Chapter Four

HOMOMORPHISM, ISOMORPHISM, COSETS AND NORMAL SUBGROUPS

Important Points from the Chapter

1. **Homomorphic Mapping** Suppose G and G' are two groups, the composition in each being denoted multiplicatively. A mapping f of G into G' is said to be a homomorphic mapping (or a homomorphism) of G into G', if $f(ab)=f(a)\,f(b), \forall\, a, b \in G$.

   If f is homomorphic mapping of a group G onto the group G', so that $f(G)=G'$, then the group G' is called a **homomorphic image** of the group G. **(2015, 10, 07, 02, 1998)**

2. **Isomorphic Mapping** Let G and G' are two groups, the composition in each group is multiplication. A mapping f of G into G' is said to be isomorphic mapping of G into G', if

   (i) f is one-to-one that means distinct elements in G have distinct f-image in G'.

   (ii) $f(ab)=f(a)\,f(b), \forall\, a, b \in G$ that means the image of the product is the product of the images.

   If f is an isomorphic mapping of a group G into a group G', then f is also called an isomorphism of G into G'. If f is an isomorphism of G onto G', then the group G is called an isomorphic image of the group G. **(2015, 12, 08, 1996)**

3. **Isomorphic Group** Suppose G and G' are two groups. Also, suppose that the compositions in both G and G' have been denoted multiplicatively. Then, group G is isomorphic to the group G', if there exists a one-to-one mapping f of G onto G' such that

$$f(ab)=f(a)\,f(b), \forall\, a, b \in G$$

   i.e. the mapping f preserves the compositions in G and G'.

   If the group G is isomorphic to the group G'. Symbolically, we write $G \cong G'$. **(2015)**

4. **Homomorphism of Groups**

   (i) **Homomorphism into** A mapping f from group G into a group G' is said to be a homomorphism of G into G', if

$$f(ab)=f(a)\,f(b), \forall\, a, b \in G.$$

(2010, 04, 01, 2000)

(ii) **Homomorphism onto** A mapping f from a group G onto a group G' is said to be a homomorphism of G onto G', if

$$f(ab) = f(a)\,f(b), \forall\, a, b \in G.$$

Also, G' is said to be homomorphic image of G.

- **Note** Isomorphism is a special type of homomorphism, if f is a homomorphism of G into G' and f is one-one, then f is an isomorphism of G into G'.
  Similarly, if f is a homomorphism of G onto G' and f is one-one, then f is an isomorphism of G onto G'.

(iii) **Endomorphism** A homomorphism of a group into itself is called an endomorphism.

5. **Conjugate Elements and Conjugacy Relation** If a and b be two elements of a group G, then b is said to be conjugate to 'a', if there exists an elements $x \in G$ such that $b = x^{-1}ax$.
   If $b = x^{-1}ax$, then b is also called transform of a by x.
   If b is conjugate to a, then symbolically we will write $b \sim a$ and this relation in G will be called the **relation of conjugacy**.
   Thus, $b \sim a$ iff $b = x^{-1}ax$ for some $x \in G$.
6. **Conjugate Class** For any element $a \in G$, the set $C(a) = \{x : x \in G, x \sim a\}$ is called the conjugate class of a in G.
   In fact, $C(a)$ consists of all elements of the type $y^{-1}ay$ as y varies over G.
   If G is a finite group, then the relation of conjugacy on G is an equivalence relation on G and therefore it will partition the set G into disjoint equivalence classes. Let $C(a_1), C(a_2), \ldots, C(a_m)$ be the totality of all conjugate classes of G. As the equivalence classes are pairwise disjoint and their union is G, we have $o(G) = \sum_{i=1}^{m} |C(a_i)|$, where $|C(a_i)|$ denotes the order of the class $C(a_i)$.
7. **Self-conjugate Elements** An element $a \in G$ is said to be self-conjugate if a is the only member of the class $C(a)$ of elements conjugate to a, i.e. if $C(a) = \{a\}$.
8. **Cosets** Suppose G is a group and H is any subgroup of G. Let a be any element of G. Then, the set $Ha = \{ha : h \in H\}$ is called a **right coset** of H in G generated by a. Similarly, the set $aH = \{ah : h \in H\}$ is called a **left coset** of H in G generated by a. **(2012, 07)**
9. **Lagrange's Theorem** The order of each subgroup of a finite group is a divisor of the order of the group. **(2015, 09, 08, 06, 1998, 96)**
10. **Normalizer of an Element of a Group** If $a \in G$, then $N(a)$, the normalizer of a in G is the set of all those elements of G which commute with a. Symbolically, $N(a) = \{x \in G : ax = xa\}$.
11. **Normalizer of a Subgroup of a Group** Let G be a group and A be its subgroup. Then, the normalizer of A in G denoted by $N(A)$ is the collection of all those elements of G which commute with A,
    i.e. $N(a) = \{x \in G : xA = Ax\}$.

12. **Centre of a Group** The set Z of all self conjugate elements of a group G is called the centre of G.

    Symbolically, $Z = \{z \in G : zx = xz, \forall x \in G\}$.

13. **Cayley's Theorem** Every finite group G is isomorphic to a permutation group.

14. **Kernel of Homomorphism** If $f : G \to G'$ is a group homomorphism, then the set K of all those elements of G which are mapped onto the identity e' of G' is called the kernel of homomorphism f.

    Thus, $\ker f = K = \{x \in G : f(x) = e'$, where e' the identity of $G'\} = f^{-1}(\{e'\})$.
    (2015, 09, 1999)

15. **Index of Subgroup in a Group** If H is a subgroup of a group G, then the number of distinct right (left) cosets of H in G is called the index of H in G and it is denoted by $[G : H]$. **(2016, 06)**

16. **Euler's Theorem** If m is a positive integers and a is relatively prime to m, i.e. $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$, where $\phi(m)$ is the Euler's-ϕ function. **(2014, 1999, 97)**

17. **Fermat's Theorem** If p is prime and a is any integer, then $a^p \equiv a \pmod{p}$. **(1998, 97)**

18. Let H and K be finite subgroups of a group G, then

$$o(HK) = \frac{o(H)\,o(K)}{o(H \cap K)}.$$

19. **Automorphism** An isomorphic mapping of a group G onto itself is called an automorphism of G.

20. **Inner Automorphism** If G is a group, the mapping $f_a : G \to G$ defined by $f_a(x) = a^{-1}xa$, $\forall\, x \in G$ where a is fixed element of G, is an automorphism of G known as inner automorphism.

21. **Normal Subgroup** A subgroup H of a group G is said to be a normal subgroup of G, if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

    Every group G possesses at least two normal subgroups, namely G itself and the subgroup consisting of identity element e alone. These are called improper normal subgroups. **(2011, 08, 1996)**

22. **Simple Group** A group having no proper normal subgroups is called a simple group.

    ■ **Note** Every group of prime order is simple.

23. **Quotient Group** If G in a group and H is a normal subgroup of G, then the set G/H of all cosets of H in G is a group with respect to multiplication of cosets. It is called the quotient group or factor group of G by H.

    ■ **Note** The identity element of the quotient group $\frac{G}{H}$ is H.

Very Short Answer Questions

Q 1. $G=(Z,+)$ and $G'=(\{2^m: m=0, \pm 1, \pm 2, \ldots\}, \cdot)$, then, prove that the mapping $f: G \to G'$ defined by $f(m)=2^m$, $\forall\, m \in Z$ is an isomorphism. **(2014)**

Sol. **f is homomorphism** Let $x, y \in Z$.

Then, we have $f(xy)=2^{xy}=2^x 2^y = f(x) f(y)$

$\therefore \quad f(xy)=f(x) f(y), \forall\, x, y \in Z$

Hence, f is a homomorphism.

f is one-one Let $x, y \in Z$. Then, we have

$$f(x)=f(y) \Rightarrow 2^x = 2^y$$

Taking log on both sides, we get

$$\log 2^x = \log 2^y \Rightarrow x \log 2 = y \log 2 \Rightarrow x = y$$

$\therefore \quad f(x)=f(y) \Rightarrow x=y$

Hence, f is one-one.

f is onto Let $2^x \in G'$, then there exists an element $x \in G$ such that $f(x)=2^x$.

$\therefore$ f is onto.

Hence, f is an isomorphism. **Hence proved.**

Q 2. Let G be a group and let e be the identity element of G, then the mapping $f: G \to G$ defined by $f(a)=e, \forall\, a \in G$ is an endomorphism of G.

Sol. Let a, b be any two elements of G, then $f(a)=e$, $f(b)=e$.

Now, we have $f(ab)=e=ee=f(a) f(b)$.

Thus, f is a homomorphism of G into G'.

Therefore, f is an endomorphism of G. **Hence proved.**

Q 3. Let H be a subgroup of a group G. If two right cosets of H in G are not disjoint. Prove that they will be identical. **(2016)**

Sol. See the solution of Q. 4 (i) of Short Answer Questions.

Q 4. Show that the additive group of integers $G=\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is isomorphic to the additive group $G'=\{\ldots, -3m, -2m, -m, 0, m, 2m, 3m, \ldots\}$, where m is any fixed integer not equal to zero.

Sol. If $x \in G$, then clearly $mx \in G'$. Let $f: G \to G'$ be defined by

$$f(x)=mx, \forall\, x \in G$$

f is one-to-one Let $x_1, x_2 \in G$. Then, $f(x_1)=f(x_2)$

$\Rightarrow \quad mx_1 = mx_2$ [by definition of f]

$\Rightarrow \quad x_1 = x_2$ [$\because m \neq 0$]

Therefore, f is one-to-one.

f is onto Suppose y is any element of G'.

Then, clearly $y/m \in G$.

Also, $\quad f(y/m) = m\,(y/m) = y.$

Thus, $\quad y \in G'$

$\Rightarrow$ there exists $y/m \in G$ such that $f(y/m) = y$.

Therefore, each element of G' is the f-image of some element of G.

Hence, f is onto.

Again, if x_1 and x_2 are any two elements of G, then

$$f(x_1 + x_2) = m\,(x_1 + x_2) \quad \text{[by definition of } f\text{]}$$
$$= mx_1 + mx_2 \quad \text{[by distributive law for integers]}$$
$$= f(x_1) + f(x_2) \quad \text{[by definition of } f\text{]}$$

Thus, f preserves compositions in G and G'. Therefore, f is an isomorphic mapping of G onto G'. Hence, G is ismorphism to G'. **Hence proved.**

Q 5. Prove that the normalizer $N(a)$ of $a \in G$ is a subgroup of G.

Sol. We have, $N\,(a) = \{x \in G : ax = xa\}$

Let $x_1, x_2 \in N(a)$, then $ax_1 = x_1a,\ ax_2 = x_2a$

First, we will show that $x_2^{-1} \in N(a)$

We have, $\quad ax_2 = x_2a \Rightarrow x_2^{-1}(ax_2)x_2^{-1} = x_2^{-1}(x_2a)x_2^{-1}$

$\Rightarrow \quad x_2^{-1}a = ax_2^{-1} \Rightarrow x_2^{-1} \in N(a)$

Now, we will show that $x_1x_2^{-1} \in N(a)$

We have, $a(x_1x_2^{-1}) = (ax_1)x_2^{-1} = (x_1a)x_2^{-1}$

$$= x_1(ax_2^{-1}) = x_1(x_2^{-1}a) = (x_1x_2^{-1})a$$

Therefore, $x_1x_2^{-1} \in N(a)$

Thus, $x_1, x_2 \in N(a) \Rightarrow x_1x_2^{-1} \in N(a)$

Hence, $N(a)$ is a subgroup of G. **Hence proved**

Q 6. Prove that the centre Z of a group of G is a normal subgroup of G.

Sol. We have, $Z = \{z \in G : zx = xz,\ \forall\ x \in G\}$

First, we will prove that Z is a subgroup of G.

Let $z_1, z_2 \in Z$. Then, $z_1x = xz_1$ and $z_2x = xz_2,\ \forall x \in G$.

We have, $\quad z_2x = xz_2, \forall\ x \in G \Rightarrow z_2^{-1}(z_2x)z_2^{-1} = z_2^{-1}(xz_2)z_2^{-1}$

$\Rightarrow \quad xz_2^{-1} = z_2^{-1}x,\ \forall\ x \in G \Rightarrow z_2^{-1} \in Z$

Now, $\quad (z_1z_2^{-1})x = z_1\ (z_2^{-1}x) = z_1\ (xz_2^{-1}) = (z_1x)z_2^{-1} = (xz_1)z_2^{-1} = x(z_1z_2^{-1})$

Therefore, $(z_1z_2^{-1})x = x(z_1z_2^{-1}), \forall\ x \in G \Rightarrow z_1z_2^{-1} \in Z$

Thus, $z_1, z_2 \in Z \Rightarrow z_1z_2^{-1} \in Z$

$\therefore$ Z is a subgroup of G.

Now, we will show that Z is a normal subgroup of G. Let $x \in G$ and $z \in Z$.
Then, $xzx^{-1} = (xz)x^{-1} = (zx)x^{-1} = z(xx^{-1}) = z(e) = z \in Z$.
Thus, $x \in G, z \in Z \Rightarrow xzx^{-1} \in Z$
Hence, Z is a normal subgroup of G. **Hence proved.**

Short Answer Questions

***Q* 1. Define group homomorphism. If $f: G \to G'$ is a group homomorphism, prove that**

(i) $f(e) = e'$, where e and e' are the identities of G and G', respectively.

(ii) $f(a^{-1}) = [f(a)]^{-1}, \forall a \in G$. **(2010, 04, 2000)**

Sol. **Part I Homomorphism of Groups**

(i) **Homomorphism into** A mapping f from group G into a group G' is said to be a homomorphism of G into G', if

$$f(ab) = f(a) f(b), \forall a, b \in G.$$

(ii) **Homomorphism onto** A mapping f from a group G onto a group G' is said to be a homomorphism of G onto G', if

$$f(ab) = f(a) f(b), \forall a, b \in G$$

Also, then G' is said to be homomorphic image of G.

(iii) **Endomorphism** A homomorphism of a group into itself is called an endomorphism.

Part II

(i) Let e be the identity of G and e' be the identity of G'. Let a be any element of G. Then, $f(a) \in G'$

Now, we have $e' f(a) = f(a)$ [$\because e'$ is the identity of G']

$= f(ea)$ [$\because e$ is the identity of G]

$= f(e) f(a)$ [$\because f$ is an isomorphic mapping]

Now in group G', we have

$$e' f(a) = f(e) f(a)$$

$\Rightarrow$ $e' = f(e)$ [by right cancellation law in G']

$\therefore$ $f(e)$ is the identity of G'.

(ii) Suppose e is the identity element of G and e' is the identity element of G'. Then, $f(e) = e'$.

Now, let a be any element of G. Then, $a^{-1} \in G$ and $aa^{-1} = e$.

We have, $e = f(e) = f(aa^{-1}) = f(a) f(a^{-1})$

[$\because f$ is composition preserving]

Therefore, $f(a^{-1})$ is the inverse of $f(a)$ in the group G'.

Thus, $f(a^{-1}) = [f(a)]^{-1}$. **Hence proved.**

Q 2. Let R be the additive group of real numbers and R_+ be mutiplicative group of positive real numbers, prove that the mapping $f: R_+ \to R$ given by $f(x) = \log x,\ \forall\ x \in R_+$ is an isomorphism. **(2012, 08, 06)**

Sol. If x is any positive real number, then $\log x$ is definitely a real number. Also, $\log x$ is unique. Therefore, if $f(x) = \log x$, then $f : R_+ \to R$.

f is one-one Let $x, y \in R_+$. Then, $f(x) = f(y)$

$\Rightarrow \quad \log x = \log y \Rightarrow e^{\log x} = e^{\log y} \Rightarrow x = y$

Therefore, f is one-to-one.

f is onto Suppose y is any element of R i.e. y is any real number. Then, e^y is definitely a positive real number, i.e. $e^y \in R_+$.

Now, $f(e^y) = \log e^y = y$. Thus, $y \in R \Rightarrow$ there exists $e^y \in R_+$ such that $f(e^y) = y$. Therefore, each element of R is the f-image of some element of R_+. Thus, f is onto.

f preserves compositions in R_+ and R

Suppose, x and y are any two elements of R_+.

Then, $f(xy) = \log (xy)$ [from definition of f]

$= \log x + \log y = g(x) + g(x)$ [from definition of f]

Thus, f preserves compositions in R_+ and R.

Here, the composition in R_+ is multiplication and the composition in R is addition. Therefore, f is an isomorphism of R_+ onto R.

Hence, $R_+ \cong R$. **Hence proved.**

Q 3. Prove that the subgroup $H = \{1, -1\}$ of multiplicative group of fourth roots of unity is a normal subgroup. **(2018)**

Sol. We have, $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$

We have to prove that H is a normal subgroup of G.

It can be easily seen that H is a subgroup of G.

Now, index of H in $G = \dfrac{o(G)}{o(H)} = \dfrac{4}{2} = 2$

And we know that every subgroup of index 2 in a group is a normal subgroup of the group.

Therefore, H is a normal subgroup of G. **Hence proved.**

Q 4. Let $f: G \to G'$ be a group homomorphism. Prove that ker f is a subgroup of G and image f is a subgroup of G'. **(2017)**

Sol. Let $f: G \to G'$ be a group homomorphism and e, e' be the identities of G and G' respectively.

Part I We have, $\ker(f) = K = \{x : x \in G, f(x) = e'\}$

we have to prove that K is a subgroup of G.

Since, $f(e) = e'$ so at least $e \in K$.

Hence, K is not empty.
Let $x, y \in K$, then $f(x) = e'$ and $f(y) = e'$.

Now, $f(xy^{-1}) = f(x)\, f(y^{-1})$ [$\because f$ is a homomorphism]
$= f(x)\, [f(y)]^{-1}$ [$\because f$ is a homomorphism]
$= e'\, e'^{-1} = e'$

Thus, $xy^{-1} \in K$ whenever $x, y \in K$.
Hence, ker f is a subgroup of G.

Part II We have

$$\text{Im}\,(f) = f(G) = \{f(a) = a' : a \in G, a' \in G'\}$$

Since, $f(e) = e'$ so $e' \in f(G)$.
Hence, $f(G)$ is not empty.
Let $a', b' \in f(G)$, so that $f(a) = a'$ and $f(b) = b'$ for some $a, b \in G$.

Now, $a'\, b'^{-1} = f(a)\, [f(b)]^{-1} = f(a)\, f(b^{-1})$
$= f\,(ab^{-1}) \in f(G)$ [$\because f$ is a homomorphism and $ab^{-1} \in G$]

Thus, $a', b' \in f(G) \Rightarrow a'\,(b')^{-1} \in f(G)$.
Hence, $f(G)$ i.e. image of f is a subgroup of G'. **Hence proved.**

***Q* 5. Define left and right cosets of a subgroup *H* of a group *G*. Prove that**
(i) two rights cosets of *H* in *G* are disjoint or identical.
(ii) $Ha = Hb\, (a, b \in G) \Leftrightarrow ab^{-1} \in H$. **(2012, 07)**

Sol. **Part I Left and Right Cosets** Suppose G is a group and H is any subgroup of G. Let a be any element of G. Then, the set $Ha = \{ha : h \in H\}$ is called a right coset of H in G generated by a. Similarly, the set $aH = \{ah : h \in H\}$ is called a left coset of H in G generated by a.

Part II

(i) Suppose H is a subgroup of a group G and let Ha and Hb be two right cosets of H in G. Suppose Ha and Hb are not disjoint. Then there exists at least one element, say c, such that $c \in Ha$ and $c \in Hb$.

Let $c = h_1 a$ and $c = h_2 b$, where $h_1, h_2 \in H$.

Then, $h_1 a = h_2 b \Rightarrow h_1^{-1} h_1 a = h_1^{-1} h_2 b$

$\Rightarrow \quad ea = (h_1^{-1} h_2) b \Rightarrow a = (h_1^{-1} h_2) b$

Since, H is a subgroup, therefore $h_1^{-1} h_2 \in H$.

Let $h_1^{-1} h_2 = h_3$. Then, $a = h_3 b$.

Now, $Ha = Hh_3 b = (Hh_3) b = Hb$. [$\because h_3 \in H \Rightarrow Hh_3 = H$]

Therefore, the two right cosets are identical if they are not disjoint.

Thus, either $Ha \cap Hb = \phi$ or $Ha = Hb$.

Similarly, we can prove that either $aH \cap bH = \phi$ or $aH = bH$.

(ii) Since, a is an element of Ha, therefore

$$Ha = Hb \Rightarrow a \in Hb \Rightarrow ab^{-1} \in (Hb)b^{-1}$$

$\Rightarrow$ $ab^{-1} \in H\ (bb^{-1}) \Rightarrow ab^{-1} \in He \Rightarrow ab^{-1} \in H$

Conversely $ab^{-1} \in H \Rightarrow Hab^{-1} = H$ $[\because h \in H \Rightarrow Hh = H]$

$\Rightarrow$ $Hab^{-1}b = Hb$

$\Rightarrow$ $Hae = Hb \Rightarrow Ha = Hb$

***Q* 6. Let $f: G \to G'$ be a group homomorphism and let H be a subgroup of G and H' be a subgroup of G'. Prove that $f(H)$ and $f^{-1}(H')$ are subgroup of G' and G, respectively.** **(2000)**

Sol. Let $h_1, h_2 \in H$. Then, $f(h_1), f(h_2) \in f(H) \subseteq G'$

Now, we have $h_1, h_2 \in H \Rightarrow h_1, h_2^{-1} \in H \Rightarrow h_1 h_2^{-1} \in H$

$\Rightarrow$ $f(h_1 h_2^{-1}) \in f(H)$ $[\because f$ is homomorphism$]$

$\Rightarrow$ $f(h_1)\, f(h_2^{-1}) \in f(H)$ $[\because f$ is homomorphism$]$

$\Rightarrow$ $f(h)[f(h_2)]^{-1} \in f(H)$

Hence, $f(H)$ is a subgroup of G'.

Again, let $a, b \in f^{-1}(H') \subseteq G$. Then, $f(a), f(b) \in H'$

$\Rightarrow$ $f(a), [f(b)]^{-1} \in H'$

$\Rightarrow$ $f(a)[f(b)]^{-1} \in H'$ $[\because H'$ is a subgroup$]$

$\Rightarrow$ $f(a)\, f(b^{-1}) \in H'$ $[\because f$ is a homomorphism$]$

$\Rightarrow$ $f(ab^{-1}) \in H'$ $[\because f$ is a homomorphism$]$

$\Rightarrow$ $ab^{-1} \in f^{-1}(H')$

Then, $a, b \in f^{-1}(H') \Rightarrow ab^{-1} \in f^{-1}(H')$

Hence, $f^{-1}(H')$ is a subgroup of G. **Hence proved.**

***Q* 7. Let g be a definite element of a group G. Prove that $\phi: G \to G'$ defined by $\phi(a) = g^{-1}ag, \forall a \in G$ is an isomorphism.** **(2005)**

Sol. (i) **ϕ is a homomorphism**

Since, if $a, b \in G$, then

$\phi(ab) = g^{-1}(ab)\, g = g^{-1}(ae\,b)g$, where e is the identity element in G.

$= g^{-1}\,(agg^{-1}b)\, g = (g^{-1}ag)\,(g^{-1}bg)$ [by associativity]

$= \phi(a)\,\phi(b)$

$\therefore \phi(ab) = \phi(a)\,\phi(b)$

(ii) **ϕ is one-one**

Let $a, b \in G$. Then, we have $\phi(a) = \phi(b)$

$\Rightarrow$ $g^{-1}ag = g^{-1}\, bg$

$\Rightarrow$ $a = b$ [by left and right cancellation laws]

$\therefore\ \phi(a) = \phi(b) \Rightarrow a = b$

Therefore, ϕ is one-one.

(iii) ϕ **is onto** Let $b \in G'$, then there exists an element $gbg^{-1} \in G$ such that

$$f(gbg^{-1}) = g^{-1}(gbg^{-1})g$$
$$= (g^{-1}g)\,b\,(g^{-1}g) \quad \text{[by associativity]}$$
$$= (e)\,b\,(e) = b$$

$\therefore$ ϕ is onto.

Hence, f is an isomorphism, i.e. $G \cong G'$. **Hence proved.**

***Q* 8. State and prove Lagrange's theorem.** **(2017)**

Or **Prove that the order of a subgroup of a finite group is a divisor of the order of the group.** **(2015)**

Or **Define index of a subgroup in a group. Prove that the order and index of a subgroup of a finite group are divisiors of the order of the group.**

Sol. **Part I Index of Subgroup in a Group** If H is a subgroup of a group G, then the number of distinct right (left) cosets of H in G is called the index of H in G and is denoted by $[G : H]$.

Part II Statement The order of each subgroup of a finite group is a divisor of the order of the group.

Proof Let G be a group of finite order n and H be a subgroup of G.

Let $o(H) = m$.

Suppose $h_1, h_2, \ldots, h_m$ are the m elements of H.

Let $a \in G$. Then, Ha is a right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, \ldots, h_m a\}$$

Ha has m distinct elements, since, $h_i a = h_j a \Rightarrow h_i = h_j$. Therefore, each right coset of H in G has m distinct elements. Any two distinct right cosets of H in G are disjoint, i.e. they have no element in common. Since, G is a finite group, the number of distinct right cosets of H in G will be finite, say equal to k. The union of these k distinct right cosets of H in G is equal to G.

Therefore, if $Ha_1, Ha_2, \ldots, Ha_k$ are the k distinct right cosets of H in G, then $G = Ha_1 \cup Ha_2 \cup Ha_3 \cup \ldots \cup Ha_k$.

Thus, the number of elements in G = the number of elements in Ha_1 + the number of elements in Ha_2 +... + number of elements in Ha_k.

[$\because$ two distinct right cosets are mutually disjoint]

$$\Rightarrow \quad o(G) = km \Rightarrow n = km$$

$$\Rightarrow \quad k = \frac{n}{m} \Rightarrow m \text{ is a divisor of } n.$$

Hence, $o(H)$ is a divisor of $o(G)$.

Here, k is the index of H in G. We have $m = \frac{n}{k}$. Thus, k is a divisor of n.

Therefore, the index of every subgroup of a finite group is a divisor of the order of the group. **Hence proved.**

Q 9. Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G. **(2012)**

Sol. Let H and K be two normal subgroups of a group G.

Since, H and K are subgroups of G, therefore $H \cap K$ is also a subgroup of G.

Now, to prove that $H \cap K$ is a normal subgroup of G.

Let x be any element of G and n be any element of $H \cap K$.

We have, $n \in H \cap K \Rightarrow n \in H, n \in K$

Since, H is a normal subgroup of G, therefore

$$x \in G, n \in H \Rightarrow xnx^{-1} \in H$$

Similarly, $xnx^{-1} \in K$

Now, we have $xnx^{-1} \in H, xnx^{-1} \in K \Rightarrow xnx^{-1} \in H \cap K$.

Thus, we have $x \in G, n \in H \cap K \Rightarrow xnx^{-1} \in H \cap K$.

Hence, $H \cap K$ is a normal subgroup of G. **Hence proved.**

Q 10. Show that every subgroup of an abelian group is normal.

Sol. Let G be an abelian group and H be a subgroup of G.

Again, let x be any element of G and h be any element of H.

We have, $xhx^{-1} = xx^{-1}h$ $[\because G$ is abelian $\Rightarrow x^{-1}h = hx^{-1}]$

$$= eh = h \in H$$

Thus, $x \in G, h \in H \Rightarrow xhx^{-1} \in H$.

Hence, H is normal in G. **Hence proved.**

Q 11. Prove that a subgroup N of a group G is normal iff the product of two right cosets of N in G is a right coset of N in G. **(2010, 06)**

Sol. Let N be a normal subgroup of a group G. Let a, b be any two elements of G.

Then, Na and Nb are two right cosets of N in G. We have,

$$(Na)(Nb) = N(aN)b = N(Na)b \quad [\because N \text{ is normal} \Rightarrow Na = aN]$$

$$= NNab = Nab \quad [\because NN = N]$$

Since, $a \in G, b \in G \Rightarrow ab \in G$, therefore Nab is also a right coset of N in G.

Thus, the product of the right cosets Na and Nb is the right coset Nab.

Conversely Let N be a subgroup of G such that the product of two right cosets of N in G is again a right coset of N in G. Let x be any element of G. Then, $x^{-1} \in G$. Therefore, Nx and Nx^{-1} are two right cosets of H in G.

Consequently, by hypothesis $NxNx^{-1}$ is also a right coset of N in G.

Since, $e \in N$, therefore $ex\, ex^{-1} = e$ is an element of the right coset $Nx\, Nx^{-1}$.

But N itself is a right coset of N in G and $e \in N$.

Also, if two right cosets have one element common they must be identical. Therefore, we must have

$$Nx\, Nx^{-1} = N, \forall\, x \in G$$

$\Rightarrow \quad n_1xnx^{-1} \in N, \forall\, x \in G$ and $\forall\, n_1, n \in N$

$\Rightarrow \quad n_1^{-1}(n_1xnx^{-1}) \in n_1^{-1}N, \forall\, x \in G$ and $\forall\, n_1, n \in N$

$\Rightarrow \quad xnx^{-1} \in N, \forall\, x \in G$ and $\forall\, n \in N$

$[\because n_1^{-1}N = N$ as $n_1^{-1} \in N$, since $n_1 \in N]$

Hence, N is a normal subgroup of G. **Hence proved.**

Q 12. Prove that a subgroup H of a group G is a normal subgroup of G iff each left coset of H in G is a right coset of H in G. **(2014)**

Sol. Let H be a normal subgroup of G.

Then, $\quad xHx^{-1} = H, \forall\, x \in G \Rightarrow (xHx^{-1})x = Hx, \forall\, x \in G$

$\Rightarrow \quad xH = Hx, \forall\, x \in G$

Therefore, each left coset xH is the right coset Hx.

Conversely Suppose that each left coset of H in G is a right coset of H in G. Let x be any element of G.

Then, $xH = Hy$ for some $y \in G$

Since, $e \in H$, therefore $xe = x \in xH$

$\therefore \quad x \in Hy \quad [\because Hy = Hx]$

But $\quad x \in Hy \Rightarrow Hx = Hy$

$\therefore \quad Hx = xH \quad [\because Hy = xH]$

Thus, we have $\quad xH = Hx, \forall\, x \in G \Rightarrow xHx^{-1} = Hxx^{-1}, \forall\, x \in G$

$\Rightarrow xHx^{-1} = H, \forall x \in G \Rightarrow H$ is a normal subgroup of G.

Hence, H is a normal subgroup of $G \Leftrightarrow xH = Hx, \forall\, x \in G$. **Hence proved.**

Q 13. Define a normal subgroup of a group. If H is a subgroup of a group G and N is a normal subgroup of G. Show that $H \cap N$ is a normal subgroup of H. **(2017, 11, 08)**

Sol. Part I Normal Subgroup A subgroup H of a group G is said to be a normal subgroup of G, if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

Part II Since, H and N are subgroups of G, therefore $H \cap N$ is also a subgroup of G. Also, we have $H \cap N \subseteq H$. Therefore, $H \cap N$ is a subgroup of H.

Now, to show that $H \cap N$ is a normal subgroup of H.

Let x be any element of H and a be any element of $H \cap N$. Then, $a \in H$ and $a \in N$. Since, N is a normal subgroup of G, therefore we have $xax^{-1} \in N$. Also, H is a subgroup of G. Therefore, we have

$$x \in H, a \in H \Rightarrow xax^{-1} \in H.$$

Thus, $xax^{-1} \in H \cap N$.

Now, we have shown that $x \in H, a \in H \cap N \Rightarrow xax^{-1} \in H \cap N$.

Consequently, $H \cap N$ is a normal subgroup of G. **Hence proved.**

Q 14. If N and M are normal subgroups of G. Prove that NM is also a normal subgroup of G.

Sol. We know that a normal subgroup is commutative with every complex. Therefore, we have $NM = MN$. Now, N and M are two subgroups of G such that $NM = MN$. Therefore, NM is a subgroup of G.

Now, to show that NM is a normal subgroup of G. Let x be any element of G and nm be any element of NM. Then, $n \in N, m \in M$ and we have

$$x(nm)x^{-1} = (xnx^{-1})(xmx^{-1}) \in NM$$

$$[\because N \text{ is normal} \Rightarrow xnx^{-1} \in N \text{ and } M \text{ is normal} \Rightarrow xmx^{-1} \in M]$$

Hence, NM is a normal subgroup of G. **Hence proved.**

***Q* 15. Define symmetric group S_n and alternating group A_n. Also, prove that A_n is a normal subgroup of S_n.** **(2015)**

Sol. **Part I Symmetric Group** Set of all permutations on n symbols forms a group with respect to permutation multiplication. This groups is called symmetric group and it is denoted by S_n. If $n \leq 2$, then S_n is abelian and if $n \geq 3$, then S_n is always non-abelian.

e.g. Symmetric group S_3 on three symbols 1, 2, 3 has $3! = 6$ elements.

$$S_3 = \{I, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

Alternating Group Set of all even permutations on n symbols forms a subgroup of symmetric group S_n. This subgroup is called alternating group and it is denoted by A_n.

e.g. Alternating group A_3 on three symbols 1, 2, 3 has $\frac{3!}{2} = 3$ elements. A_3 is the set of all even permutations on three symbols.

Hence, $A_3 = \{I, (1\ 3\ 2), (1\ 3\ 2)\}$.

Part II Let α be any element of S_n and β any element of A_n.

Then, β is an even permutation and α may be odd or even. We claim that $\alpha\beta\alpha^{-1}$ is an even permutation.

If α is odd, then α^{-1} is also odd. Now, $\alpha\beta$ is odd and consequently $\alpha\beta\alpha^{-1}$ is even.

If α is even, then α^{-1} is also even. Now, $\alpha\beta$ is even and consequently $\alpha\beta\alpha^{-1}$ is even.

Thus, $\alpha \in S_n,\ \beta \in A_n \Rightarrow \alpha\beta\alpha^{-1} \in A_n$.

Hence, A_n is a normal subgroup of S_n. **Hence proved.**

***Q* 16. Define index of a subgroup in a group *G*. If *G* is a group and *H* is a subgroup of index 2 in *G*. Prove that H is a normal subgroup of *G*.** **(2016)**

Sol. **Part I** See the Part I of Q. 8.

Part II Let H be a subgroup of index 2 in a group G. Then, the number of distinct right (left) cosets of H in G is 2. Let x be any element of G.

If $x \in H$, then we have $xH = H = Hx$.

If $x \notin H$, then the right coset Hx is distinct from H and the left coset xH is distinct from H. But H is of index 2, therefore the number of distinct right (left) cosets in right (left) coset decomposition of G will be 2.

Therefore, the cosets H, Hx, xH are such that $G = H \cup Hx = H \cup xH$. But there is no element common to H and Hx and also there is no element common to H and xH. Therefore, we must have $Hx = xH$. Thus, we have $Hx = xH, \forall\, x \in G$.

Hence, H is a normal subgroup of G. **Hence proved.**

***Q* 17. If *G* is a group and *N* is a normal subgroup of *G*, prove that the set of all cosets of *N* in *G* is a group with respect to the multiplication of cosets.** **(2016)**

***Or* Let *N* be a normal subgroup of a group *G*. Prove that the collection of all right cosets of *N* in *G* is a group with respect to multiplication of cosets defined by $(Na)(Nb) = Nab, \forall\, a, b \in G$.**

(2008, 04)

Sol. We have, $\frac{G}{N} = \{Na : a \in G\}$.

Closure property Let $a, b \in G$. Then,

$$(Na)(Nb) = N(aN)b = N(Na)b = NNab = Nab.$$

Since, $ab \in G$, therefore Nab is also a coset of N in G. So, $Nab \in G/N$.
Thus, G/N is closed with respect to coset multiplication.

Associativity Let $a, b, c \in G$, then $Na, Nb, Nc \in G/N$

We have, $Na\,[(Nb)(Nc)] = Na\,(Nbc)$

$$= Na(bc) = N(ab)c = (Nab)\,Nc = [(Na)(Nb)]Nc$$

Thus, $[(Na)(Nb)]\,Nc = Na\,[(Nb)(Nc)], \forall\, Na, Nb, Nc \in G/N$

Therefore, the product in G/N satisfies the associative law.

Existence of identity We have, $N = Ne \in G/N$.

Also, if Na is any element of G/N, then

$$N\,(Na) = (Ne)(Na) = Nea = Na$$

and $$(Na)N = (Na)(Ne) = Nae = Na$$

Thus, $$N\,(Na) = Na = (Na)\,N, \forall\, Na \in G/N$$

Therefore, the coset N is the identity element of G/N.

Existence of inverse Let $Na \in G/N$, then $Na^{-1} \in G/N$.

We have $$(Na)(Na^{-1}) = Naa^{-1} = Ne = N$$

and $$(Na^{-1})(Na) = Na^{-1}a = Ne = N$$

Thus, $$(Na^{-1})(Na) = N = (Na)(Na^{-1})$$

Therefore, the coset Na^{-1} is the inverse of Na.

Thus, each element of G/N possesses inverse.

Hence, G/N is a group with respect to product of cosets. **Hence proved.**

Q 18. Define a normal subgroup. Prove that the kernel of the homomorphism $f : G \rightarrow G'$ is a normal subgroup of G.

Or **Let f be a homomorphism of a group G into a group G'. Define kernel of f and prove that it is a normal subgroup of G.** **(2009)**

Or **If $f : G \rightarrow G$ is a group homomorphism and $\ker f = K$, then prove that K is a normal subgroup of G.** **(2005)**

Sol. **Part I** See the Part I of Q. 13.

Part II Let f be a homomorphism of a group G into a group G'. Let e, e' be the identities of G and G', respectively.

Let K be the kernel of f. Then, $K = \{ x \in G : f(x) = e' \}$.

Since, $f(e) = e'$, therefore at least $e \in K$. Thus K is not empty.

Let $a_1, a_2 \in K$. Then, $f(a_1) = e', f(a_2) = e'$.

We have, $f(a_1 a_2^{-1}) = f(a_1)\, f(a_2^{-1})$

$$= f(a_1)\, [f(a_2)]^{-1} = e'\, e'^{-1} = e'\, e' = e'.$$

$\therefore a_1\, a_2^{-1} \in K$.

Thus, $a_1, a_2 \in K \Rightarrow a_1 a_2^{-1} \in K$.

Therefore, K is subgroup of G. Now, to prove that K is normal in G. Let g be any element of G and k be any element of K, then $f(k) = e'$.

We have $f(gkg^{-1}) = f(g)\, f(k)\, f(g^{-1}) = f(g) e' [f(g)]^{-1}$

$$= f(g)\, [f(g)]^{-1} = e'$$

$\therefore gkg^{-1} \in K$

Thus, $g \in G, k \in K \Rightarrow gkg^{-1} \Rightarrow K$

Hence, K is a normal subgroup of G. **Hence proved.**

Q 19. If Z denote the centre of a group G and G/Z is cyclic, then prove that G is abelian.

Sol. It is given that $\frac{G}{Z}$ is cyclic. Let Zg be a generator of cyclic group $\frac{G}{Z}$, where $g \in G$. Let $a, b \in G$. Then, we have to show that $ab = ba$.

Since, $$a \in G \Rightarrow Za \in \frac{G}{Z}$$

But $\frac{G}{Z}$ is cyclic having Zg as a generator, so there exists some integer m such that

$$Za = (Zg)^m = Zg^m \qquad [\because Z \text{ is a normal subgroup of } G]$$

Now, $a \in Za$, therefore $Za = Zg^m$

$\Rightarrow$ $$a \in Zg^m \Rightarrow a = z_1 g^m \text{ for some } z_1 \in Z$$

Similarly, $b = z_2 g^n$, where $z_2 \in Z$ and n is some integer.

Now, $ab = (z_1g^m)(z_2g^n) = z_1g^mz_2g^n$

$= z_1z_2g^mg^n$ $\quad [\because z_2 \in Z \Rightarrow z_2g^m = g^mz_2]$

$= z_1z_2g^{m+n}$

Again, $ba = z_2g^nz_1g^m = z_2z_1g^ng^m = z_1z_2g^{m+n}$ $\quad [\because z_1 \in Z \Rightarrow z_1z_2 = z_2z_1]$

$\therefore \quad ab = ba$

Since, $ab = ba, \forall\, a, b \in G$, therefore G is abelian. **Hence proved.**

Long Answer Questions

Q 1. State and prove Cayley's theorem. **(2007, 05)**

Sol. **Statement** Every finite group G is isomorphic to a permutation group.

Proof Let G be a finite group. If $a \in G$, then for every x in G the product ax is also an element of group G. Now, consider the function f_a from G into G defined by

$$f_a(x) = ax, \forall\, x \in G$$

The function f_a is one-one, because if $x, y \in G$, then

$$f_a(x) = f_a(y) \Rightarrow ax = ay$$

$\Rightarrow \quad x = y$ $\quad$ [by using left cancellation law in G]

The function f_a is also onto because if x is any element of G, then there exists an element $a^{-1}x$ in G such that

$$f_a(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = ex = x.$$

Thus, f_a is a one-one function from G onto G. Therefore, f_a is a permutation on G. Let G' denotes the set of all such one-one onto functions defined on G corresponding to every element of G, i.e.

$G' = \{f_a : a \in G\}$.

Now, first we will show that G' is a group with respect to the operation known as composite or product of two functions.

(i) **Closure property** Let $f_a, f_b \in G'$, where $a, b \in G$. From our definition of product of two functions, we have

$$(f_af_b)(x) = f_a[f_b(x)] = f_a(bx) = a(bx) = (ab)x$$
$$= f_{ab}(x), \forall\, x \in G.$$

Therefore, by the definition of equality of two functions, we have $f_af_b = f_{ab}$.

Since, $ab \in G$, therefore $f_{ab} \in G'$ and thus G' is closed with respect to the product of functions.

(ii) **Associativity** let $f_a, f_b, f_c \in G'$, where $a, b, c \in G$.

Then, $\quad f_a(f_bf_c) = f_a f_{bc}$ $\quad [\because f_bf_c = f_{bc}]$

$$= f_{a(bc)} = f_{(ab)c} = f_{(ab)}f_c = (f_af_b)f_c.$$

Therefore, the operation in G' is associative.

(iii) **Existence of identity** If e is the identity of G, then f_e is the identity of G', because for every f_a in G', we have

$$f_e f_a = f_{ea} = f_a \text{ and } f_a f_e = f_{ae} = f_a.$$

(iv) **Existence of inverse** If a^{-1} is the inverse of a in G, then $f_{a^{-1}}$ is the inverse of f_a in G', because

$$f_{a^{-1}} f_a = f_{a^{-1}a} = f_e \text{ and } f_a f_{a^{-1}} = f_{aa^{-1}} = f_e.$$

Thus, G' is a group.

Now, we will show that $G \cong G'$.

Consider the function ϕ from G into G' defined by $\phi(a) = f_a, \forall a \in G$.

ϕ is one-one If $a, b \in G$, then

$$\phi(a) = \phi(b) \Rightarrow f_a = f_b$$

$$\Rightarrow \quad f_a(x) = f_b(x), \forall x \in G$$

$$\Rightarrow \quad ax = bx, \forall x \in G \Rightarrow a = b$$

$\therefore$ ϕ is one-one.

ϕ is onto Let f_a be any element of G'.

Then, $a \in G$ and we have $\phi(a) = f_a$. Therefore, ϕ is onto.

ϕ preserves compositions in G and G' If $a, b \in G$, then

$$\phi(ab) = f_{ab} = f_a f_b = \phi(a)\phi(b)$$

$\therefore$ ϕ preserves compositions in G and G'.

Hence, $G \cong G'$. **Hence proved.**

***Q* 2. Prove that in the set of all groups the relation of isomorphism is an equivalence relation.**

***Or* Prove that the relation of being isomorphic in the set of all groups is an equivalence relation.** **(2016, 11)**

Sol. We will prove that the relation of isomorphism denoted by $\cong$ in the set of all groups is reflexive, symmetric and transitive.

Reflexive If G is any group, then $G \cong G$. Let f be the identity mapping on G i.e. let $f : G \to G$ such that $f(x) = x, \forall x \in G$. Clearly, f is one-one onto. Also if x, y are any elements of G, then

$$f(x) = x \text{ and } f(y) = y.$$

Also, $\quad f(xy) = xy = f(x)f(y) \quad$ [$\because f$ is identity mapping]

$\therefore f$ is composition preserving also. Thus, f is an isomorphism of G onto G. Hence, $G \cong G$.

Symmetric Suppose a group G is isomorphic to another group G'. Let f be an isomorphism of G onto G'. Then, f is one-one onto and preserves compositions in G and G'. Since, f is one-one onto, therefore, it is invertible, i.e. f^{-1} exists. Also, we know that the mapping f^{-1} is also one-one onto.

Now, we will show that the mapping $f^{-1}:G' \to G$ is also composition preserving.

Let x', y' be any elements of G'. Then, there exist elements $x, y \in G$ such that

$$f^{-1}(x') = x, f^{-1}(y') = y \quad \text{...(i)}$$

and $$f(x) = x', f(y) = y' \quad \text{...(ii)}$$

Now, $$f^{-1}(x'y') = f^{-1}[f(x) f(y)] \quad \text{[from Eq. (ii)]}$$
$$= f^{-1}[f(xy)] \quad [\because f(xy) = f(x)f(y)]$$
$$= xy$$
$$= f^{-1}(x') f^{-1}(y')$$

Therefore, f^{-1} preserves compositions in G' and G.

Hence, $G' \cong G$

Transitive Suppose G is isomorphic to G' and G' is isomorphic to G''. Further suppose that $f: G \to G'$ and $g: G' \to G''$ are the respective isomorphic mappings.

We know that the composite mapping $gof : G \to G''$ defined by

$$(gof)(x) = g[f(x)], \forall x \in G$$

is also one-one onto if both f and g are one-one onto.

Further, if x and y are any elements of G, then

$$(gof)(xy) = g[f(xy)]$$
$$= g[f(x) f(y)] \quad [\because f \text{ is composition preserving}]$$
$$= g[f(x)] g[f(y)] \quad [\because g \text{ is an isomorphism}]$$
$$= [(gof)(x)][(gof)(y)]$$

Hence, gof preserves compositions in G and G''.

Therefore, gof is an isomorphism of G onto G'', i.e. $G \cong G''$.

Hence, the relation of isomorphism in the set of all groups is an equivalence relation. **Hence proved.**

Q 3. Show that the mapping $f: G \to G$ given by $f(x) = x^{-1}, \forall x \in G$ is an isomorphism iff G is abelian. **(1996)**

Sol. Let G be an abelian group.

(i) **f is one-one** Let $x, y \in G$. Then, we have

$$f(x) = f(y)$$

$\Rightarrow$ $$x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1} \Rightarrow x = y$$

Thus, $$f(x) = f(y) \Rightarrow x = y$$

$\therefore$ f is one-one.

(ii) **f is onto** If $x \in G$, then $x^{-1} \in G$ and we have $f(x^{-1}) = (x^{-1})^{-1} = x$.

$\therefore$ f is onto.

(iii) Let $x, y \in G$. Then, we have

$$f(xy) = (xy)^{-1} = y^{-1} x^{-1}$$
$$= f(y) f(x) = f(x) f(y) \quad [\because G \text{ is an abelian group}]$$

Thus, $f(xy)=f(x)\,f(y),\ \forall\ x, y \in G$

$\therefore$ f is homomorphism.

Hence, f is an isomorphism.

Conversely Let f is an isomorphism. Then, for all $x, y \in G$, we have

$$f(xy)=f(x)\,f(y)$$

$$\Rightarrow \quad (xy)^{-1}=x^{-1}y^{-1} \Rightarrow y^{-1}x^{-1}=x^{-1}y^{-1}$$

$$\Rightarrow \quad (y^{-1}x^{-1})^{-1}=(x^{-1}y^{-1})^{-1} \Rightarrow xy=yx, \forall\ x, y \in G$$

Hence, G is an abelian group. **Hence proved.**

***Q* 4. Define homomorphism, kernel of a homomorphism and an isomorphism of groups. State and prove the fundamental theorem of group homomorphism. (2015)**

***Or* Prove that every homomorphic image of a group G is isomorphic to the quotient group G/K, where K is the kernel of the homomorphism. (2013)**

***Or* State and prove fundamental theorem of group homomorphism. (2018)**

***Sol.* Part I**

(i) **Homomorphism**

(a) **Homomorphism into** A mapping f from group G into a group G' is said to be a homomorphism of G into G', if

$$f(ab)=f(a)\,f(b), \forall\ a, b \in G.$$

(b) **Homomorphism onto** A mapping f from a group G onto a group G' is said to be a homomorphism of G onto G', if

$$f(ab)=f(a)\,f(b), \forall\ a, b \in G$$

Also, then G' is said to be homomorphic image of G.

(c) **Endomorphism** A homomorphism of a group into itself is called an endomorphism.

(ii) **Kernel of Homomorphism** If $f: G \to G'$ is a group homomorphism, then the set K of all those elements of G which are mapped onto the identity e' of G' is called the kernel of homomorphism f.

Thus, $\ker f = K = \{x \in G : f(x)=e'$ where e' the identity of $G'\}$
$= f^{-1}(\{e'\})$.

(iii) **Isomorphic Group** Suppose G and G' are two groups. Also, suppose that the compositions in both G and G' have been denoted multiplicatively. Then, group G is isomorphic to the group G', if there exists a one-to-one mapping f of G onto G' such that

$$f(ab)=f(a)\,f(b), \forall\ a, b \in G$$

i.e. the mapping f preserves the compositions in G and G'.

If the group G is isomorphic to the group G'. Symbolically, we write $G \cong G'$.

Part II Statement Every homomorphic image of a group G is isomorphic to some quotient group of G.

Proof Let G' be the homomorphic image of a group G and f be the corresponding homomorphism. Then, f is a homomorphism of G onto G'. Let K be the kernel of this homomorphism.

Then, K is a normal subgroup of G. We will prove that $\frac{G}{K} \cong G'$.

If $a \in G$, then $Ka \in \frac{G}{K}$ and $f(a) \in G'$.

Consider the mapping $\phi : \frac{G}{K} \to G'$ such that

$$\phi(Ka) = f(a), \forall a \in G.$$

First we will show that the mapping ϕ is well-defined, i.e. if $a, b \in G$ and $Ka = Kb$, then $\phi(Ka) = \phi(Kb)$.

We have, $Ka = Kb \Rightarrow ab^{-1} \in K$

$\Rightarrow \quad f(ab^{-1}) = e'$ (identity of G') $\Rightarrow f(a)\, f(b^{-1}) = e'$

$\Rightarrow \quad f(a)\,[f(b)]^{-1} = e' \Rightarrow f(a)\,[f(b)]^{-1} f(b) = e'\, f(b)$

$\Rightarrow \quad f(a)\, e' = f(b) \Rightarrow f(a) = f(b)$

$\Rightarrow \quad \phi(Ka) = \phi(Kb)$

$\therefore$ ϕ is well-defined.

ϕ is one-one We have, $\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$

$\Rightarrow \quad f(a)\,[f(b)]^{-1} = f(b)\,[f(b)]^{-1}$

$\Rightarrow \quad f(a)\, f(b^{-1}) = e' \Rightarrow f(ab^{-1}) = e'$

$\Rightarrow \quad ab^{-1} \in K \Rightarrow Ka = Kb$

$\therefore$ ϕ is one-one.

ϕ is onto G' Let y be any element of G'. Then, $y = f(a)$ for some $a \in G$, because f is onto G'.

Now, $Ka \in \frac{G}{K}$ and we have

$$\phi(Ka) = f(a) = y$$

$\therefore$ ϕ is onto G'.

Finally, we have $\phi[(Ka)\,(Kb)] = \phi(Kab) = f(ab)$

$$= f(a)\, f(b) = \phi(Ka)\, \phi(Kb)$$

$\therefore$ ϕ is an isomorphism of $\frac{G}{K}$ onto G'.

Hence, $\frac{G}{K} \cong G'$. **Hence proved.**