Chapter Fourteen

NUMERICAL INTEGRATION

Important Points from the Chapter

1. Numerical integration is used to obtain the approximate answers for definite integrals that cannot be solved analytically. It is a process of finding the numerical value of definite integral $\int_a^b f(x)\,dx$, when a function $y = f(x)$ is not known explicitly. But we given only a set of values of the function $y = f(x)$ corresponding to the some values of x. To evalute the integral, we fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it within the desired limits. Here, we integrate an approximate interpolation formula instead of $f(x)$. When this teachnique is applied on a function of single variable, the process is called **Quadrature.**

2. **Some Quadrature Formulae for Equidistant Values of x**

   (i) **General quadrature formula**

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \ldots\right]$$

(2005)

   (ii) **Trapezoidal rule**

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \ldots + (y_{n-1} + y_n)\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots + y_{n-1})\right]$$

(2005)

   (iii) **Simpson's one-third rule**

$$\int_{x_0}^{x_0+nh} x dx = \frac{h}{3}\left[(y_0 + y_n) + 4\,(y_1 + y_3 + \ldots + y_{n-1}) + 2\,(y_2 + y_4 + \ldots + y_{n-2})\right]$$

   (iv) **Simpson's three-eighth rule**

$$\int_{x_0}^{x_0+nh} x dx = \frac{3h}{8}\left[(y_0 + y_n) + 3\,(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2\,(y_3 + y_6 + \ldots + y_{n-3})\right]$$

   (v) **Weddle's rule**

$$\int_{x_0}^{x_0+nh} x dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + \ldots\right]$$

Very Short Answer Questions

Q 1. Show that 1.62 is an approximate value of $\int_1^5 \frac{1}{x}\,dx$, if $h = 1$ in the Simpson's one-third rule. **(2012)**

Sol. In this case, divide the interval of integration into four parts by taking $h = \frac{5-1}{4} = 1$.

x	1	2	3	4	5
$y = 1/x$	1	1/2	1/3	1/4	1/5
	y_0	y_1	y_2	y_3	y_4

From Simpson's one-third rule,

$$\int_1^5 \frac{1}{x}\,dx = \frac{h}{3}\,[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{3}\left[1 + \frac{1}{5} + 4\left(\frac{1}{2} + \frac{1}{4}\right) + 2\cdot\frac{1}{3}\right] = \frac{1}{3}\left[\frac{73}{15}\right] = 1.62$$

Q 2. Prove that Simpson's formula

$$\int_a^b f(x)\,dx = \frac{b-a}{6n}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + f(x_{2n})]$$

where, $x_0 = a$ and $x_{2n} = b$.

Sol. We know that, Simpson's one-third rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[f(x_0) + f(x_0 + nh) + 4\,\{f(x_0 + h) + f(x_0 + 3h) + \ldots\}$$
$$+2\,\{f(x_0 + 2h) + f(x_0 + 4h) + \ldots\}]$$

Now, putting $x_0 = a$, $n = 2n$, $x_0 + 2nh = x_{2n} = b$ and $h = \frac{b-a}{2n}$, we get the required form of Simpson's rule.

Short Answer Questions

Q 1. Compute the integral $\int_5^{12} \frac{1}{x}\,dx$ by applying Gauss-Legendre quadrature formula. **(2005)**

Sol. Let $n = 3$

First, we transform the variable x to u by substitution,

i.e. $$x = \frac{1}{2}(b-1)u + \frac{1}{2}(b+a)$$

$$\Rightarrow \qquad x = \frac{1}{2}(12-5)u + \frac{1}{2}(12+5) = \frac{7u+17}{2}$$

$$\therefore \qquad f(x) = \frac{1}{x} = \frac{2}{7u+17} = \phi(u)$$

The corresponding abscissa and weights are given below

u_i	w_i	$\phi(u_i)$
$-\sqrt{3}/5$	5/9	$\frac{2}{(-7\sqrt{3}/5)+17}$
0	8/9	2/17
$\sqrt{3}/5$	5/9	$\frac{2}{(7\sqrt{3}/5)+17}$

$$\therefore \quad \int_{-1}^{1} \phi(u)du = \frac{5}{9}\left[\phi\left(-\frac{\sqrt{3}}{5}\right) + \phi\left(\frac{\sqrt{3}}{5}\right)\right] + \frac{8}{9}\phi(0)$$

$$= \frac{5}{9}\left[\frac{2\sqrt{5}}{17\sqrt{5}-7\sqrt{3}} + \frac{2\sqrt{5}}{17\sqrt{5}+7\sqrt{3}}\right] + \frac{8}{9}\cdot\frac{2}{17}$$

$$= \frac{5}{9}\left(\frac{2\sqrt{5}\cdot 2\cdot 17\sqrt{5}}{1298}\right) + \frac{16}{153}$$

$$= 0.145523 + 0.1045751 = 0.2500981$$

Hence, by using $\int_a^b f(x)dx = \frac{1}{2}(b-a)\int_{-1}^{1} \phi(u)dx$, we get

$$\int_5^{12} \frac{dx}{x} = \frac{1}{2}(12-5)\int_{-1}^{1} \phi(u)\,du = \frac{7}{2} \times 0.2500981 = 0.8753433$$

***Q* 2. Calculate (upto 3 places of decimals) $\int_2^{10} \frac{dx}{1+x}$ by dividing the range into eight equal parts.**

Sol. Here, $h = \frac{10-2}{8} = 1$ and $y = \frac{1}{1+x}$

The required values are given in the following table

x	y	
2	1/3	y_0
3	1/4	y_1
4	1/5	y_2
5	1/6	y_3
6	1/7	y_4
7	1/8	y_5
8	1/9	y_6
9	1/10	y_7
10	1/11	y_8

By Simpson's one-third rule,

$$\int_2^{10} \frac{1}{1+x}\,dx = \frac{h}{3}\left[y_0 + y_8 + 4(y_1 + y_3 + \ldots) + 2\,(y_2 + y_4 + \ldots)\right]$$

$$= \frac{1}{3}\left[\frac{1}{3} + \frac{1}{11} + 4\left\{\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right\} + 2\left\{\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right\}\right]$$

$$= \frac{1}{3}\left[\frac{11+3}{33} + 4\left\{\frac{30+20+15+12}{120}\right\} + 2\left\{\frac{63+45+35}{315}\right\}\right]$$

$$= \frac{1}{3}\left[\left(\frac{14}{33} + \frac{308}{120} + \frac{286}{315}\right)\right]$$

$$= \frac{1}{3}\,(0.424 + 2.567 + 0.908)$$

$$= \frac{1}{3} \times 3.889 = 1.29$$

***Q* 3. Show that $\int_0^1 \dfrac{dx}{1+x} = \log_e 2 = 0.69315.$**

Sol. On dividing the whole range (0, 1) into 10 equal parts making use of Simpson's one-third rule, we have

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{3}\left[1 + \frac{1}{2} + 4\left\{\frac{1}{1\cdot1} + \frac{1}{1\cdot3} + \frac{1}{1\cdot5} + \frac{1}{1\cdot7} + \frac{1}{1\cdot9}\right\}\right.$$

$$\left. + 2\left\{\frac{1}{1\cdot2} + \frac{1}{1\cdot4} + \frac{1}{1\cdot6} + \frac{1}{1\cdot8}\right\}\right]$$

$$= \frac{1}{30}\,(1.5 + 4 \times 3.45955 + 2 \times 2.72818)$$

$$= \frac{20.79456}{30} = 0.69315$$

But $\int_0^1 \dfrac{1}{1+x}\,dx = [\log\,(1+x)]_0^1 = \log_e 2$

Hence, $\int_0^1 \dfrac{1}{1+x}\,dx = \log_e 2 = 0.69315$

***Q* 4. Find the value of $\log_e 5$ from $\int_0^5 \dfrac{dx}{4x+5}$ using Simpson's one-third rule to four decimal places by dividing the range into ten equal parts.** **(2011)**

Sol. Here, $h = \dfrac{5-0}{10} = 0.5,\; y = \dfrac{1}{4x+5}$

The table is as follows

x	y	
0	0.2	y_0
0.5	0.14286	y_1
1	0.11111	y_2
1.5	0.09091	y_3
2.0	0.07692	y_4
2.5	0.06667	y_5
3.0	0.05882	y_6
3.5	0.05263	y_7
4.0	0.04762	y_8
4.5	0.04348	y_9
5.0	0.04	y_{10}

By Simpson's one-third rule,

$$\int_0^5 y dx = \frac{h}{3}[y_0 + y_{10} + 2(y_2 + y_4 + \dots) + 4(y_1 + y_3 + y_5 + \dots)]$$

$$= \frac{0.5}{3}[0.2 + 0.04 + 2(0.11111 + 0.07692 + 0.05882 + 0.04761)$$

$$+ 4(0.14286 + 0.09099 + 0.06667 + 0.05263 + 0.04348)]$$

$$= \frac{1}{6}(0.40252) \qquad \dots(i)$$

and $\int_0^5 \frac{1}{4x+5} dx = \frac{1}{4}[\log(4x+5)]_0^5 = \frac{1}{4}(\log 25 - \log 5) = \frac{1}{4}\log 5 \qquad \dots(ii)$

From Eqs. (i) and (ii), $\frac{1}{4}\log 5 = 0.40252 \Rightarrow \log 5 = 4 \times 0.40252 = 1.61008$

Q 5. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by dividing the interval of integration into six equal parts by Weddle's rule. **(2015)**

Sol. Divide the range (0, 6) into six equal parts each of width $h = \frac{1-0}{6} = \frac{1}{6}$.

The values of $y = \frac{1}{1+x^2}$ at the end points of intervals are given below

x		$y = \frac{1}{1+x^2}$	
0	x_0	1	y_0
1/6	$x_0 + h$	0.97297	y_1
2/6	$x_0 + 2h$	0.9	y_2
3/6	$x_0 + 3h$	0.8	y_3
4/6	$x_0 + 4h$	0.69231	y_4
5/6	$x_0 + 5h$	0.59016	y_5
1	$x_0 + 6h$	0.5	y_6

By Weddle's rule,

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{10}\,[y_0 + y_6 + 5\,(y_1 + y_5) + (y_2 + y_4) + 6y_3]$$

$$= \frac{3}{10} \times \frac{1}{6}\,[1 + 0.5 + 5(0.97297 + 0.59016) + (0.9 + 0.69235) + 6 \times 0.8]$$

$$= \frac{1}{20}\,(1.5 + 7.81565 + 1.59231 + 4.8) = \frac{1}{20}\,(15.70796) = 0.785398$$

Long Answer Questions

Q 1. State and prove general quadrature formula. Deduce Trapezoidal rule from the formula. **(2005)**

Sol. Let $I = \int_a^b y\,dx$, where $y = f(x)$ and $f(x)$ be the given for certain equally distant values of arguments, say $x_0, x_0 + h, x_0 + 2h, \ldots$.

Again, let the range $(b - a)$ be divided into n equal parts, each of which is of width 'h', i.e. $b - a = nh$.

Suppose, $y_0, y_1, \ldots, y_n$, where $y_i = f(x_i)$ are the entries corresponding to arguments $x_0 = a, x_1 = a + h, \ldots, x_n = a + nh = b$ respectively.

Clearly, from the given $(n + 1)$ values, we can find the differences upto nth order and hence higher order differences will be zero.

$$\therefore \quad I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \int_0^n y_{x_0+uh} \cdot h\,du \qquad \ldots\text{(i)}$$

[where, new variate $u = \dfrac{x - x_0}{h}$ and $dx = h\,du$]

$$= h\int_0^n \left[y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{2!}\,\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \ldots + \frac{u\,(u-1)(u-n+1)}{n!}\,\Delta^n y_0 \right] dx$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^7}{7} - \frac{5n^6}{2} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2\right)\frac{\Delta^6 y_0}{6!} + \ldots \right] \ldots\text{(ii)}$$

$\therefore$ Eq. (ii) is the required general quadrature formula.

In this, we have integrated function $f(x)$ over all the n-intervals at the same time. In other words, we have found the area of all the n-strips at the same time.

Part II On putting $n = 1$ in Eq. (ii) and neglecting the second and higher order differences, we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}[y_0 + y_1] \qquad [\because \Delta y_0 = y_1 - y_0] \quad \text{...(iii)}$$

Here, Eq. (iii) gives the area of one strip bounded by the ordinates $x = x_0$ and $x = x_0 + h$.

Now, from Eq. (iii), we get

$$\int_{x_0+h}^{x_0+2h} y\,dx \approx \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_0+2h}^{x_0+3h} y\,dx \approx \frac{h}{2}[y_2 + y_3] \quad \text{...(iv)}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx \approx \frac{h}{2}(y_{n-1} + y_n)$$

Hence,

$$I = \int_a^b ydx = \int_{x_0}^{x_0+nh} ydx$$

$$= \int_{x_0}^{x_0+h} ydx + \int_{x_0+h}^{x_0+2h} ydx + \ldots + \int_{x_0+(n-1)h}^{x_0+nh} ydx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}[(y_0 + y_n] + 2(y_1 + y_2 + \ldots + y_{n-1})] \quad \text{...(v)}$$

Here, Eq. (v) gives the approximate value of the integral I.

A formula which is obtained by integrating over the subintervals and then adding, is called composite numerical quadrature formula.

Thus, Eq. (v) gives the composite formula known as Trapezoidal rule.

Q 2. Derive Simpson's one-third formula.

Sol. On putting $n = 2$ in Eq. (ii) of above question and neglecting the third and higher order differences, we get

$$\int_{x_0}^{x_0+2h} ydx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{1}{2}\cdot\frac{\Delta^2 y_0}{1!}\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{2}{3}\cdot\frac{1}{2}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2) \quad \text{...(vi)}$$

Clearly, Eq. (vi) determines the area of two strips, bounded by the ordinates at $x_0, x_0 + h$ and $x_0 + 2h$ at a time, now by using, Eq. (vi), if n is a multiple of 2, i.e. is an even positive integer, we get

$$\int_{x_0+2h}^{x_0+4h} y\,dx \approx \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_0+4h}^{x_0+6h} y\,dx \approx \frac{h}{3}[y_4 + 4y_5 + y_6]$$

... ...

... ...

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx \approx \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n] \quad \text{...(vii)}$$

Hence, $I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$

$$= \int_{x_0}^{x_0+2h} y\,dx + \int_{x_0+2n}^{x_0+4h} f(x)\,dx + \ldots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)dx$$

$$\approx \left(\frac{h}{3}\right)(y_0 + 4y_1 + y_2) + \left(\frac{h}{3}\right)(y_2 + 4y_3 + y_4) + \ldots + \left(\frac{h}{3}\right)(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \left(\frac{h}{3}\right)[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})] \quad \text{...(viii)}$$

Q 3. Derive Simpson's three-eighth rule.

Sol. On putting $n = 3$ in Eq. (ii) of Q. 1 and neglecting fourth and higher order differences, we get

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!}\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] \quad \text{...(ix)}$$

[put the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ and then simplifying]

Clearly, Eq. (ix) determines the area of three strips at a time, which are bounded by the ordinates $x = x_0, x = x_0 + h, x = x_0 + 2h, x = x_0 + 3h$. Now, using Eq. (ix), if n is a multiple of three, we get

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

....

....

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n] \quad \text{...(x)}$$

Hence, $I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y\,dx$

$$= \int_{x_0}^{x_0+3h} y\,dx = \int_{x_0+3h}^{x_0+6h} y\,dx + \ldots + \int_{x_0+(n-3)h}^{x_0+nh} y\,dx$$

$$\approx (3h/8)[y_0 + 3y_1 + 3y_2 + y_3] + \ldots + (3h/8)$$
$$[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$
$$\approx (3h/8)[y_0 + y_n] + 3[y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-2} + y_{n-1})$$
$$+ 2(y_3 + y_6 + \ldots + y_{n-3})] \quad \ldots(\text{xi})$$

Here, Eq. (xi) is known as Simpson's three-eighth rule.

Q 4. Derive Weddle's rule.

Sol. On putting $n = 6$ in Eq. (ii) of Q. 1 and neglecting the differences of orders higher than six, we get

$$\int_{x_0}^{x_0+6h} ydx = h\,[6y_0 + 18\Delta y_0 + \left(\frac{6^3}{3} - \frac{6^2}{2}\right)\frac{\Delta^2 y_0}{2}$$
$$+ \left(\frac{6^3}{3} - 6^3 + 6^2\right)\frac{\Delta^3 y_0}{6} + \left(\frac{6^6}{5} - \frac{3\cdot 6^4}{2} - \frac{11\cdot 6^3}{3} - 3\cdot 6^2\right)\frac{\Delta^4 y_0}{24}$$
$$+ \left(\frac{6^6}{6} - 2\cdot 6^5 + \frac{35}{4}6^4 - \frac{50}{3}6^3 + 12n^2\right)\frac{\Delta^5 y_0}{15}$$
$$+ \left(\frac{6^7}{6} - \frac{15}{6}\cdot 6^6 + 17\cdot 6^5 - \frac{225.6^4}{4} + \frac{274}{3}6^3 - 60\cdot 6^2\right)\frac{\Delta^6 y_0}{6!}$$
$$\approx h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0\right.$$
$$\left. + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$
$$= \frac{3h}{10}[6y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \quad \ldots(\text{xii})$$

Here, we have found the area of six strips at a time.

Now, from Eq. (xii), if n is a multiple of six, we get

$$\int_{x_0+6h}^{x_0+12n} ydx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

....

....

$$\int_{x_0(n-6)h}^{x_0+nh} ydx \approx \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2}$$
$$+ 5y_{n-1} + y_n] \quad \ldots(\text{xiii})$$

Hence, $I = \int_{x_0}^{x_0+nh} ydx = \int_{x_0}^{x_0+6h} ydx + \ldots + \int_{x_0+(n-6)h}^{x_0+nh} ydx$

$$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + \ldots] \quad \ldots(\text{xiv})$$

Here, Eq. (xiv) gives the Weddle's rule of numerical integration.
In this, we integrate the function over six subintervals at a time. The drawback of this method is that it reuires at least seven consecutive values of $f(x)$. Also, this method is applicable when n is a mulple of six.

Q 5. Evaluate $\int_4^{5.2} \log x \, dx$ by taking $h = 0.2$ or by dividing the interval of integration into six equal parts and by using

(i) Trapezoidal rule.
(ii) Simpson's one-third rule.
(iii) Simpson's three-eighth rule.
(iv) Weddle's rule.

After finding actual value of the above integral by direct integrating, compare the values obtained by using the above four rules and decide which of the four rules is most accurate? **(2014)**

Sol. Here, $h = 0.2$, $x_0 = 4.0$, $y = \log_e x$

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5260	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) **By Trapezoidal rule**

$$\int_4^{5.2} \log_e x dx = \frac{(0.2)}{2} [1.3863 + 1.6486 + 2\,(1.4351 + 1.4816 + 1.5260 + 1.5686 + 1.6094]$$

$$= 1.8276$$

Also, $\int_4^{5.2} \log_e x dx = [x \log x - x]_4^{5.2} = 5.2 \log 5.2 - 2.2 - 4 \log 4 + 4 = 1.8280$

$\therefore$ Error term $= 1.8280 - 1.8276 = 0.0004$

(ii) **By Simpson's one-third rule**

$$\int_4^{5.2} \log_e x dx = \frac{h}{3} [y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2\,(y_2 + y_4)]$$

$$= \frac{0.2}{3} [1.3863 + 1.6486 + 4(1.4351 + 1.5260 + 1.6094) + 2(1.4816 + 1.5686)]$$

$$= \frac{0.2}{3} [3.0349 + 4 \times 4.5704 + 2 \times 3.0502]$$

$$= \frac{0.2}{3} [3.0349 + 18.2816 + 6.1004]$$

$$= \frac{0.2}{3} \times 27.4169 = 1.8278$$

$\therefore$ Error term $= 1.8280 - 1.8278 = 0.0002$

(iii) **By Simpson's three-eighth rule**

$$\int_4^{5.2} \log_e x dx = \frac{3h}{8} [y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 0.2}{8} [1.3863 + 1.6486 + 3(1.4351 + 1.4816 + 1.5286 + 1.6094) + 2 \times 1.5260]$$

$$= 0.075 [3.0349 + 3(6.0547) + 3.052] = 0.075 \times 24.251 = 1.8279$$

$\therefore$ Error term $= 1.8280 - 1.8279 = 0.0001$

(iv) **By Weddle's rule**

$$\int_4^{5.2} \log_e x dx = \frac{3h}{10}[y_0 + y_6 + 5(y_1 + y_5) + (y_2 + y_4) + 6y_3]$$

$$= \frac{3 \times 0.2}{10} [1.3863 + 1.6486 + 5(1.4351 + 1.6094) + 1.4816 + 1.5686 + 6 \times 1.5260]$$

$$= 0.06 (3.0349 + 5 \times 3.0445 + 3.0502 + 9.156)$$

$$= 0.06 (3.0349 + 15.2225 + 3.0502 + 9.152)$$

$$= 0.06 (30.4596) = 1.8275$$

$\therefore$ Error term $= 1.8280 - 1.8285 = 0.0005$

***Q* 6. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval of integration into 6 equal parts and by using**

(i) Trapezoidal rule.

(ii) Simpson's one-third rule.

(iii) Simpson's three-eighth rule.

Compare the values so obtained with the exact value of integral. Hence, obtain the approximate value of π in each case. **(2016, 13, 12, 09)**

Sol. Divide the range (0, 1) into six equal parts each of width $h = \frac{1}{6}$.

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = \frac{1}{1+x^2}$	1	0.97297	0.9	0.8	0.69231	0.59016	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) **By Trapezoidal rule**

$$\int_0^1 \frac{1}{1+x^2} dx = h\left[\frac{y_0 + y_6}{2} + y_1 + y_2 + y_3 + y_4 + y_5\right]$$

$$= \frac{1}{6}\left[\frac{1 + 0.5}{2} + 0.97297 + 0.9 + 0.8 + 0.69231 + 0.59016\right]$$

$$= \frac{1}{6} [0.75 + 3.95544] = \frac{4.70544}{6} = 0.78424$$

Actual value of integral

$$\int_0^1 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4} = \frac{11}{14} = 0.785714$$

$\therefore$ Error term $= 0.785714 - 0.78424 = 0.001474$

(ii) **By Simpson's one-third rule**

$$\int_2^1 f(x)\, dx = \frac{h}{3} [y_0 + y_6 + 4\,(y_1 + y_2 + y_3) + 3(y_1 + y_4)]$$

$$= \frac{1}{18} [1.5 + 4\ (0.97297 + 0.8 + 0.59016 + 2\ (0.9 + 0.69231)] = 0.75397$$

$\therefore$ Error term $= 0.785714 - 0.75397 = 0.031744$

(iii) **By Simpson's three-eighth rule**

$$\int_0^1 f(x)\, dx = \frac{3h}{8} [y_0 + y_6 + 3\,(y_1 + y_2 + y_3 + y_4 + y_5) + 2y_3]$$

$$= \frac{1}{16} [1.5 + 3\ (0.97297 + 0.9 + 0.69231 + 0.59016 + 2 \times 0.8)]$$

$$= \frac{1}{16} [1.5 + 3 \times 3.15544 + 1.6] = \frac{1}{16} [1.5 + 9.46632 + 1.6]$$

$$= \frac{12.56632}{16} = 0.785395$$

$\therefore$ Error term $= 0.785714 - 0.785395 = 0.000319$

Q 7. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using Simpson's one-third rule. Divide the interval [0, 1] into 5 equal parts. **(2017)**

Sol. Divide the interval [0, 1] into six equal parts, each of width $h = \frac{1-0}{6}$, i.e. $h = \frac{1}{6}$.

The values of $f(x) = \frac{1}{1+x}$ are given below

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	6/7	6/8	6/9	6/10	6/11	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$\therefore$ By Simpson's one-third rule,

$$\int_0^1 \frac{1}{1+x}\, dx = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{18}\left[\left(1 + \frac{1}{2}\right) + 2\left(\frac{6}{6} + \frac{6}{10}\right) + 4\left(\frac{6}{7} + \frac{6}{9} + \frac{6}{11}\right)\right] = \frac{12.4771}{18} = 0.6937$$