

Q.30 : Two Simple Harmonic motions are represented by the following equation

$y_1 = 10\sin(3\pi + \pi/4)$, $y_2 = 5\left(\sin 3\pi t + \sqrt{3}\cos 3\pi t\right)$ **(2014)**

Solⁿ : The equation of first S.H.M. is -

$$y_1 = 10\sin(3\pi + \pi/4) \quad \text{......... (1)}$$

The equation of second S.H.M. is

$$y_2 = 5\left(\sin 3\pi t + \sqrt{3}\cos 3\pi t\right)$$

or, $$y_2 = 10\left(\frac{1}{2}\sin 3\pi t + \frac{\sqrt{3}}{2}\cos 3\pi t\right)$$

or, $$y_2 = 10\left(\cos\frac{\pi}{3}\sin 3\pi t + \sin\frac{\pi}{3}\cos 3\pi t\right)$$

or, $$y_2 = 10\sin\left(3\pi t + \frac{\pi}{3}\right)$$

Comparign it with first equation, we see that the amplitudes of oscillation are equal to (10:10).

Q.31 : The simple harmonic motion is represented by $x = 5\sin(10t + \pi/6)$, where x is in c,m. and t is in sec. Calculate displacement and acceleration of the particle at $\pi/10$ second. **(2015)**

Solⁿ : The given equation is $x = 5\sin(10t + \pi/6)$ .... (1)

Now, the displacement at $\pi/10$ sec. is

$$x = 5\sin\left(10 \times \frac{\pi}{10} + \frac{\pi}{6}\right) \quad \text{or} \quad x = 5\sin\left(\pi + \frac{\pi}{6}\right)$$

or, $$x = -5\sin\left(\frac{\pi}{6}\right) \quad \text{or,} \quad x = -5/2 \text{ cm.}$$ **Ans.**

Now differentiating equation (1) twice, then we get the acceleration

$$f = \frac{d^2x}{dt^2} = -500\sin(10t + \pi/6)$$

Now acceleration at $\pi/10$ sec. is

$$f = \frac{d^2x}{dt^2} = -500\sin\left(10 \times \frac{\pi}{10} + \frac{\pi}{6}\right)$$

$$f = -500\sin\left(\pi + \frac{\pi}{6}\right) \Rightarrow f = 500\sin\left(\frac{\pi}{6}\right) \Rightarrow f = 500 \times \frac{1}{2} = 250 \text{ cm/sec}^2$$ **Ans.**

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