Chapter Ten

SOME SPECIAL CURVES ON A SURFACE

Important Points from the Chapter

1. **Lines of Curvature** A curve on a surface is called a line of curvature if the tangent at any point of it is along the principal direction at that point.
2. **Equation of Line of Curvature** Since, the direction of line of curvature at any point is along the direction at that point, therefore the differential equation of the two system of line of curvature, is the same as the equation of the principal direction, so their equation is
$$\varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta} du^{\gamma} du^{\delta} = 0.$$
3. **Conjugate Directions** Let P and Q be two neighbouring points of curve on a surface and PR be a line parallel to the line of intersection L of tangent planes at P and Q, then the limiting position of the directions PQ and PR as Q tends to P are called conjugate direction at P. **(2008, 06, 04, 1999)**
4. **Equation of Conjugate Direction** The equation of conjugate direction is given by $d_{\alpha\beta} du^{\alpha} \delta u^{\beta} = 0$.
5. **Asymptotic Lines** The directions which are self conjugate, are called asymptotic direction and the curves whose tangents are along asymptotic directions, are called asymptotic lines.
   The differential equation of asymptotic line is $d_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0$. **(2007)**
6. **Null Lines** (Minimal Lines) A curve on a surface of zero length is called null lines or minimal lines.
   The differential equation of the null lines is $g_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0$ which is obtained by equating to zero the square of the line element.
7. **Isometric Lines** When the metric of a surface assumes the form
   $ds^2 = \lambda\, [(du^1)^2 + (du^2)^2]$
   The parameters u^{α} are known as the isometric parameter, λ being a function of u^{α} or a constant. Such parametric curves are called isometric lines.
8. **Fundamental Equation of Surface Theory**
   (i) $N^i_{,\alpha} = -d_{\alpha\beta}\, g^{\gamma\beta}\, x^i_{,\gamma}$ is known as Weingarten equation.
   (ii) $d_{\alpha\beta,\gamma} - d_{\alpha\gamma,\beta} = 0$ is known as Mainardi-Codazzi equation.
   (iii) $R_{\sigma\alpha\beta\gamma} = d_{\alpha\gamma}\, d_{\beta\sigma} - d_{\alpha\beta}\, d_{\gamma\sigma}$ is known as Gauss characteristic equation.

Very Short Answer Questions

Q 1. Find the necessary and sufficient condition that the parametric curves at a point of a surface have conjugate direction. **(2010, 06)**

Or **Prove that the necessary and sufficient condition that the parametric curves at a point of a surface be conjugate is that $d_{12} = 0$.**

Sol. We have the combined equation of the parametric curves is

$$du^1\, du^2 = 0 \qquad \ldots(\text{i})$$

Now, if the parametric curves are along the conjugate direction, then

$$d^{\alpha\beta} P_{\alpha\beta} = 0 \qquad \ldots(\text{ii})$$

We know that, $P_{\alpha\beta}\, du^\alpha\, du^\beta = 0$ ...(iii)

From Eqs. (i) and (ii), we get $P_{11} = 0, P_{12} + P_{21} \neq 0$ and $P_{22} = 0$

The parametric curves is conjugate iff $d^{\alpha\beta} P_{\alpha\beta} = 0$

$$\Leftrightarrow \quad d^{11} P_{11} + d^{12}(P_{12} + P_{21}) + d^{22} P_{22} = 0 \qquad [\because d^{\alpha\beta} = \alpha^{\beta\alpha}]$$

$$\Leftrightarrow \quad d^{12}(P_{12} + P_{21}) = 0 \qquad [\because P_{12} + P_{21} \neq 0]$$

$$\Leftrightarrow \quad d^{12} = 0 \qquad [\because P_{12} + P_{21} \neq 0]$$

$$\Leftrightarrow \quad -\frac{d_{12}}{\alpha} = 0 \qquad \left[\because d^{12} = \frac{-d_{12}}{\alpha}\right]$$

$$\Leftrightarrow \quad d_{12} = 0$$

Hence proved.

Q 2. Define null lines on a surface and show that at a given point of a surface, there are two imaginary null lines. **(2007)**

Sol. **Part I Null Lines** A curve on a surface of zero length is called null lines or minimal lines. The differential equation of the null lines is $g_{\alpha\beta}\, du^\alpha\, du^\beta = 0$ which is obtained by equating to zero the square of the line element.

Part II We know that the equation of null lines are

$$g_{\alpha\beta}\, du^\alpha\, du^\beta = 0$$

$$\Rightarrow \quad g_{11}(du^1)^2 + 2\, g_{12}\, du^1\, du^2 + g_{22}(du^2)^2 = 0 \qquad [\because g_{12} = g_{21}]$$

$$\Rightarrow \quad g_{11}\left(\frac{du^1}{du^2}\right)^2 + 2\, g_{12}\frac{du^1}{du^2} + g_{22} = 0$$

Now, we have two null lines given by

$$\frac{du^1}{du^2} = -\frac{2\, g_{12} \pm \sqrt{4(g_{12})^2 - 4(g_{11})g_{12}}}{2\, g_{11}} \qquad [\because g = g_{11}\, g_{12} - (g_{12})^2 \neq 0]$$

$$\Rightarrow \quad \frac{du^1}{du^2} = -\frac{g_{12} \pm \sqrt{-g}}{g_{12}} \Rightarrow \text{Null lines are imaginary.}$$

Hence, at a given point of a surface, there are two imaginary null lines.

Short Answer Questions

Q 1. Prove that the necessary and sufficient condition that the lines of curvature is along parametric curves, is that

$$d_{12} = 0, g_{12} = 0 \text{ and } \frac{d_{11}}{g_{11}} \neq \frac{d_{22}}{g_{22}}. \qquad \textbf{(2004, 01)}$$

Sol. The differential equation of line of curvature is

$$\varepsilon^{\alpha\beta}\, g_{\alpha\gamma}\, d_{\beta\gamma}\, du^{\gamma}\, du^{\beta} = 0$$

$$\Rightarrow \quad (g_{11}\, d_{12} - g_{12}\, d_{11})\,(du^1)^2 + (g_{11}\, d_{22} - g_{22} d_{11})\, du^1 du^2 + (g_{12}\, d_{22} - g_{22} d_{12})\,(du^2)^2 = 0 \qquad \text{...(i)}$$

The combined equation of the parametric curves is

$$du^1 du^2 = 0 \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$g_{11}\, d_{12} - g_{12}\, d_{11} = 0 \qquad \text{...(iii)}$$

$$g_{12}\, d_{22} - g_{22}\, d_{12} = 0 \qquad \text{...(iv)}$$

and $$g_{12}\, d_{22} - g_{22}\, d_{11} = 1 \neq 0 \qquad \text{...(v)}$$

On adding Eqs. (iii) and (iv), we get

$$g_{11}\, d_{12}\, d_{22} - g_{22}\, d_{11} d_{12} = 0 \Rightarrow d_{12}(g_{11} d_{22} - g_{22} d_{11}) = 0$$

Hence, $$d_{12} = g_{12} = 0$$

From Eq. (v), $g_{11}\, d_{22} \neq g_{22} d_{11} \Rightarrow \dfrac{d_{11}}{g_{11}} \neq \dfrac{d_{22}}{g_{22}}$

Hence, $$d_{12} = g_{12} = 0 \text{ and } \frac{d_{11}}{g_{11}} \neq \frac{d_{22}}{g_{22}}$$

Conversely Suppose the following condition holds

i.e. $$g_{12} = d_{12} = 0, \frac{d_{11}}{g_{11}} \neq \frac{d_{22}}{g_{22}}$$

Using this in Eq. (i), we get $du^1\, du^2 = 0$

which is the differential equation of parametric curves.

Q 2. Define conjugate directions and prove that if the directions given by $P_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0$ are conjugate, then $d^{\alpha\beta}\, P_{\alpha\beta} = 0$.

(2008, 06, 1999)

Sol. **Part I Conjugate Directions** Let P and Q be two neighbouring points of curve on a surface and PR be a line parallel to the line of intersection L of tangent planes at P and Q, then the limiting position of the directions PQ and PR as Q tends to P are called conjugate direction at P.

Part II Here, the given directions are $P_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0$.

$$\Rightarrow \quad P_{11}\,(du^1)^2 + (P_{12} + P_{21})\, du^1\, du^2 + P_{22}(du^2)^2 = 0$$

$$\Rightarrow \qquad P_{11}\left(\frac{du^1}{du^2}\right)^2 + (P_{12}+P_{21})\frac{du^1}{du^2} + P_{22} = 0$$

Let the two directions be $\frac{du^1}{du^2}$ and $\frac{\delta u^1}{\delta u^2}$.

Then, $$\frac{du^1}{du^2} + \frac{\partial u^1}{\partial u^2} = -\frac{(P_{12}+P_{21})}{P_{11}} \qquad \text{...(i)}$$

and $$\frac{du^1}{du^2}\cdot\frac{\partial u}{\delta u^2} = \frac{P_{22}}{P_{11}} \qquad \text{...(ii)}$$

The two directions $\frac{du^1}{du^2}$ and $\frac{\partial u^1}{\delta u^2}$ (i.e. du^α and δu^α) are conjugate iff

$$d_{\alpha\beta}\, du^\alpha\, \delta u^\beta = 0$$

$$\Rightarrow d_{11}\, du^1\, \delta u^1 + d_{12}\,(du^1\,\delta u^2 + du^2\,\delta u^1) + d_{22} du^2\,\delta u^2 = 0 \qquad [\because d_{12} = d_{21}]$$

$$\Rightarrow \qquad d_{11}\frac{du^1}{du^2}\frac{\delta u^1}{\delta u^2} + d_{12}\left(\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2}\right) + d_{22} = 0$$

$$\Rightarrow \qquad \frac{\partial_{11}P_{22}}{P_{11}} + d_{12}\left[-\frac{(P_{12}+P_{21})}{P_{11}}\right] + d_{22} = 0$$

$$\Rightarrow \qquad d_{11}\frac{P_{22}}{P_{11}} - d_{12}(P_{12}+P_{21}) + d_{22}P_{11} = 0$$

$$\left[\because d^{22} = \frac{d_{11}}{d_1},\ d^{12} = -\frac{d_{12}}{d},\ d^{11} = \frac{d_{22}}{d}\right].$$

$$\Rightarrow \qquad d^{22}P_{22} + d^{12}\,(P_{12}+P_{21}) + d^{11}P_{11} = 0$$

$$\Rightarrow \qquad d^{11}P_{11} + d^{12}\,(P_{12}+P_{21}) + d^{22}P_{22} = 0$$

$$\therefore \qquad d^{\alpha\beta}\, P_{\alpha\beta} = 0$$ **Hence proved.**

***Q 3.* State and prove Mainardi-Codazzi equation for three dimensional space.** **(2009, 06, 2000)**

Sol. **Statement** (i) $d_{\alpha\beta\gamma} - d_{\alpha\gamma,\beta} = 0$

(ii) $R\sigma_{\alpha\beta\gamma} = d_{\alpha\gamma}\, d_{\beta\sigma} - d_{\alpha\beta}\, d_{\gamma\sigma}$, where $R\,\sigma_{\alpha\beta\gamma} = g\,\sigma \in R^{\varepsilon}_{\alpha\beta\gamma}$ is Riemann Christoffel curvature tensor of first kind.

Proof We know that three vectors X_1, X_2 and N form a basis for three dimensional vector space. Therefore, any vector can be expressed as linear combination of these three vectors. The components of vectors X_α $(\alpha = 1, 2)$ are $x^i_{,\alpha}$ and the components of N are N^i.

By Gauss equation, $x^i_{,\alpha\beta} = d_{\alpha\beta}\cdot N^i$ ...(i)

By Weingarten equation, $N^i_{,\alpha} = -\, d_{\alpha\beta}\cdot g^{\beta\gamma}\, x^i_{,\gamma}$ ...(ii)

Taking covariant derivative of Eq. (i) w.r.t. u^γ, we get

$$x^i_{,\alpha\beta\gamma} = -\, d_{\alpha\beta,\,\gamma} N^i + d_{\alpha\,\beta}\, N^i_{,\gamma}$$

$$\Rightarrow \qquad x^i_{,\alpha\beta\gamma} = d_{\alpha\beta,\gamma}\, N^i - d_{\alpha\beta}\, d_{\gamma\delta}\, g^{\delta\rho}\, x^i_{,\rho} \qquad \text{[from Eq. (ii)] ...(iii)}$$

Interchanging $\beta \leftrightarrow \gamma$ in Eq. (iii), we get

$$x^i_{,\alpha\gamma\beta} = d_{\alpha\gamma\beta}N^i - d_{\alpha\gamma}\, d_{\beta\delta} g^{\delta\rho}\, x^i_{,\rho} \quad \text{...(iv)}$$

Now, subtracting Eq. (iii) from Eq. (iv), we get

$$x^i_{,\alpha\beta\gamma} - x^i_{,\alpha\gamma\beta} = (d_{\alpha\beta\gamma} - d_{\alpha\gamma\beta})\, N^i + (d_{\alpha\gamma}\, d_{\beta\delta} - d_{\alpha\beta}\, d_{\gamma\delta})\, g^{\delta\rho}\, x^i_{,\rho}$$

$$\Rightarrow \quad x^i_{,\varepsilon}\, R^{\varepsilon}_{\alpha\beta\gamma} = (d_{\alpha\beta\gamma} - d_{\alpha\gamma\beta})\, N^i + (d_{\alpha\gamma}\, d_{\beta\delta} - d_{\alpha\beta} d_{\gamma\beta\delta})\, g^{\delta\rho} x^i_{,\rho} \quad \text{...(v)}$$

$$\Rightarrow \quad N^i x^i_{,\varepsilon} R^{\varepsilon}_{\alpha\beta\gamma} = (d_{\alpha\beta\gamma} - d_{\alpha\gamma\beta})\, N^i \cdot N^i + (d_{\alpha\beta}\, d_{\beta\delta} - d_{\alpha\beta} d_{\gamma\beta\delta}) g^{\delta\rho} N^i x^i_{,\rho}$$

$$[\because N^i N^i = 1,\ N^i\, x^i_{,\rho} = 0,\ N^i x^i_{,\varepsilon} = 0]$$

$$\Rightarrow \quad 0 = (d_{\alpha\beta\gamma} - d_{\alpha\gamma\beta}) + 0$$

$$\Rightarrow \quad d_{\alpha\beta,\,\gamma} - d_{\alpha\gamma,\,\beta} = 0$$

which is the required Mainardi-Codazzi equation.

Q 4. State and prove Gauss characteristic equation for three dimensional space. **(2001, 1999)**

Sol. Proceed as above Q. 3.

Now, multiplying Eq. (v) of Q. 3 by $x^i_{,\sigma}$, we get

$$x^i_{\sigma}\, x^i_{,\varepsilon}\, R^{\varepsilon}_{\alpha\beta\gamma} = (d_{\alpha\beta,\,\gamma} - d_{\alpha\gamma,\,\beta})$$

$$x^i_{,\varepsilon}\, N^i + (d_{\alpha\gamma}\, d_{\beta\delta} - d_{\alpha\beta}\, d_{\gamma\delta})\, (g^{\delta\rho}\, x_{i,\rho}\, x^i_{,\sigma})\ [\because x^i_{,\sigma}\, x^i_{,\varepsilon} = g_{\sigma\varepsilon} \text{ and } x^i_{,\sigma} N^i = 0]$$

$$\Rightarrow \quad g_{\sigma\varepsilon}\, R^{\varepsilon}_{\alpha\beta\gamma} = 0 + (d_{\alpha\,\beta}\, d_{\beta\delta} - d_{\alpha\beta}\, d_{\gamma\delta})\, g^{\delta\rho}\, g_{\rho\sigma}$$

$$\Rightarrow \quad R_{\sigma\,\alpha\beta\gamma} = (d_{\alpha\gamma}\, d_{\beta\delta} - d_{\alpha\beta}\, d_{\gamma\delta})\, \delta^{\delta}_{\sigma}$$

$$\Rightarrow \quad R_{\sigma\alpha\beta\gamma} = d_{\alpha\gamma}\, d_{\beta\sigma} - d_{\alpha\beta}\, d_{\gamma\sigma}$$

which is the required Gauss characteristic equation.

Q 5. Define asymptotic lines and prove that the necessary and sufficient condition that the asymptotic lines are parametric curves, is that $d_{11} = d_{22} = 0$ and $d_{12} \neq 0$. **(2007)**

Sol. **Part I Asymptotic Lines** The directions which are self conjugate, are called asymptotic direction and the curves whose tangents are along asymptotic directions, are called asymptotic lines.

The differential equation of asymptotic line is $d_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = 0$.

Part II The differential equation of the parametric curves is

$$du^1\, du^2 = 0 \quad \text{...(i)}$$

and the differential equation of asymptotic line is

$$d_{\alpha\,\beta}\, du^{\alpha} du^{\beta} = 0$$

i.e. $\quad d_{11}(du^1)^2 + 2d_{12}\, du^1 du^2 + d_{22}(du^2)^2 = 0 \qquad [\because d_{12} = d_{21}]$ ...(ii)

If asymptotic lines are parametric curves, then by comparison, we have

$$d_{11} = d_{12} = 0 \text{ and } d_{12} \neq 0$$

Conversely Suppose that $d_{11} = d_{22} = 0$ and $d_{12} \neq 0$

Using this in Eq. (ii), we find that

$$du^1\, du^2 = 0$$

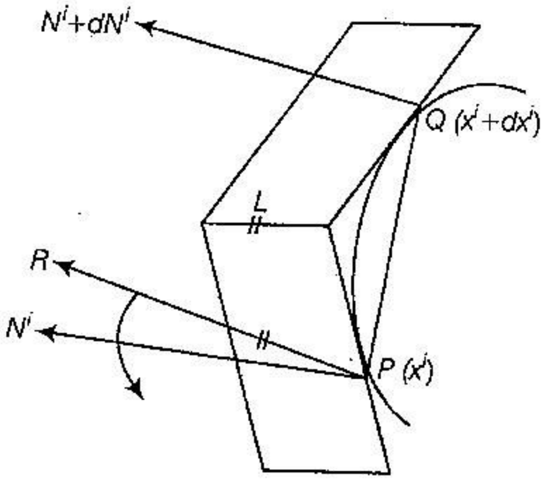
which is differential equation of parametric curves i.e. asymptotic lines are parametric curves.

Q 6. Define conjugate directions and find the equations of conjugate directions. **(2004)**

Or **Find analytical expression for two directions to be conjugate.**

Sol. **Part I Conjugate Directions** Let P and Q be two neighbouring points of curve on a surface and PR be a line parallel to the line of intersection L of tangent planes at P and Q, then the limiting position of the directions PQ and PR as Q tends to P are called conjugate direction at P.

Part II Let x^i, $x^i + dx^i$ be the coordinates of two neighbouring points P and Q of a curve on a surface and N^i, $N^i + dN^i$ be the components of normals at P and Q, respectively.



From the above figure, it is clear that $\delta x^i \parallel PR$.

Since, δx^i lies in tangent plane of the surface at P, then we have

$$N^i, \delta x^i = 0 \quad \text{...(i)}$$

and

$$(N^i + dN^i)\,\delta x^i = 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$dN^i, \delta x^i = 0 \Rightarrow \frac{\delta N^i}{\delta u^\alpha} du^\alpha \frac{\partial x^i}{\partial u^\beta} \delta u^\beta = 0$$

$$\left[\because \delta u^i = \frac{\partial x^i}{\partial u^\beta} \delta u^\beta, dN^i = \frac{\partial N^i}{\partial u^\alpha} du^\alpha\right]$$

$$\Rightarrow \quad N^i_\alpha, x^i_\beta \, du^\alpha \, \delta u^\beta = 0 \quad \text{...(iii)}$$

We know that,

$$d_{\alpha\beta} = -N^i_\alpha \, x^i_\beta \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$-d_{\alpha\beta} \, du^\alpha \, \delta u^\beta = 0$$

$$\Rightarrow \quad d_{\alpha\beta} \, du^\alpha \, \delta u^\beta = 0$$

which is the required equation of conjugate directions.

Long Answer Questions

Q 1. Find the asymptotic lines of the paraboloid of revolution $z = x^2 + y^2$. **(2010)**

Sol. Consider x, y as parameters, then the parametric equation of given surface is

$$x = x, y = y, z = x^2 + y^2 \qquad [\text{here, } x^1 = x, x^2 = y, x^3 = z]$$

$\therefore$ $$X_1 = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) = (1, 0, 2x)$$

and $$X_2 = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) = (0, 1, 2y)$$

Now, $$\frac{\partial^2 x^i}{\partial x\, \partial x} = (\partial_1\partial_1 x, \partial_1\partial_1 y, \partial_1\partial_1 z) = (0, 0, 2),$$

$$\frac{\partial^2 x^i}{\partial x\, \partial y} = (\partial_1\partial_2 x, \partial_1\partial_2 y, \partial_1\partial_2 z) = (0, 0, 0)$$

and $$\frac{\partial^2 x^i}{\partial y\, \partial y} = (\partial_2\partial_2 x, \partial_2\partial_2 y, \partial_2\partial_2 z) = (0, 0, 2)$$

$\therefore$ $$g_{11} = X_1 \cdot X_1 = 1 + 4x^2,$$

$$g_{12} = X_1 \cdot X_2 = 4xy \text{ and } g_{22} = X_2 \cdot X_1 = 1 + 4y^2$$

$\therefore$ $$g = g_{11}g_{22} - (g_{12})^2 = (1 + 4x^2)(1 + 4y^2) - (4xy)^2 = 1 + 4x^2 + 4y^2$$

$$\sqrt{g} = \sqrt{1 + 4x^2 + 4y^2} = |x_1 + x_2|$$

$\therefore$ $$N^i = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \left(\frac{-2x_1, -2y, 1}{\sqrt{1 + 4x^2 + 4y^2}}\right)$$

$$= \left(\frac{-2x}{\sqrt{1 + 4x^2 + 4y^2}}, \frac{-2y}{\sqrt{1 + 4x^2 + 4y^2}}, \frac{1}{\sqrt{1 + 4x^2 + 4y^2}}\right)$$

Now, $$d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i$$

$\therefore$ $$d_{11} = N^1\partial_1\partial_2 x + N^2\partial_1\partial_1 y + N^3\, \partial_1\partial_1 z = 21\sqrt{1 + 4x^2 + 4y^2}$$

$$d_{12} = d_{21} = N^1\partial_1\partial_2 x + N^2\partial_1\partial_2 y + N^3\partial_1\partial_2 z = 0$$

and $$d_{22} = N^1\partial_2\partial_2 x + N^2\partial_2\partial_2 y + N^3\partial_2\partial_2 z = \frac{2}{\sqrt{1 + 4x^2 + 4y^2}}$$

The differential equation of asymptotic line is $d_{\alpha\beta}\, du^\alpha du^\beta = 0$.

$\therefore$ $$d_{11}(du^1)^2 + 2d_{12}du^1du^2 + d_{22}(du^2)^2 = 0 \qquad [\text{here, } u^1 = x, u^2 = y]$$

$\Rightarrow$ $$\frac{2}{\sqrt{11 + 4x^2 + 4y^2}}(dx)^2 + 0 + \frac{2}{\sqrt{1 + 4x^2 + 4y^2}}(dy)^2 = 0$$

$\Rightarrow$ $$(dx)^2 + (dy)^2 = 0 \Rightarrow dx = \pm\, i\, dy \Rightarrow x = \pm\, iy + \lambda$$

Hence, $x = \pm\, iy + \lambda$ are asymptotic lines.

Q 2. State and prove Beltrami and Enneper's theorem.

(2011, 2000)

Sol. **Statement** At a point on a surface where the Gaussian curvature is negative and equal to K, the torsion of the asymptotic line is $\pm\sqrt{-K}$.

Proof The torsion τ of an asymptotic line is given by

$$\tau = d_{\alpha\gamma}\, e_{\beta\delta}\, g^{\gamma\delta} u^{1\alpha}\, u^{1\beta} \qquad \text{...(i)}$$

We suppose that the asymptotic lines are taken as parametric curve.

Then, $\quad d_{12} = 0, d_{22} = 0$

We know that, $\quad e_{12} = -e_{21}, e_{11} = 0 = e_{22}$

$\Rightarrow \quad d_{21} = d_{12}$

Now, from Eq. (i),

$$\tau = d_{12} l_{12} g^{22} u'^1 \cdot u'^1 + d_{21} l_{21} g^{11} u'^2 \cdot u'^2$$

$$\Rightarrow \quad \tau = d_{12} l_{12}\, g^{22} \left(\frac{du^1}{d\delta}\right)^2 - l_{12} d_{12}\, g^{11} \left(\frac{du^1}{d\delta}\right)^2 \qquad \text{...(ii)}$$

We know that, $\quad l_{\beta\delta} = \varepsilon_{\beta\delta}\sqrt{g} \Rightarrow l_{12} = \sqrt{g} \qquad [\because \varepsilon_{12} = 1]$

and $\quad g^{22} = \dfrac{g_{11}}{g}, g^{11} = \dfrac{g_{22}}{g}$

On putting this value in Eq. (ii), we get

$$\tau = d_{12} l_{12} \left[g^{22} \left(\frac{du^1}{ds}\right)^2 - g^{11} \left(\frac{du^2}{ds}\right)^2 \right]$$

$$\Rightarrow \quad \tau = d_{12}\sqrt{g} \left[\frac{g_{11}}{g} \left(\frac{du^1}{ds}\right)^2 - \frac{g^{22}}{g} \left(\frac{du^2}{ds}\right)^2 \right]$$

$$\Rightarrow \quad \tau = \frac{d_{12}}{\sqrt{g}} \left[g_{11} \left(\frac{du^1}{ds}\right)^2 - g_{22} \left(\frac{du^2}{ds}\right)^2 \right] \qquad \text{...(iii)}$$

For a parametric curve, $u^2 = c^2$ and $\dfrac{du^2}{ds} = 0$

$$\therefore \quad g_{11} \left(\frac{du^1}{ds}\right) = 1$$

Therefore, the torsion τ_1 along the parametric curve $u^2 = c^2$ is

$$\tau_1 = \frac{d_{12}}{\sqrt{g}}$$

Similarly, the torsion τ_2 along the parametric curve $u^1 = c^1$ is

$$\tau_2 = -\frac{d_{12}}{\sqrt{g}}$$

Hence, $\tau_1^2 = \tau_2^2 = \left(\frac{d_{12}}{\sqrt{g}}\right)^2 \Rightarrow \tau_1^2 = \tau_2^2 = \frac{d_{12}^2}{g}$ ...(iv)

We know that, the Gaussian curvature K is

$$K = \frac{d}{g} = \frac{d_{11}d_{22} - (d_{12})^2}{g} \Rightarrow K = -\frac{d_{12}^2}{g} \qquad [\because d_{11} = 0, d_{22} = 0] \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$\tau_1^2 = \tau_2^2 = -K \Rightarrow \tau_1 = \pm\sqrt{-K}, \tau_2 = \pm\sqrt{-K}$$

$$\Rightarrow \qquad \tau_1 = \tau_2 = \pm\sqrt{-K}$$

Q 3. State and prove Rodrigue's formula. (1998)

Or Prove that the necessary and sufficient condition that a curve on a surface be line of curvature is that $dN^i + K_n dx^i = 0$ at each of its point. (1992)

Or What do you mean by line of curvature? Prove that a curve on a surface be line of curvature iff at each point of it $dN^i + K_n dx^i = 0$, where K_n denotes the normal curvature. (2014)

Sol. Statement A necessary and sufficient condition that a curve on a surface be line of curvature is that

$$dN^i + K_n dx^i = 0 \qquad \text{...(i)}$$

at each of its points, where K_n denotes the normal curvature.

Proof Let a curve on a surface be line of curvature.

Then, we have $(d_{\alpha\beta} - K_n g_{\alpha\beta})\, du^\beta = 0$

But, we know that, $d_{\alpha\beta} = -x^i, dN^i_{,\beta}$ and $g_{\alpha\beta} = -x^i, dx^i_{,\beta}$ therefore

Then, $(x^i_{,\alpha}\, dN^i_{,\beta} + K_n x^i_{,\alpha}\, x^i_{,\beta})\, du^\beta = 0$

$\Rightarrow \qquad x^i_{,\alpha}\,(dN^i_{,\beta} + K_n\, x^i_{,\beta}) = 0$

Since, $\qquad N^i \cdot N^i = 1$

On differentiating, we get $2N^i dN^i = 0$

i.e. dN^i is perpendicular to N^i.

i.e. dN^i is tangential to the surface.

Also, dx^i is tangent to the surface.

Therefore, $dN^i + Kdx^i$ are components to tangent vector to the surface. Also, $x^i_{,\alpha}$ are the components of tangent vector x_α. Therefore, from Eq. (i), we have

$$dN^i_{,} + K_n\, dx^i = 0$$

Conversely Let us suppose that $dN^i + K_n dx^i = 0$, along any curve on the surface, where K_n is any function, then

$$x^i_{,\alpha}\,(dN^i + K_n dx^i) = 0 \Rightarrow x^i_{,\alpha}(N^i_{,\beta} + K_n x^i_{,\beta})du^\beta = 0$$

$\Rightarrow$ $$(-\,d_{\alpha\beta} + K_n g_{\alpha\beta})\,du^\beta = 0 \Rightarrow (d_{\alpha\beta} - K_n g_{\alpha\beta})\,du^\beta = 0$$

Hence, the curve is a line of surface in case K_n is normal curvature to the surface.

Also, $$(K_n dx^i + dN^i) = 0 \Rightarrow K_n dx^i = -\,dN^i$$

$\Rightarrow$ $$K_n x^i_{,\beta}\,du^\beta = -\,N^i_{,\beta}\,du^\beta$$

Taking inner product with $x^i_{,\alpha}\,du^\alpha$ on both sides, we get

$$K_n x^i_{,\alpha}\,x^i_{,\beta}\,du^\alpha\,du^\beta = -\,N^i_{,\beta}\,x^i_{,\alpha}\,du^\alpha\,du^\beta$$

$\Rightarrow$ $$K_n g_{\alpha\beta}\,du^\alpha du^\beta = d_{\alpha\beta}du^\alpha\,du^\beta$$

$\Rightarrow$ $$K_n = \frac{d_{\alpha\beta}\,du^\alpha\,du^\beta}{g_{\alpha\beta}\,du^\alpha\,du^\beta}$$

Hence, K_n is a normal curvature at the point u^α in the direction du^α.

Q 4. State and prove Euler's theorem. **(2018, 11, 03, 01, 1999)**

Sol. **Statement** If ψ is the angle between a direction at a point P and the principal direction at P corresponding to principal curvature k_1, then the normal curvature K_n in the direction is given by

$$K_n = K_1 \cos^2\psi + K_2 \sin^2\psi$$

Proof Let the line of curvature be taken as parametric curves, then $d_{12} = 0$, $g_{12} = 0$ and normal curvature is

$$K_n = d_{\alpha\beta}\,\frac{du^\alpha}{ds}\cdot\frac{du^\beta}{ds}$$

$\Rightarrow$ $$K_n = d_{11}\left(\frac{du^1}{ds}\right)^2 + d_{22}\left(\frac{du^2}{ds}\right)^2 \qquad \text{...(i)}$$

Then, the principal curvatures K_1 and K_2 are the roots of K_n given by the equation

$$K_n^2 - K_n\,g^{\alpha\beta}\,d_{\alpha\beta} + \frac{d}{g} = 0$$

$\Rightarrow$ $$K_n^2 - K_n\left(\frac{d_{11}}{g_{11}} + \frac{d_{22}}{g_{22}}\right) + \frac{d_{11}d_{22}}{g_{11}g_{22}} = 0$$

[$\because d_{12} = 0,\ g_{12} = 0$]

$\Rightarrow$ $$K_n^2 - K_n\left(\frac{d_{11}}{g_{11}} + \frac{d_{22}}{g_{22}}\right) + \frac{d_{11}d_{22}}{g_{11}g_{22}} = 0$$ [$\because d_{12} = 0,\ g_{12} = 0$]

$\Rightarrow$ $$\left(K_n - \frac{d_{11}}{g_{11}}\right)\left(K_n - \frac{d_{22}}{g_{22}}\right) = 0$$

$\Rightarrow$ $$K_1 = \frac{d_{11}}{g_{11}},\ K_2 = \frac{d_{22}}{g_{22}} \qquad \text{...(ii)}$$

Let C be a curve on the surface in the given direction along which components of unit tangent vector is x'^{i}.

We know that, $\partial_1 x^i$ are the components of tangent vector x_1 along the parametric curve $du^2 = 0$.

Since, ψ is the angle between given direction and parametric curve $du^2 = 0$ whose magnitude of vector x_1 is $\sqrt{g_{11}}$.

$$\therefore \quad \partial_1 x^i \frac{dx^i}{ds} = \sqrt{g_1} \cos \psi$$

$$\Rightarrow \quad x^i_{,1} x^i_{,\alpha} \frac{du^\alpha}{ds} = \sqrt{g_{11}} \cos \psi$$

$$\Rightarrow \quad g_{1\alpha} \frac{du^\alpha}{ds} = \sqrt{g_{11}} \cos \psi \qquad [\because x^i_{,1} x^i_{,\alpha} = g_{1\alpha}]$$

$$\Rightarrow \quad g_{11} \frac{du^1}{ds} = \sqrt{g_{11}} \cos \psi \qquad [\because g_{12} = 0]$$

$$\therefore \quad \cos \psi = \sqrt{g_{11}} \frac{du^1}{ds} \qquad \ldots \text{(iii)}$$

Similarly, if ψ is the angle between the given direction and the parametric curve $du^1 = 0$, then

$$\cos \psi = \sqrt{g_{22}} \frac{du^2}{ds}$$

Since, the parametric curves are orthogonal, then we have

$$\psi = \frac{\pi}{2} - \psi$$

$$\therefore \quad \sin \psi = \sqrt{g_{22}} \frac{du^2}{ds} \qquad \ldots \text{(iv)}$$

Using Eqs. (ii), (iii) and (iv) in Eq. (i), we get

$$K_n = K_1 \cos^2 \psi + K_2 \sin^2 \psi$$