Chapter Eleven

GEODESICS

⏻ Important Points from the Chapter

1. **Geodesics on a Surface** A curve on a surface is called a geodesic on the surface, if its osculating plane at every point contains the normal to the surface point.

Or

Let P and Q be any two points on any surface S. These points are jointed by a number of curves lying on S, then the curve which possesses a stationary length for small variations is called a geodesic. These geodesics are curves of stationary length.

2. **Differential Equation of Geodesics on a Surface**

$$\frac{d^2u^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \frac{du^\beta}{ds} \cdot \frac{du^\gamma}{ds} = 0 \Rightarrow \mu''^\alpha + \Gamma(\beta\gamma)^\alpha \mu'^\beta \mu'^\alpha = 0$$

which is the differential equation of geodesic on a surface. **(2004, 01)**

3. **Torsion of a Geodesic** $\tau = e_{\alpha\delta}\, \alpha_{\beta\gamma}\, g^{\gamma\delta} \frac{du^\alpha}{ds} \cdot \frac{du^\beta}{ds} = \frac{1}{2}(k_2 - k_1)\sin 2\theta$

4. **Geodesics Tangent** If P is a point of the curve C, then geodesic tangent of curve C at point P is called as the geodesic which touches the curve at P. Thus, geodesic tangent at any point on a curve is the geodesic which touches the curve at the point.

5. **Geodesic Curvature** For any curve on a surface, the components of curvature vector at a point P is $x''^i = kp^i$, where k is the curvature and p^i is the components of the principal normal of the curve at P, its resolved parts along the normal to the surface and tangential to the surface P are called the normal curvature vector and geodesic curvature vector of the curve at P. Their magnitudes are called the normal curvature and the geodesics curvature of the curve at P.

6. **Geodesic Coordinates** If the parametric curve are orthogonal and one of the families of parametric curves are geodesics on a surface, the coordinates on a surface can be introduced in infinite number of ways as one family of parametric curves can be chosen arbitrarily.

7. **Geodesic Triangle** A curvilinear triangle on a surface whose three sides are geodesics, are called geodesic triangle.

8. **Some Important Theorems**

   (i) These passes a unique geodesic through any two points of the surface.

(ii) If a surface deforms into another surface without tearing or stretching, a geodesic of the surface transforms into a geodesic of the deformed surface.

(iii) The necessary and sufficient condition that the parametric curves $u^\alpha =$ constant is geodesic, is that $\begin{Bmatrix} \alpha \\ \beta\beta \end{Bmatrix} = 0$ for $\alpha \neq \beta$.

(iv) At an umbilic point, the torsion of the geodesic is zero.

(v) The necessary and sufficient condition that geodesic is a line of curvature, is that it is a plane curve.

(vi) The torsion of an asymptotic line and geodesic tangent are equal.

(vii) The torsion of the two perpendicular geodesics at a point of a surface are equal in magnitude but opposite in sign.

(viii) According to Bonnett's theorem, the relation between the torsion of a given curve with the torsion of its geodesic tangent is $\tau_g = \tau + \frac{d\theta}{ds}$.

(ix) The geodesic curvature vector of any curve is orthogonal to the curve, is given by $\kappa_g = e\sqrt{g}\, \varepsilon_{\alpha\beta} u^\alpha \lambda^\beta$, where $e = \pm 1$, according as geodesic curvature vector and the curve make angle $\pm \frac{\pi}{2}$.

(x) $\kappa^2 = \kappa_g^2 + \kappa_n^2$

(xi) The metric in a geodesic coordinate system on a surface is given by

$$ds^2 = (du^1)^2 + g_{22}(du^2)^2.$$

Very Short Answer Questions

Q 1. Prove that the necessary and sufficient condition that geodesic be a line of curvature is that it is a plane curve. **(2011, 2000)**

Sol. Let the geodesics be a line of curvature. Then, at all points of geodesic, we have

$$\varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta} du^\gamma du^\delta = 0$$

$$\therefore \quad g^{\alpha\beta} e_{\alpha\gamma} d_{\beta\delta} \frac{du^\gamma}{dS} \cdot \frac{du^\delta}{ds} = 0 \qquad [\because e^{\alpha\beta} g_{\alpha\gamma} = g^{\alpha\beta} e_{\alpha\gamma}]$$

Changing the dummy indices $\alpha, \gamma, \beta, \delta$ by $\delta, \alpha, \gamma, \beta$, we get

$$e_{\delta\alpha} \alpha_{\beta\gamma} g^{\gamma\delta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0, \text{ i.e. } \tau = 0$$

Hence, $\tau = 0$ at all points is a plane curve.

Conversely Let $\tau = 0$ at all points of the geodesic.

Then, at all points, $e_{\alpha\delta} \alpha_{\beta\gamma} g^{\gamma\delta} \mu'^\alpha \mu'^\beta = 0$ which can be written as

$$\varepsilon^{\alpha\beta} g_{\alpha y} \alpha_{\beta\delta} du^\gamma du^\delta = 0$$

Hence, geodesic is a line of curvature.

Q 2. Prove that at an umbilic point, the torsion of the geodesic is zero.

Sol. We know that at an umbilic point, $d_{\alpha\beta} = \lambda_{g\alpha\beta}$

Therefore, $\tau = e_{\alpha\delta}\, d_{\beta\gamma}\, g^{\gamma\delta}\, \mu'^{\alpha}\, \mu'^{\beta} = \lambda\, e_{\alpha\delta}\, \delta^{\delta}_{\beta}\, \mu'^{\alpha}\, \mu'^{\beta}$

$= \lambda\, e_{\alpha\delta}\, \delta^{\delta}_{\beta}\, \mu'^{\alpha}\, \mu'^{\beta} = \lambda\, e_{\alpha\beta}\, \mu'^{\alpha}\, \mu'^{\beta} = 0$ **Hence proved.**

Short Answer Questions

Q 1. Prove that two geodesics at right angles have their torsion equal in magnitude but opposite in sign. **(2009)**

Sol. Let τ_1 and τ_2 be torsions of two perpendicular geodesics through a point P of a surface and θ be the angle which makes the first geodesic with the principal direction for which principal curvature k_1.

Then, the second geodesic will make angle $\left(\frac{\pi}{2} + \theta\right)$ with it.

We know that torsion of a geodesic in terms of principal curvatures k_1 and k_2 is

$$\tau = (k_2 - k_1) \sin\theta \cos\theta$$

Therefore, $\tau_1 = (k_2 - k_1) \sin\theta \cos\theta$ ...(i)

and $\tau_2 = (k_2 - k_1) \sin\left(\frac{\pi}{2} + \theta\right) \cos\left(\frac{\pi}{2} + \theta\right)$

i.e. $\tau_2 = -(k_2 - k_1) \cos\theta \sin\theta$

$\Rightarrow$ $-\tau_2 = (k_2 - k_1) \sin\theta \cos\theta \Rightarrow -\tau_2 = \tau_1$

$\therefore$ $\tau_1 = -\tau_2$ **Hence proved.**

Q 2. Show that the straight line in three dimensional Euclidean space is the example of geodesics. **(2014)**

Sol. Consider the Euclidean space s_n of n-dimensions. In this case, the metric tensor g_{ij} is denoted by a_{ij}

and $$g_{ij} = a_{ij} = \delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Thus, $\Gamma^k_{ij} = 0 = [k, ij]$ relatives to s_n. ...(i)

Then, differential equation of a geodesic in s_n is

$$\frac{d^2y^j}{ds^2} + \Gamma^i_j \frac{dy^j}{ds} \cdot \frac{dy^k}{ds} = 0 \quad \text{...(ii)}$$

Then, Eqs. (i) and (ii) becomes $\frac{d^2y^i}{ds^2} = 0$ ...(iii)

On integrating, we get $\frac{d^2y^i}{ds} = a^i$ ...(iii)

Again integrating, we get $y^i = a^i s + b^i$ ...(iv)

which is of the form $y = mx + c$, which also represents straight line and Eq. (iv) is the solution of Eq. (ii).

Therefore, the geodesics relation to s_n are given by Eq. (i) which proves that the geodesics in s_n are straight line.

Q 3. Prove that the necessary and sufficient condition that the parametric curve u^α = (constant) be geodesic, is that

$$\Gamma^\alpha_{\beta\beta} = \begin{Bmatrix} \alpha \\ \beta\beta \end{Bmatrix} = 0 \text{ for } \alpha \neq \beta.$$ **(2018, 16, 14, 12, 07)**

Sol. Here, we will prove that, if $u_1 = c_1$ (constant) is geodesic, then $\Gamma^1_{22} = 0$.

For $u^1 = c^1, \dfrac{du^1}{ds} = 0 \Rightarrow \dfrac{d^2u^1}{ds^2} = 0$

The equation of geodesic is $\dfrac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\dfrac{du^\beta}{ds}\dfrac{du^\gamma}{ds} = 0$ ...(i)

Reduces for $\alpha = 1, 2$, $\quad \Gamma^1_{\beta\gamma}\dfrac{du^\beta}{ds}\cdot\dfrac{du^\gamma}{ds} = 0$ $\quad [\because \beta, \gamma = 1, 2]$

$$\Rightarrow \quad \Gamma^1_{11}\frac{du^1}{ds}\frac{du^1}{ds} + \Gamma^1_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^1_{21}\frac{du^2}{ds}\frac{du^1}{dS} + \Gamma^1_{22}\frac{du^2}{ds}\frac{du^2}{ds} = 0 \quad \left[\because \frac{du^1}{ds} = 0, \frac{d^2u^1}{ds^2} = 0\right]$$

$$\Rightarrow \quad \Gamma^1_{22}\frac{du^2}{ds}\cdot\frac{du^2}{ds} = 0 \Rightarrow \Gamma^1_{22} = 0 \quad \left[\because \frac{du^2}{ds} \neq 0 \text{ i.e.} \left(\frac{du^2}{ds}\right)^2 \neq 0\right]$$

$$\Rightarrow \quad \Gamma^\alpha_{\beta\beta} = 0 \text{ for } \alpha \neq \beta$$

Conversely Suppose that $\Gamma^\alpha_{\beta\beta} = 0 \Rightarrow \Gamma^1_{22} = 0$. Then, for $u^1 = c^1$, we see that Eq. (i) for $\alpha = 1$ is satisfied.

Now, we have to prove that Eq. (i) is satisfied for $\alpha = 2$.

For $u^1 = c^1, \dfrac{du^1}{ds} = 0$

$$\because \quad ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta \Rightarrow g_{\alpha\beta}\frac{du^\alpha}{ds}\cdot\frac{du^\beta}{ds} = 1$$

$$\Rightarrow \quad g_{11}\left(\frac{du^1}{ds}\right)^2 + 2\, g_{12}\frac{du^1}{ds}\cdot\frac{du^2}{ds} + g_{22}\left(\frac{du^2}{ds}\right)^2 = 0 \quad [\because g_{12} = g_{21}]$$

$$\Rightarrow \quad g_{22}\left(\frac{du^2}{ds}\right)^2 = 1 \quad \left[\because \frac{du^1}{ds} = 0\right]$$ ...(ii)

On differentiating Eq. (ii) covariantly with respect to μ^{β} and making use of $g_{22}, \beta = 0$, we have

$$g_{22}\left(\frac{du^2}{ds}\right)\frac{du^2}{ds} = 0 \Rightarrow g_{22}\left[\left(\frac{du^2}{ds}\right)\frac{du^{\beta}}{ds}\right]\frac{du^2}{ds} = 0$$

$\because$ $$g_{22} \neq 0, \frac{du^2}{ds} \neq 0$$

We have, $$\left(\frac{du^2}{ds}\right)\frac{du^{\beta}}{ds} = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial\mu\beta}\left(\frac{du^2}{ds}\right) + \frac{du^{\gamma}}{ds}\Gamma^2_{\gamma\beta}\right]\frac{du^{\beta}}{ds} = 0$$

$$\Rightarrow \frac{du^2}{ds^2} + \Gamma^2_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0$$

Thus, $u^1 = c^1$ is a geodesic.

Similarly, we can show that $u^2 = c^2$ is a geodesic iff $\Gamma^2_{11} = 0$.

Hence, $u^{\alpha} = c^{\alpha}$ = constant is geodesic.

Q 4. Prove that a curve for which $\frac{\kappa}{\tau}$ is constant is geodesic on a cylinder. **(2018)**

Sol. Suppose, the generators of the cylinder are parallel to the constant unit vector **a**. Let the curve C be a helix on the cylinder. At any point P on C let **t**, **n** be the unit vectors along the tangent and principal normal to C and N be the unit vector along normal to the surface of the cylinder at P. Since, the curve C is a helix, therefore

$$\mathbf{t}\cdot\mathbf{a} = \text{Constant} \quad \text{...(i)}$$

On differentiating both sides of Eq. (i) with respect to arc length S of C, we get

$$\frac{d\mathbf{t}}{dS}\cdot\mathbf{a} + \mathbf{t}\cdot\mathbf{0} = 0$$

$$\Rightarrow k\mathbf{n}\cdot\mathbf{a} = 0$$

$$\Rightarrow \mathbf{n}\cdot\mathbf{a} = 0 \quad \text{...(ii)}$$

Also, $$\mathbf{n}\cdot\mathbf{t} = 0 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we see that the vector **n** is parallel to the vector $\mathbf{a}\times\mathbf{t}$. But both the vectors **a** and **t** are tangential to the surface of the cylinder at P. Therefore, $\mathbf{a}\times\mathbf{t}$ is parallel to **N**. Thus, **n** is parallel to **N**. Therefore, by normal property of geodesics C is a geodesic on the cylinder.

Hence, the curve for which $\frac{\kappa}{\tau}$ is constant is a helix on a cylinder. So, the curve for $\frac{\kappa}{\tau}$ is constant is a geodesic on a cylinder.

Long Answer Questions

Q 1. Find the equation of the geodesics of the surfaces $x^1 = u^1 \cos u^2,\ x^1 = u^1 \sin u^2,\ x^3 = f(u^1)$. (2010)

Sol. We have, $\mathbf{X}_1 = \dfrac{\partial x^i}{\partial u'} = \left(\dfrac{\partial x^1}{\partial u^1}, \dfrac{\partial x^2}{\partial u^1}, \dfrac{\partial x^3}{\partial u^1}\right) = (\cos u^2, \sin u^2, f^1)$

$$\mathbf{X}_2 = \frac{\partial x^i}{\partial u^2} = \left(\frac{\partial x^1}{\partial u^2}, \frac{\partial x^2}{\partial u^2}, \frac{\partial x^3}{\partial u^3}\right) = (-u^1 \sin u^2, u^1 \cos u^2,\ 0)$$

$$\therefore \quad g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = \cos^2 u^2 + \sin^2 u^2 + (f')^2 = 1 + (f')^2$$

$$g_{21} - g_{12} = \mathbf{X}_1 \cdot \mathbf{X}_2 = -u^1 \cos u^2 - \sin u^2 + u^1 \cos u^2 + \sin u^2 + 0 = 0$$

$$g_{22} = \mathbf{X}_2 \cdot \mathbf{X}_2 = (u^1)^2 \sin^2 u^1 + (u^1)^2 \cos^2 u^2 = (u^1)^2$$

and $g^{11} = \dfrac{1}{1+(f')^2},\ g^{12} = g^{21} = 0,\quad g^{22} = \dfrac{1}{(u^1)^2}$

Also, $[1, 11] = \dfrac{1}{2}\left[\dfrac{\partial g_{11}}{\partial u^1} + \dfrac{\partial g_{12}}{\partial u^1} - \dfrac{\partial g_{11}}{\partial u^1}\right] = f' f''$

and $[1, 12] = \dfrac{1}{2}\left[\dfrac{\partial g_{11}}{\partial u^2} + \dfrac{\partial g_{12}}{\partial u^1} - \dfrac{\partial g_{12}}{\partial u^1}\right] = 0$

Similarly, $[1, 22] = -u^1,\ [2, 11] = 0$ and $[2, 12] = u^1,\ [2, 22] = 0$

$$\therefore \quad \Gamma^1_{11} = g^{1\delta}\,[\delta, 11] = g^{11}\,[1, 11] + g^{12}\,[2, 11]$$

$$= \frac{g_{22}}{g}\,[1, 11] \qquad \left[\because g^{11} = \frac{g_{22}}{g}\right]$$

$$= \frac{(u^1)^2 f^1 f^2}{g} = \frac{f^1 f^2}{1+(f^1)^2}$$

$$\Gamma^1_{12} = g^{11}\,[1, 12] = 0$$

$$\Gamma^1_{22} = g^{11}[1, 22] = \frac{g_{22}}{g}\,[1, 22] = -\frac{(u^1)^3}{g} = \frac{-u^1}{1+(f')^2}$$

$$\Gamma^1_{11} = g^{22}\,[2, 11] = 0$$

$$\Gamma^2_{12} = g^{22}\,[2, 12] = \frac{1}{u^1} \text{ and } \Gamma^2_{22} = g^{22}\,[2, 22] = 0$$

Then, the equation of geodesic are

$$\frac{d^2u^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\cdot\frac{du^\gamma}{ds} = 0$$

For $\alpha = 1,\ \dfrac{d^2u^1}{ds} + \Gamma^1_{11}\left(\dfrac{du^1}{ds}\right)^2 + 2\Gamma^1_{12}\dfrac{du^1\,du^2}{ds\,ds} + \Gamma^1_{22}\left(\dfrac{du^2}{ds}\right)^2 = 0$

$$\Rightarrow \qquad \frac{d^2u^1}{ds} + \frac{f^1 f^{11}}{1+(f^1)^2}\left(\frac{du^1}{ds}\right)^2 + 0 - \frac{u^1}{1+(f')^2}\left(\frac{du^2}{ds}\right)^2 = 0$$

$$\Rightarrow \qquad \frac{d^2u^1}{ds^2} + \frac{f^1 f^{11}}{1+(f^1)^2}\left(\frac{du^2}{ds}\right)^2 - \frac{u^1}{1+(f^1)^2}\left(\frac{du^2}{ds}\right)^2 = 0$$

and for $\alpha = 2$, $\frac{d^2u^2}{ds} + \Gamma_{11}^2\left(\frac{du^1}{ds}\right)^2 + 2\,\Gamma_{12}^2 \frac{du^1}{ds}\cdot\frac{du^2}{ds} + \Gamma_{22}^2\left(\frac{du^2}{ds}\right)^2 = 0$

$$\frac{d^2u^2}{ds} + 0 + \frac{2}{u^1}\frac{du^1}{ds}\cdot\frac{du^2}{ds} + 0 = 0$$

$$\Rightarrow \qquad (u^1)^2 \frac{d^2u^2}{\alpha s} + 2u^1 \frac{du^1}{ds}\cdot\frac{du^2}{ds} = 0$$

$$\Rightarrow \qquad \frac{d}{ds}[(u^1)^2 (u^2)^1] = 0 \qquad \left[\because (u^2)^1 = \frac{du^2}{ds}\right]$$

On integrating, we get

$$(u^1)^2 \frac{du^2}{ds} = h \text{ (constant)}$$

$$\Rightarrow \qquad h\, ds = (u^1)^2\, du^2 \qquad \ldots(i)$$

The metric on the surface of revolution is given by

$$ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta$$

$$\Rightarrow \qquad ds^2 = g_{11}(du^1)^2 + 2\, g_{12}\, du^1 du^2 + g_{22}\,(du^1)^2.$$

$$\Rightarrow \qquad ds^2 = (1+(f^1)^2)\,(du^1)^2 + 0 + (u^1)^2\,(du^1)^2$$

$$\Rightarrow \qquad h^2\, ds^2 = h^2\,(1+(f^1)^2\,(du^1)^2 + h^2(u^1)^2\,(du^2)^2$$

$$\Rightarrow \qquad (u^1)^4\,(du^2)^2 = h^2\,(1+(f^1)^2)\,(du^1)^2 + h^2(u^1)^2\,(du^2)^2$$

$$\Rightarrow \qquad \frac{1}{h^2} = \frac{1+(f^1)^2)}{(u^1)^4}\left(\frac{du^1}{du^2}\right)^2 + \frac{1}{(u^1)^2}$$

$$\Rightarrow \qquad \frac{(u^1)^2 - h^2}{h^2}\,(du^2)^2 = \frac{1+(f^1)^2}{(u^1)^2}\,(du^1)^2$$

$$\Rightarrow \qquad \left(\frac{du^2}{du^1}\right) = \frac{h^2}{(u^1)^2}\left(\frac{1+(f^1)^2}{(u^1)^2 - h^2}\right)$$

$$\Rightarrow \qquad \frac{du^2}{du^1} = \pm\frac{h}{u^1}\sqrt{\frac{1+(f^1)^2}{(u^1)^2 - h^2}} \Rightarrow du^2 = \pm\frac{h}{u^1}\sqrt{\frac{1+(f^1)^2}{(u^1)^2 - h^2}}$$

On integrating both the sides, we get

$$u^2 = c \pm h \int \frac{1}{u^1}\frac{(1+(f^1)^2}{(u^1)^2 - h^2}\, du^1$$

where, c is a constant.

Q 2. Show that on the right circular cone $x^1 = u \cos v$, $x^2 = u \sin v$, $x^3 = u \cot \alpha$, the geodesics are given by $u = h \sec (v \sin \alpha + \beta)$, where h, α, β are constants. (2008)

Sol. We have, $\mathbf{X}_1 = \frac{\partial x^i}{\partial u} = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right) = (\cos v, \sin v, \cot \alpha)$

$$\mathbf{X}_2 = \frac{\partial x^i}{\partial V} = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial}\right) = (-u \sin v, u \cos v, 0)$$

$\therefore$ $g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = 1 + \cot^2 \alpha = \operatorname{cosec}^2 \alpha$

$g_{12} = g_{21} = \mathbf{X}_1 \cdot \mathbf{X}_2 = 0$

$g_{22} = u_2$

$g = g_{11} \cdot g_{22} - (g_{12})^2 = u^2 \operatorname{cosec}^2 \alpha$

$\therefore$ $g_{11} = \frac{1}{\operatorname{cosec}^2 \alpha} = \sin^2 \alpha$

$g_{12} = g_{21} = 0, g_{22} = \frac{1}{u^2}$

Also, $[1, 11] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}\right) = 0$

$[1, 12] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}\right) = 0$

$= -u, [2, 11] = 0$ and $[2, 12] = u, [2, 22] = 0$

$\therefore$ $\Gamma^1_{11} = g^{1\delta} [\delta, 11] = 0 + 0 = 0$

$\Gamma^1_{12} = g^{1\delta} [\delta, 12] = -0 + 0 = 0$

$\Gamma^1_{22} = g^{1\delta} [\delta, 22] = -u \sin^2 \alpha$

$\Gamma^2_{11} = g^{2\delta} [\delta, 11] = 0$

$\Gamma^2_{12} = g^{2\delta} [\delta, 12] = \frac{1}{u}$ and $\Gamma^2_{22} = g^{2\delta} [\delta, 22] = 0$

The equation of geodesic are $-\frac{d^2 u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \cdot \frac{du^\gamma}{ds} = 0$

Since, $u^1 = u, u^2 = v$ for $\alpha = 1$,

$$\frac{d^2 u^\alpha}{ds^2} + \Gamma^1_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^1_{12} \frac{du}{ds} \cdot \frac{dv}{ds} + \Gamma^1_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

$\Rightarrow$
$$\frac{d^2 u}{ds^2} - u \sin^2 \alpha \left(\frac{dv}{ds}\right)^2 = 0$$

and $\alpha = 2, \frac{d^2v}{ds} + \Gamma_{11}^2 \left(\frac{du}{ds}\right)^2 + 2\,\Gamma_{12}^2 \frac{du}{ds} \cdot \frac{dv}{ds} + \Gamma_{22}^2 \left(\frac{dv}{ds}\right)^2 = 0$

$$\Rightarrow \quad \frac{d^2v}{ds^2} + \frac{2}{u}\frac{du}{ds}\frac{dv}{ds} = 0$$

$$\Rightarrow \quad u^2 \frac{d^2v}{ds^2} + 2u \frac{du}{ds} \cdot \frac{dv}{ds} = 0$$

$$\Rightarrow \quad \frac{d}{ds}(u^2 v^1) = 0 \qquad \left[\because v^1 = \frac{dv}{ds}\right]$$

$$\Rightarrow \quad u^2 v^1 = h \text{ (constant)}$$

$$\Rightarrow \quad v^2 \frac{dv}{ds} = h \Rightarrow h\, ds = u^2\, dv \qquad \ldots\text{(i)}$$

The metric on the surface of revolution is given by

$$ds^2 = g_{11}\,(du)^2 + 2g_{12}\,du dv + g_{22}(dv)^2$$

$$\Rightarrow \quad ds^2 = \operatorname{cosec}^2 \alpha\ (du)^2 + u^2 (dv)^2$$

$$\Rightarrow \quad h^2 ds^2 = h^2 \operatorname{cosec} \alpha (du)^2 + h^2 u^2 (dv)^2$$

$$\Rightarrow \quad u^4 (dv)^2 - h^2 u^2 (dv)^2 = h^2 \operatorname{cosec}^2 \alpha (du)^2 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad u^4 [u^2 - h^2]\,(dv)^2 = h^2 \operatorname{cosec}^2 \alpha\ (du)^2$$

$$\Rightarrow \quad \sin \alpha\ dv = \frac{h\, du}{u\sqrt{u^2 - h^2}}$$

On integrating, we get $v \sin \alpha + \beta = \frac{h}{h} \sec^{-1}\left(\frac{u}{h}\right)$

$$\Rightarrow \quad \sec^{-1}\left(\frac{u}{h}\right) = v \sin \alpha + \beta$$

$$\Rightarrow \quad \frac{u}{h} = \sec\,(v \sin \alpha + \beta)$$

$$\Rightarrow \quad u = h \sec\,(v \sin \alpha + \beta)$$

Q 3. Find the differential equation of geodesic on a surface.

(2017, 15, 09, 04, 01)

Or Define geodesics with example and also find the differential equation of geodesics on a surface. (2013)

Sol. Let the equation of surface be $x^i = x^i(u^\alpha)$ and equation of a curve on this surface be $u^\alpha = u^\alpha(s)$ and if the curve is a geodesic, then $p^i = N^i$.

By Frenet's formula, we have

$$t'^i = X''^i = kp^i$$

$$\Rightarrow \quad x''^i = kN^i \qquad \ldots\text{(i)}$$

On multiplying Eq. (i) by $\partial_\delta x^i$, we get

$$x''^i \partial_\delta x^i = k \partial_\delta x^i N^i \qquad [\because \partial_\delta x^i \cdot N^i = 0]$$

$$\Rightarrow \quad x'' \partial_\delta x^i = 0 \qquad \left[\because \partial \delta x^i = \frac{\partial x^i}{\partial u^\delta}\right] \quad \text{...(ii)}$$

Since, $$x'^i = \partial_\gamma x^i u'^\gamma \Rightarrow x''^i = \frac{d}{ds}\left(\frac{dx^i}{ds}\right)$$

$$\Rightarrow \quad x'^i = \frac{d}{du^\beta}\left(\frac{dx^i}{ds}\right)\frac{du^\beta}{ds} = \frac{\partial}{du^\beta}(\partial_\gamma x^i u'^\gamma) u'^\beta$$

$$\Rightarrow \quad x''^i = \partial_\beta \partial_\gamma x^i \mu'^\gamma \mu'^\beta + \partial_\gamma x^i \mu''^\gamma \quad \text{...(iii)}$$

By using Gauss equation, it is given by

$$x^i, \beta\gamma = d_{\beta\gamma} N^i \qquad [\because x^i, \beta\gamma = \partial_\beta \partial_\gamma x^i - \partial_p x^i \Gamma^\rho_{\beta\gamma}]$$

$$\Rightarrow \quad \partial_\beta \partial_\gamma x^i - \partial_\rho x^i \Gamma^P_{\beta\gamma} = d_{\beta\gamma} N^i$$

$$\Rightarrow \quad \partial_\beta \partial_\gamma x^i = d_{\beta\gamma} N^i + \partial_\rho x^i \Gamma^P_{\beta\gamma} \quad \text{...(iv)}$$

Using this in Eq. (iii), we get

$$x''^i = (\partial_{\beta\gamma} N^i + \partial_\rho x^i \Gamma^\rho_{\beta\gamma} \mu'^\beta \mu'^\gamma + \partial_y x^i \mu''^\gamma)$$

$$\partial_\delta x^i = 0 \Rightarrow 0 + g_{\rho\delta} \Gamma^\rho_{\beta\gamma} \mu'^\beta \mu'^\gamma + g_{\gamma\delta} \mu''^\gamma = 0 \quad \text{...(v)}$$

$$[\because \partial_\delta x^i N^i = 0, \delta_\rho x^i \partial_\delta x^i = g_{\rho\delta}]$$

Now, multiplying Eq. (v) by $g^{\alpha\delta}$ and using $g^{\alpha\delta} g_{\rho\delta} = \delta^\alpha_\rho$, we get

$$g^\alpha_\rho \Gamma^\rho_{\beta\gamma} \mu'^\beta \mu'^\gamma + \partial^\alpha_\gamma u''^\gamma = 0$$

$$\Rightarrow \quad u''^\alpha + \Gamma^\alpha_{\beta\gamma} u'^\beta u'^\gamma = 0$$

$$\therefore \quad \frac{d^2\mu^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \cdot \frac{d\mu^\gamma}{ds} = 0 \qquad [\because \alpha, \beta, \gamma = 1, 2, \ldots]$$