Chapter Two

POINSOT'S CENTRAL AXIS, WRENCHES, NULL LINES & PLANES

Important Points from the Chapter

1. **Central Axis or Poinsot's Central Axis** The line along which the single resultant force of magnitude R acts and which is also the axis of the single couple of moment $G \cos \theta$ to which the system of forces acting on a body is reduced, is the called the Poinsot's central axis of the system of forces. **(2018, 17, 14, 11, 09, 07, 01)**
2. **Characteristics of a Central Axis**
   (i) Central axis for a system of forces acting on a rigid body is unique.
   (ii) The moment $G \cos \theta$ of the resultant couple about central axis is less than the moment G of the resultant couple corresponding to any other point O (not lying on the central axis) as $G \cos \theta < G$.
3. **Wrench** Let a system of forces acting at different point of a rigid body be reduced to a single force R and a couple of moment K. Then, R and K taken together are called wrench of the system of forces and written as (R, K). The single force, R is often called the intensity of wrench. **(2017, 16, 15, 13, 10)**
4. **Pitch of Wrench** The ratio $\frac{K}{R}$ is called the pitch of the wrench and is denoted by p. Thus, $\frac{K}{R} = p \Rightarrow K = Rp$. **(2013)**
5. **Screw** The straight line along which the single force acts when considered together with the pitch, is called screw. Thus, screw is a definite straight line associated with a definite pitch.

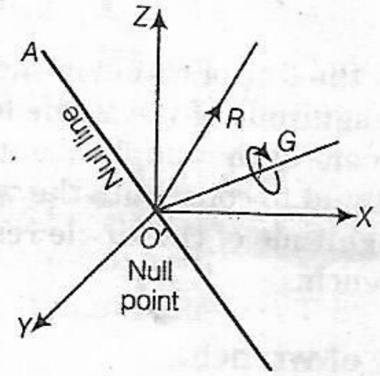
   ■ Note

   (i) The condition that a general system of forces in space should reduce to a single force, is $LX + MY + NZ = O$, where X, Y, Z are the algebraic sums of the forces along coordinate axes and L, M, N are the algebraic sums of the moments about the axes.

   (ii) Whatever origin and the axes are chosen the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariants for any given system of forces acting on a rigid body.
6. **Equation of the Central Axis** Let a given system of forces $F_1 = (X_1, Y_1, Z_{1)}$, $F_2 = (X_2, Y_2, Z_2)$, ... act at the different points $P_1 (x_1, y_1, z_1)$, $P_2 (x_2, y_2, z_2)$, ... of a rigid body.

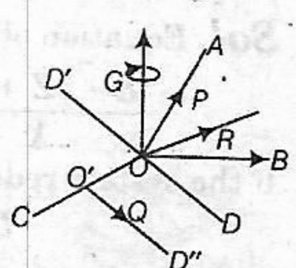
Then, $\frac{L-(yZ-zY)}{X}=\frac{M-(zX-xZ)}{Y}=\frac{N-(xY-yX)}{Z}=\frac{K}{R}=p$ is the equation of central axis and $p=\frac{K}{R}$ is called the **pitch** of the wrench.

7. **Null Lines** Null lines of a given system of forces, referred to any origin or base point O', are those lines about which the moment of the system vanishes.



(2018, 16, 14, 09, 05, 03, 01)

8. **Null Plane** The plane in which all these null lines lie, is called the null plane of the point O'. **(2009, 05, 03, 01)**
9. **Equation of Null Plane** Equation of a null plane of a given point (f, g, h) referred to any axis OX, OY, OZ will be
$$x(L-gZ+hY)+y(M-hX+fZ)+z(N-fY+gX)=Lf+Mg+Nh.$$
10. **Null Point** The point O' itself is called the null point. **(2001, 03, 05, 09)**
11. **To find the Null Point** Let (f, g, h) be the null point of the given plane $lx+my+nz=1$. Then, the null point (f, g, h) of the given plane can be obtained by
$$\frac{f}{X-\begin{vmatrix} M & N \\ m & n \end{vmatrix}}=\frac{g}{Y-\begin{vmatrix} N & L \\ n & l \end{vmatrix}}=\frac{h}{Z-\begin{vmatrix} L & M \\ l & m \end{vmatrix}}=\frac{1}{lX+mY+nZ}.$$
12. **Condition for a Straight Line to be a Null Line** The condition for a straight line $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ to be a null line for the system of forces (X, Y, Z, L, M, N) is $\begin{vmatrix} X & Y & Z \\ l & m & n \\ f & g & h \end{vmatrix}=lL+mM+nN.$
13. **Conjugate Forces and Conjugate Lines** A given system of forces may be replaced by two forces P and Q, one of which acts along a given line. Such forces P and Q are called conjugate forces and their lines of action are called conjugate lines. **(2016, 11, 07)**



■ **Note** A given system of forces may be replaced by two forces, one of which acts along a given line OA. This replacement can be in an infinite number of ways and the tetrahedron formed by two forces is of constant volume.

Very Short Answer Questions

Q 1. Define wrench for a system of forces acting on a rigid body. **(2012)**

Sol. Suppose, a system of forces acting at different points of a rigid body reduces to a single force R acting at an arbitrarily chosen point O and a couple of moment G.

Let θ be an angle between the line of action of the force R and the axis of the couple G. If R is the magnitude of the single force and $K = G\cos\theta$ be the magnitude of the moment of the couple about the central axis, then R and K taken together are said to constitute the wrench of the system and written as (R, K). The magnitude of the single resultant force R is known as the intensity of the wrench.

Q 2. Define intensity of wrench. **(2018, 16)**

Sol. **Wrench** Let a system of forces acting at different point of a rigid body be reduced to a single force R and a couple of moment K. Then, R and K taken together are called wrench of the system of forces and written as (R, K). The single force, R is often called the intensity of wrench.

Q 3. What is the condition for a straight line to be a null line? **(2015)**

Sol. Let system of forces be equivalent to a dyname $(X, Y, Z; L, M, N)$ and (f, g, h) be the coordinates of the null point O'.

Again, let $\dfrac{x-f}{l} = \dfrac{y-g}{m} = \dfrac{z-h}{n}$

be the equation of straight line through $O'(f, g, h)$, then the condition for a straight line to be null line through $O'(f, g, h)$ will be a

$$\begin{vmatrix} X & Y & Z \\ l & m & n \\ f & g & h \end{vmatrix} = lL + mM + nN$$

Q 4. Write the equation of the line of action of single resultant. **(2008)**

Sol. Equation of the line of action of single resultant is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = p$$

If the system reduces to a single force, then

$$LX + MY + NZ = 0 \qquad \text{...(i)}$$

and $$p = \frac{K}{R} = \frac{G\cos\theta}{R} = \frac{GR\cos\theta}{R^2} = \frac{LX + MY + NY}{X^2 + Y^2 + Z^2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $p = 0$

$$\Rightarrow \quad \frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = 0$$

which are the equations of the line of action of single resultant force.

Q 5. Write down the equation of null plane through (3, 5, 7) for the system $(L, M, N; X, Y, Z)$. **(2006)**

Sol. As we know that the equation of a null plane of a point (f, g, h) for the given system (L, M, N, X, Y, Z) is

$$x(L - gZ + hY) + y(M - hX + fZ) + z(N - fY + gX) = Lf + Mg + Nh$$

Since, the given point is (3, 5, 7).

$$\therefore \quad f = 3, g = 5, h = 7$$

Therefore, the equation of the null plane will be

$$x(L - 5Z + 7Y) + y(M - 7X + 3Z) + z(N - 3Y + 5X) = 3L + 5M + 7N$$

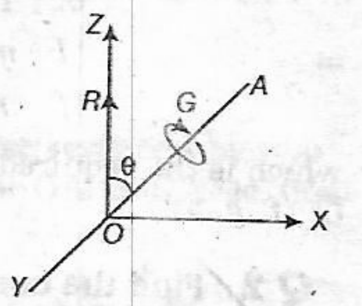
$$\Rightarrow \quad L(x - 3) + M(y - 5) + N(z - 7) + X(5z - 7y) + Y(7x - 3z) + Z(3y - 5x) = 0$$

which is the required equation of a null plane of (3, 5, 7) for the system (L, M, N, X, Y, Z).

Q 6. Show that $LX + MY + NZ = RK$, where $K = G \cos\theta$ **(2006)**

Sol. For any given system of forces,

$$R^2 = X^2 + Y^2 + Z^2 \text{ and } G^2 = L^2 + M^2 + N^2$$



where, R is the resultant of forces acting at an arbitrary point O and G is the resultant couple of moment about an axis through O. The direction cosines of the line of action R and the axis of G are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$ respectively,

where X, Y and Z are algebraic sums of forces along coordinate axes and L, M, N are algebraic sums of moment about axes.

If θ is the angle between the line of action of R and the axis of G, then

$$\cos\theta = \frac{LX + MY + NZ}{GR} \qquad [\because \cos\theta = ll' + mm' + nn']$$

$$\Rightarrow \quad G\cos\theta = \frac{LX + MY + NZ}{R}$$

$$\Rightarrow \quad K = \frac{LX + MY + NZ}{R}$$

$$\Rightarrow \quad LX + MY + NZ = RK \qquad [\text{where, } K = G\cos\theta]$$

Short Answer Questions

Q 1. Find the condition that $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ may be a null line for some system of forces. **(2010, 2008)**

Or **Find the condition under which a given line is a null line.**
(2007, 04, 02, 2000, 1998, 97, 94, 92)

Sol. Let the system is reduced to a dyname $(X, Y, Z; L, M, N)$.

Again, let the coordinates of O' be (f, g, h) and $O'X'$, $O'Y'$, $O'Z'$ be the lines through O' and parallel to OX, OY, OZ, respectively.

The moments of couples about these lines, then

$$L' = L - (gZ - hY),\ M' = M - (hX - fZ) \text{ and } N' = N - (fY - gX)$$

Since, the direction ratios of the axis of the resultant couple at O' are L', M' and N'.

The line $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ with direction ratios l, m, n will be null line through O', if it is perpendicular to the axis of the couple at O'.

Therefore, the condition of perpendicularity gives

$$lL' + mM' + nN' = O$$

$$\Rightarrow \quad l(L - gZ + hY) + m(M - hX + fZ) + n(N - fY + gX) = 0$$

$$\Rightarrow \quad lL + mM + nN = X(-ng + mh) - Y(-nf + hl) + Z(-mf + lg)$$

$$\Rightarrow \quad \begin{vmatrix} X & Y & Z \\ l & m & n \\ f & g & h \end{vmatrix} = lL + mM + nN$$

which is the required condition for the given line to be a null line through $O'(f, g, h)$.

Q 2. Find the condition under which the given line $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ is a null line. **(2010)**

Sol. Do same as Q. 1.

Ans. The required condition is $\begin{vmatrix} X & Y & Z \\ l & m & n \\ x_0 & y_0 & z_0 \end{vmatrix}$.

Q 3. Find the null point of the plane $x + y + z = 0$ for the system of forces given by $(X, Y, Z; L, M, N)$. **(2011, 08, 04, 1996)**

Sol. Let (f, g, h) be the null point of the plane

$$x + y + z = 0 \qquad \text{...(i)}$$

As (f, g, h) lies on the plane (i), then we have

$$f + g + h = 0 \quad \text{...(ii)}$$

But the null plane of (f, g, h) is

$$x(L - gZ + hY) + y(M - hX + fZ) + z(N - fY + gX) = Lf + Mg + Nh$$

On comparing this with the plane (i), we get

$$\frac{L - gZ + hY}{1} = \frac{M - hX + fZ}{1} = \frac{N - fY + gX}{1} = \frac{Lf + Mg + Nh}{0}$$

Taking first, second ratios and second, third ratios, and third and first ratios, we get

$$L - gZ + hY = M - hX + fZ$$
$$M - hX + fZ = N - fY + gX$$
$$N - fY + gX = L - gZ + hY$$

or $\quad L - M = -hY - hX + (f + g)Z = -hY - hX - hZ \quad$ [from Eq. (ii)]

$M - N = (g + h)X - fZ - fY = -fX - fZ - fY \quad$ [from Eq. (ii)]

$N - L = (f + h)Y - gX - gZ$

$= -gY - gX - gZ \quad$ [from Eq. (ii)]

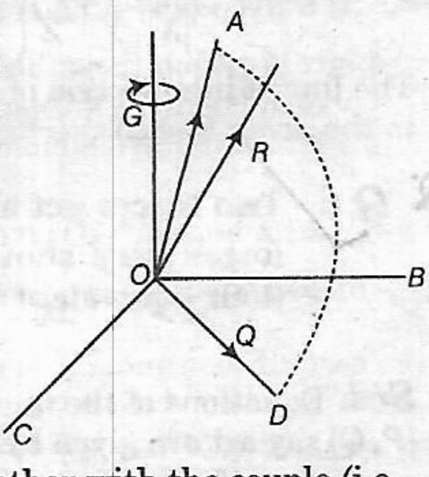
$\Rightarrow \quad h = -(L - M)k, f = -(M - N)k, g = -(N - L)k$

where, $\quad k = \dfrac{1}{X + Y + Z}$

Thus, $\left(\dfrac{N - M}{X + Y + Z}, \dfrac{L - N}{X + Y + Z}, \dfrac{M - L}{X + Y + Z}\right)$ are the coordinates of the null point.

Q 4. Show that a given system of forces may be replaced by two forces, one of which acts along a given line *OA*. (2009, 2000, 1995)

Sol. Let the system reduces to the resultant force R and couple G at the point O. Suppose BOC is the plane of the couple, so that the axis of the couple is perpendicular to the plane BOC.



Now, drawing a plane through the given line OA and the line of action of R and suppose that this plane cuts the plane BOC in a line OD. Consequently, line of action of R, OA, OD all are coplanar.

Resolve R along OA and OD. Let the components of R along OA and OD be respectively P and Q. The force Q along OD together with the couple (i.e. two forces of the couple) will give rise to a force Q (in the plane BOC) parallel to OD. The plane BOC is the null plane of O.

Hence, the system is reduced to a force P along OA and other force Q is in null plane of O but parallel to OD. **Hence proved.**

Q 5. Find the equation of a line which is conjugate to a given line.

(2011)

Sol. Let the system of forces with respect to O as the origin be

$$(X, Y, Z; L, M, N)$$

In order to find the equations of the conjugate line of the line

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n} \qquad \text{...(i)}$$

We have to find out the equations of null planes of any two chosen points on the line (i). Let one point be (f, g, h) and the other point $(l\rho, m\rho, n\rho)$ at infinity where $\rho \to \infty$. These two point lie on Eq. (i).

Equation of the null plane through (f, g, h) is

$$\begin{vmatrix} x & y & z \\ f & g & h \\ X & Y & Z \end{vmatrix} = L(x-f) + M(y-g) + N(z-h) \qquad \text{...(ii)}$$

and the null plane through $(l\rho, m\rho, n\rho)$ is

$$\begin{vmatrix} x & y & z \\ l\rho & m\rho & n\rho \\ X & Y & Z \end{vmatrix} = L(x - l\rho) + M(y - m\rho) + N(z - n\rho)$$

On dividing by ρ, we get

$$\begin{vmatrix} x & y & z \\ l & m & n \\ X & Y & Z \end{vmatrix} = L\left(\frac{x}{\rho} - l\right) + M\left(\frac{y}{\rho} - m\right) + N\left(\frac{z}{\rho} - n\right) \qquad \text{...(iii)}$$

As $\rho \to \infty$, then Eq. (iii) reduces to

$$\begin{vmatrix} x & y & z \\ l & m & m \\ X & Y & Z \end{vmatrix} = -(Ll + Mn + Nn)$$

The line of intersection of null planes Eqs. (ii) and (iii) is a conjugate line to the given line (i).

imp **Q 6. Two forces act along lies $y = 0, z = 0$ and $x = 0, z = c$. As the forces vary, show that the surface, generated by the axis of their equivalent wrench is, $(x^2 + y^2)z = cy^2$.**

(2016, 10, 07, 05, 04, 02, 1997, 95, 94, 92)

Sol. Equations of the lines along which forces of different magnitudes (P, Q) say act are given by

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0} \text{ and } \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-c}{0}$$

From above, it is obvious that the force P acts at the point $(0, 0, 0)$ along the line whose direction cosines are $(1, 0, 0)$.

The force Q acts at the point $(0, 0, c)$ along the line whose direction cosines are $(0, 1, 0)$.

Now, if the dyname are $(X, Y, Z; L, M, N)$, then

$$X = X_1 + X_2 = Q \cdot 0 + P \cdot 1 = P$$
$$Y = Y_1 + Y_2 = P \cdot 0 + Q \cdot 1 = Q$$
$$Z = Z_1 + Z_2 = P \cdot 0 + Q \cdot 0 = 0$$
$$L = \Sigma(y_1 Z_1 - z_1 Y_1) = -cQ$$
$$M = \Sigma(z_1 X_1 - x_1 Z_1) = 0$$

and
$$N = \Sigma(x_1 Y_1 - y_1 X_1) = 0$$

$\therefore$ Equation of the central axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

i.e.
$$\frac{-cQ + zQ}{P} = \frac{-zP}{Q} = \frac{-xQ + yP}{0}$$

We are to eliminate P and Q from first and second equations, we get

$$(z - c)\frac{Q}{P} = -z\frac{P}{Q} \Rightarrow (z - c)\frac{y}{x} = -z\frac{x}{y} \qquad \left[\because \frac{P}{Q} = \frac{x}{y}\right]$$

$$\Rightarrow \quad (c - z)y^2 = zx^2$$

$$\Rightarrow \quad cy^2 = z(x^2 + y^2)$$

Hence proved.

Q 7. What is the condition of single resultant of a system of forces acting on a rigid body? **(2013)**

***Or* Describe the condition of single resultant force.** **(2007)**

***Or* Show that the condition that a general system of forces in the space should reduce to a single force is $LX + MY + NZ = 0$.**

Sol. We know that a system of forces acting at different points of rigid body in space can be reduced to a single force R acting at an arbitrary chosen point O together with a single couple of moment G about an axis through point O.

Let R acts along a line OA and G acts about an axis OD making an angle θ with OA. Then, $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$ are direction cosines of OA and OD respectively, where X, Y, Z are components of R along coordinates axes and L, M, N are components of G about axes.

Now,
$$\cos\theta = \frac{XL + YM + ZN}{GR}$$

$$\Rightarrow \quad XL + YM + ZN = GR\cos\theta \qquad \text{...(i)}$$

If $\theta = \frac{\pi}{2}$, then $G\cos\theta = 0$

Then, the system is reduced to a single force R and the condition for this is $\cos\theta = 0$.

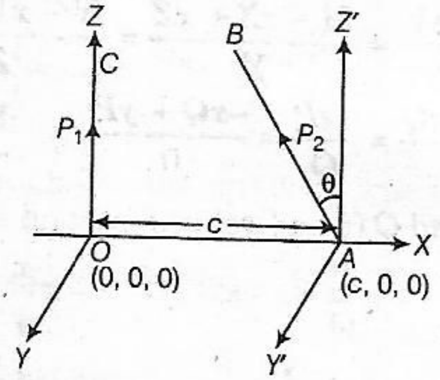
Hence, Eq. (i) becomes

$LX + MY + NZ = 0$, which is the required condition.

Q 8. Find the invariant of two forces P_1 and P_2 inclined at an angle θ and at a distance 'c' apart. **(2007)**

Or **Find the invariant I in terms of forces P_1 and P_2 inclined at an angle θ, at a distance 'c' apart.** **(2002)**

Sol. Let OC and AB be the axes of forces P_1 and P_2, respectively and let θ be the angle between OC and AB.



If (X, Y, Z, L, M, N) be the dyname of the given system, then $X = 0 + 0 = 0$.

$$Y = 0 + P_2 \cos\left(\frac{\pi}{2} - \theta\right) = P_2 \sin\theta \text{ and } Z = P_1 + P_2\cos\theta$$

Now, for finding L, M, N, we make the following scheme

Coordinates of the point of application of the force	0	0	0	c	0	0
Components of forces	0	0	P_1	0	$P_2\sin\theta$	$P_2\cos\theta$

$\therefore \quad L = 0 + 0 = 0, \quad M = 0 - cP_2\cos\theta = -cP_2\cos\theta$

and $N = 0 + cP_2\sin\theta = cP_2\sin\theta$

$\therefore$ Resultant force $R = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{P_1^2 + P_2^2 + 2P_1P_2\cos\theta}$

and resultant couple $K = \sqrt{L^2 + M^2 + N^2}$, $K = \sqrt{c^2P_2^2} = cP_2$

Hence, invariant $I = LX + MY + NZ$

i.e. $I = 0 - cP_2^2\cos\theta\sin\theta + P_1P_2\, c\sin\theta + cP_2^2\cos\theta\sin\theta \Rightarrow I = P_1P_2\, c\sin\theta$

Q 9. A system of forces given by (X, Y, Z, L, M, N) is replaced by two forces, one acting along the axis of X and another force. Show that the magnitude of the forces are

$$\frac{LX + MY + NZ}{L} \text{ and } \frac{[(MY + NZ)^2 + L^2(Y^2 + Z^2)]^{1/2}}{L}.$$

Sol. Suppose the given system of forces reduces to a dyname $(X, Y, Z; L, M, N)$. Let a force P acts at $(0,0,0)$ along the axis of X whose equations are

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0} \qquad \text{...(i)}$$

The components of this force parallel to axes are P, 0, 0. Also, components of couple due to this force at $(0,0,0)$ about the lines parallel to axes are 0, 0, 0.
[as $L = yZ - zY = 0 \cdot 0 - 0 \cdot 0 = 0$]

Therefore, the components of other force parallel to axes are $X - P, Y, Z$ and the components of couple due to this force parallel to axes are L, M, N.

Let this second force acts at $(f, g, 0)$.

Then, $\quad L = gZ, M = -fZ$ and $N = fY - g(X-P)$

$$\Rightarrow \quad N = -\frac{M}{Z}Y - \frac{L}{Z}(X-P) \Rightarrow NZ + MY + LX - LP = 0$$

$$\Rightarrow \quad P = \frac{LX + MY + NZ}{L} \Rightarrow X - P = X - \frac{LX + MY + NZ}{L}$$

$$= -\frac{(MY + NZ)}{L}$$

$\therefore$ Magnitude of the second force

$$= [(X-P)^2 + Y^2 + Z^2]^{1/2} = \left[\left(\frac{MY + NZ}{L}\right)^2 + Y^2 + Z^2\right]^{1/2}$$

$$= \left(\frac{1}{L}\right)[(MY + NZ)^2 + L^2(Y^2 + Z^2)]^{1/2}$$

Thus, magnitude of the first force, $P = \dfrac{(LX + MY + NZ)}{L}$

and magnitude of second force $= \left(\dfrac{1}{L}\right)[(MY + NZ)^2 + L^2(Y^2 + Z^2)]^{1/2}$

Hence proved.

Long Answer Questions

Q 1. Derive equation of centrals axis for a system of forces.
(2018, 14, 12, 09, 07, 05, 03, 01)

Or **Define central axis and derive its equation. (2010)**

Or **Find the equation of central axis. (2018, 16)**

For the central axis, establish the equation

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}. \qquad \textbf{(2008)}$$

Sol. **Part I Central Axis** The line along which the single resultant force of magnitude R acts and which is also the axis of the single couple of moment $G\cos\theta$ to which the system of forces acting on a body is reduced, is the called the Poinsot's central axis of the system of forces.

Part II Let a given system of forces $F_1 = (X_1, Y_1, Z_1)$, $F_2 = (X_2, Y_2, Z_2)$, ... act at different points $P_1\,(x_1, y_1, z_1)$, $P_2\,(x_2, y_2, z_2)$, ... of a rigid body.

Then, the given system of forces is equivalent to

$$R = X^2 + Y^2 + Z^2 \quad\text{and}\quad G = L^2 + M^2 + N^2$$

where, $X = \Sigma X_1, Y = \Sigma Y_1, Z = \Sigma Z_1$

and $L = \Sigma(y_1 Z_1 - z_1 Y_1),\ M = \Sigma\,(z_1 X_1 - x_1 Z_1),\ N = \Sigma(x_1\,Y_1 - y_1 X_1)$

Taking any point $O'\,(f, g, h)$ on the central axis. At point O', R remains invariant.

In order to find new values (L', M', N') of (L, M, N) of the components of G about a line through O', replace x_1, y_1, z_1 by $x_1 - f,\ y_1 - g,\ z_1 - h$, respectively in the values of L, M, N.

Therefore,
$$\begin{aligned}L' &= \Sigma[(y_1 - g)Z_1 - (z_1 - h)Y_1]\\ &= \Sigma[(y_1 Z_1 - gZ_1 - z_1 Y_1 + hY_1)]\\ &= \Sigma[(y_1 Z_1 - z_1 Y_{1)} - gZ_1 + hY_1]\\ &= \Sigma(y_1 Z_1 - z_1 Y_1) - g\,\Sigma Z_1 + h\Sigma Y_1\\ &= L - gZ + hY\end{aligned}$$

Similarly, $M' = M - hX + fZ$ and $N' = N - fY + gX$

Since, the axis of the couple and the line of action of P coincide and O' lies on the central axis, therefore $\dfrac{L'}{X} = \dfrac{M'}{Y} = \dfrac{N'}{Z}$

On putting the values of L', M', N', we get

$$\frac{L - gZ + hY}{X} = \frac{M - hX + fZ}{Y} = \frac{N - fY + gX}{Z}$$

$$\therefore\ \frac{L - gZ + hY}{X} = \frac{M - hX + fZ}{Y} = \frac{N - fY + gX}{Z}$$

$$= \frac{LX - gZX + hYX + MY - hXY + fZY + NZ - fYZ + gXZ}{X^2 + Y^2 + Z^2}$$

$$= \frac{LX + MY + NZ}{R^2} = \frac{G\,R\cos\theta}{R^2} = \frac{G\cos\theta}{R} = \frac{K}{R}$$

$$\Rightarrow \frac{L - gZ + hY}{X} = \frac{M - hX + fZ}{Y} = \frac{N - fY + gX}{Z} = \frac{K}{R} = P$$

Hence, the locus of (f, g, h) is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{K}{R}$$

$$\Rightarrow \frac{L - (yZ - zY)}{X} = \frac{M - (zX - xZ)}{Y} = \frac{N - (xY - yX)}{Z} = \frac{K}{R} = P$$

which is the required equation of central axis.

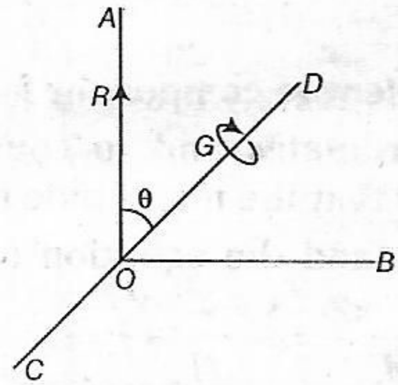
Q 2. Discuss the forces in three dimensions and obtain Poinsot's central axis. **(2008, 1995, 94)**

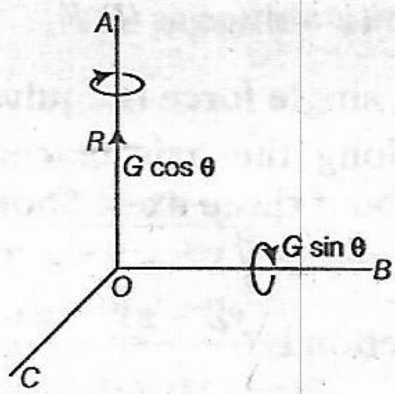
Or **Show that a given system of forces acting on a rigid body can be reduced to a force together with a couple whose axis coincides with the direction of the force.**

Or **Show that every given system of forces acting on a rigid body can be reduced to a wrench.**

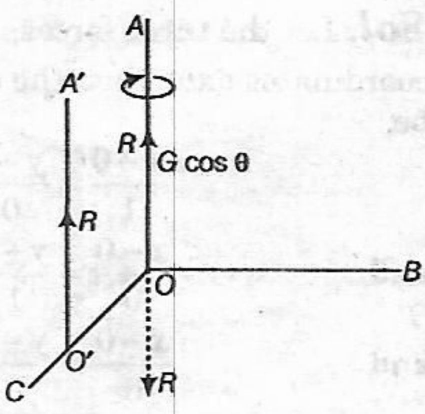
Sol. Let a system of forces acting on a rigid body be reduced to a single force R acting at O along OA and a couple of moment G whose axis is the line OD passing through O and $\angle AOD = \theta$.

Now, drawing a line OB perpendicular to OA such that OA, OB and OD lie in one plane. Also, drawing a line OC perpendicular to the plane AOB.





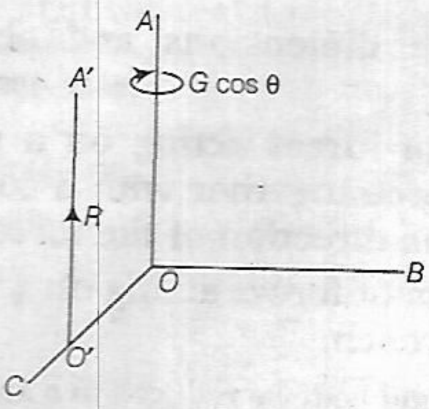
The couple of moment G acting about OD is equivalent to a couple of moment $G \cos\theta$ about OA and a couple of moment $G \sin\theta$ about OB. The line OB is perpendicular to the plane AOC. Therefore, the couple of moment $G \sin\theta$ about OB acts in this plane AOC and hence it can be replaced by two equal and unlike parallel forces in the plane AOC. Let one of these forces be R at O in the direction opposite to OA, then the other force must be equal to R acting parallel to OA at some point say O' in OC, such that

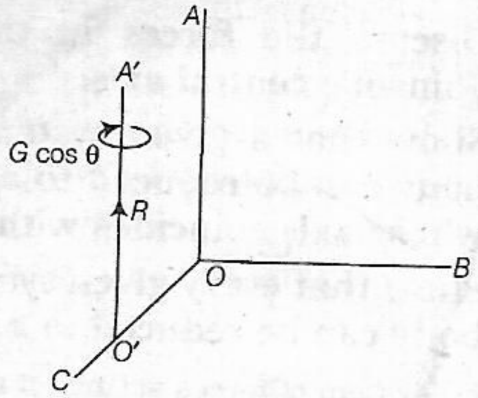


$R, OO' = G \sin\theta$, i.e. $OO' = \dfrac{G \sin\theta}{R}$

Now, two equal forces of magnitude R acting at O in opposite directions balance each other and thus we are left with a force R at O' acting along $O'A'$ and a couple of moment $G \cos\theta$ about the parallel line OA.

Since, the axis of couple can be transferred to any parallel axis, therefore we transfer the axis of couple $G \cos\theta$ from OA to $O'A'$.

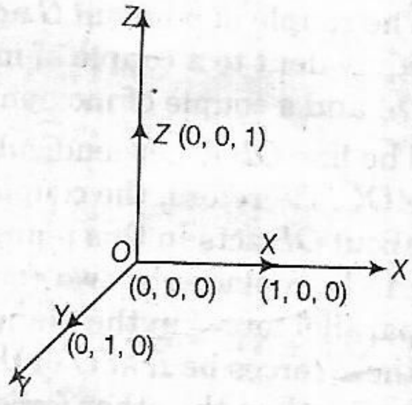




Hence, the system of forces acting on the rigid body is equivalent to a single force R acting along the line $O'A'$ and a single couple of moment $G\cos\theta$ about the same line $O'A'$. The axis $O'A'$ is called poinsot's central axis and force R and couple of moment $G\cos\theta = K$ together are called wrench and is written as (R, K).

Q 3. A single force is equivalent to component forces X, Y, Z along the axis of coordinates and to couple L, M, N about three axes. Show that the magnitude of the single force is $\sqrt{X^2+Y^2+Z^2}$ and the equation of its line of action is $\dfrac{yZ-zY}{L}=\dfrac{zX-xZ}{M}=\dfrac{xY-yX}{N}=1.$ **(2005, 03)**

Sol. Let the three forces, X, Y, Z act along the coordinates axis, then the equation of line will be



$$\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0} \quad \ldots\text{(i)}$$

and $$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0} \quad \ldots\text{(ii)}$$

and $$\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1} \quad \ldots\text{(iii)}$$

Let the given system of forces reduced to $(X, Y, Z; L, M, N)$, then

$$X = X+0+0 = X, Y = 0+Y+0 = Y$$

and $$Z = 0+0+Z = Z$$

$\therefore$ $$R = \sqrt{X^2+Y^2+Z^2}$$

Since, the given system is reduced to a single force, then

$$LX+MY+NZ=0 \quad \ldots\text{(iv)}$$

and the equation of the central axis is

$$\frac{L-(yZ-zY)}{X}=\frac{M-(zX-xZ)}{Y}=\frac{N-(xY-yX)}{Z}=\frac{LX+MY+NZ}{R^2}$$

$$\Rightarrow \frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = 0 \quad \text{[from Eq. (iv)]}$$

$$\Rightarrow L - yZ + zY = 0 \Rightarrow L = yZ - zY \Rightarrow \frac{yZ - zY}{L} = 1 \quad \text{...(v)}$$

$$M - zX + xZ = 0 \Rightarrow \frac{zX - xZ}{M} = 1 \quad \text{...(vi)}$$

and $$N - xY + yX = 0 \Rightarrow \frac{xY - yX}{N} = 1 \quad \text{...(vii)}$$

From Eqs. (v), (vi) and (vii), we have

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1$$ **Hence proved.**

Q 4. Equal forces act along the axes and along the straight line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\lambda}{n}$. Find the equation of the central axis of the system. **(2005, 03)**

Sol. Let the equal forces F act along the lines

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}, \quad \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$$

$$\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1} \text{ and } \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

The components of forces along the axes and straight line are

$$(F \cdot 1, F \cdot 0, F \cdot 0); (F \cdot 0, F \cdot 1, F \cdot 0); (F \cdot 0, F \cdot 0, F \cdot 1) \text{ and } (F \cdot l, F \cdot m, F \cdot n)$$

$\Rightarrow (F, 0, 0); (0, F, 0); (0, 0, F)$ and (Fl, Fm, Fn).

Therefore, $X = F + 0 + 0 + Fl = F(1 + l)$

$$Y = 0 + F + 0 + Fm = F(1 + m)$$

$$Z = 0 + 0 + F + Fn = F(1 + n)$$

$$L = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2) + (y_3 Z_3 - z_3 Y_3) + (y_4 Z_4 - z_4 Y_4)$$

$$= F(\beta n - \gamma m)$$

Similarly, $M = F(\gamma l - \alpha n)$ and $N = F(\alpha m - \beta l)$

On putting the values of X, Y, Z and L, M, N in the equation of central axis, we get

$$\frac{F(\beta n - \gamma m) - \{yF(1+n) - zF(1+m)\}}{F(1+l)}$$

$$= \frac{F(\gamma l - \alpha n) - \{zF(1+l) - xF(1+n)\}}{F(1+m)}$$

$$= \frac{F(\alpha m - \beta l) - \{xF(1+m) - yF(1+l)\}}{F(1+n)}$$

or $$\frac{(\beta n - \gamma m) - y(1+n) + z(1+m)}{1+l}$$

$$= \frac{(\gamma l - \alpha n) - z(1+l) + x(1+n)}{1+m}$$

$$= \frac{(\alpha m - \beta l) - x(1+m) + y(1+l)}{(1+n)}$$

which is the required equation of central axis.

Q 5. Find the resultant of two wrenches (R_1, K_1) and (R_2, K_2) acting a distance 'c' apart inclined at an angle θ.

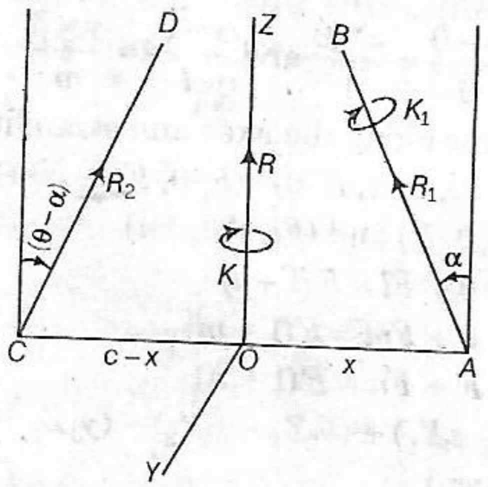
(2013, 11, 06)

***Or* Find the resultant wrench of two given wrenches.**

(2009, 05, 03, 01, 2000, 1998, 96, 94, 93, 92)

Sol. Let AB be the axis of one wrench (R_1, K_1) and CD be the axis of another wrench (R_2, K_2).

Let the angle between AB and CD be θ and $AC = c$ be the distance between the acting points of given two wrenches. The resultant wrench (R_1, K) is along OZ perpendicular to AC such that $OA = x$, $CO = c - x$ If axis AB makes an angle α to OZ, then CD will make an angle $\theta - \alpha$ to OZ.



On resolving forces along and perpendicular to OZ, i.e. along OY, we get

$$R = R_1 \cos\alpha + R_2 \cos(\theta - \alpha) \quad \text{...(i)}$$

and

$$0 = R_1 \sin\alpha - R_2 \sin(\theta - \alpha) \quad \text{...(ii)}$$

Taking moments about the above lines OZ and OY

$$K = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + xR_1 \sin\alpha + (c - x) R_2 \sin(\theta - \alpha) \quad \text{...(iii)}$$

$$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) - xR_1 \cos\alpha + (c - x) R_2 \cos(\theta - \alpha) \quad \text{...(iv)}$$

On squaring and adding Eqs. (i) and (ii), we get

$$R^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos(\alpha + \theta - \alpha)$$

$$\Rightarrow \quad R^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos\theta \quad \text{...(v)}$$

Writing Eqs. (iii) and (iv) with the help of Eqs. (i) and (ii), we get,

$$K = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + xR_1 \sin\alpha + (c - x) R_1 \sin\alpha$$

and $$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) + xR_2 \cos(\theta - \alpha) - xR + cR_2 \cos(\theta - \alpha) - xR_2 \cos(\theta - \alpha)$$

$\Rightarrow$ $$K = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + cR_1 \sin\alpha \quad \text{...(vi)}$$

and $$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) - xR + cR_2 \cos(\theta - \alpha) \quad \text{...(vii)}$$

Now, from Eq. (ii), $R_1 \sin\alpha - R_2 (\sin\theta \cos\alpha - \cos\theta \sin\alpha) = 0$

$\Rightarrow$ $$(R_1 + R_2 \cos\theta) \sin\alpha - R_2 \sin\theta \cos\alpha = 0$$

By using cross-multiplication,

$$\frac{\sin\alpha}{R_2 \sin\theta} = \frac{\cos\alpha}{R_1 + R_2 \cos\theta} = \frac{\sqrt{\sin^2\alpha + \cos^2\alpha}}{\sqrt{\{R_2^2 \sin^2\theta + (R_1 + R_2 \cos\theta)^2\}}}$$

$$= \frac{1}{\sqrt{R_1^2 + R_2^2 + 2R_1R_2 \cos\theta}} = \frac{1}{R}$$

Therefore, $$\frac{\sin\alpha}{R_2 \sin\theta} = \frac{\cos\alpha}{R_1 + R_2 \cos\theta} = \frac{1}{R} \quad \text{...(viii)}$$

From Eq. (vi), we have

$$K = K_1 \cos\alpha + K_2 (\cos\theta \cos\alpha + \sin\theta \sin\alpha) + c R_1 \sin\alpha.$$
$$= (K_1 + K_2 \cos\theta) \cos\alpha + (K_2 \sin\theta + cR_1) \sin\alpha$$

On multiplying by R and then putting the values of $R \sin\alpha$ and $R \cos\alpha$ from Eq. (viii), we get

$$RK = (K_1 + K_2 \cos\theta)(R_1 + R_2 \cos\theta) + (K_2 \sin\theta + cR_1) R_2 \sin\theta$$
$$= K_1R_1 + K_2R_2 + K_1R_2 \cos\theta + R_1K_2 \cos\theta + cR_1R_2 \sin\theta$$

$\Rightarrow$ $$RK = R_1K_1 + R_2K_2 + (R_1K_2 + R_2 K_1) \cos\theta + cR_1R_2 \sin\theta \quad \text{...(ix)}$$

Since, θ is given and so R can be obtained from Eq. (v). Now, putting the values of R and θ in Eq. (ix), we will have the value of K.

Now, we have to find the value of x, i.e. position of central axis.

$\therefore$ From Eq. (vii),

$$xR = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) + cR_2 \cos(\theta - \alpha)$$
$$= K_1 \sin\alpha - K_2 (\sin\theta \cos\alpha - \cos\theta \sin\alpha) + cR_2 (\cos\theta \cos\alpha + \sin\alpha \sin\theta)$$
$$= (K_1 + K_2 \cos\theta + cR_2 \sin\theta) \sin\alpha + (cR_2 \cos\theta - K_2 \sin\theta) \cos\alpha$$

On multiplying by R and using Eq. (viii), we get

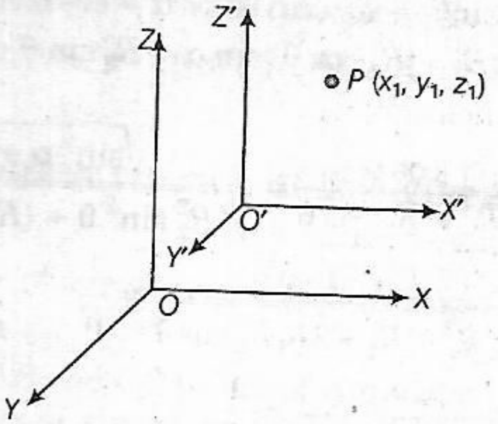
$$xR^2 = (K_1 + K_2 \cos\theta + cR_2 \sin\theta) R_2 \sin\theta + (cR_2 \cos\theta - K_2 \sin\theta)(R_1 + R_2 \cos\theta)$$
$$= (R_2K_1 - R_1K_2) \sin\theta + cR_2^2 \sin^2\theta + cR_1R_2 \cos\theta + cR_2^2 \cos^2\theta$$
$$= (R_2K_1 - R_1K_2) \sin\theta + cR_2^2 + cR_1R_2 \cos\theta$$
$$= (R_2K_1 - R_1 K_2) \sin\theta + cR_2(R_2 + R_1 \cos\theta)$$

On putting the values of R and θ, x gives the position of central axis.

Q 6. Find the null plane of a point (x_0, y_0, z_0); and also the condition for a line through (x_0, y_0, z_0) to be a null line.

(2012)

Sol. To find the null plane of a point (x_0, y_0, z_0) referred to the axes OX, OY, OZ Let the system of forces be equivalent to a dyname $(X, Y, Z; L, M, N)$ referred to origin O.



Let (x_0, y_0, z_0) be the coordinates of null point O'.
Then, the coordinates of any point $P(x_1, y_1, z_1)$ are changed to $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ referred to O' as origin.
Hence, the moment about a line through O' parallel to OX-axis is

$$L' = \Sigma[(y_1 - y_0) Z_1 - (z_1 - z_0)Y_1]$$
$$= \Sigma(y_1 Z_1 - z_1 Y_1) - y_0 \Sigma Z_1 + z_0 \Sigma Y_1$$
$$= L - y_0 Z + z_0 Y$$

Similarly, moment about lines through O' parallel to OY and OZ-axes are

$$M' = M - z_0 X + x_0 Z$$
$$N' = N - x_0 Y + y_0 X$$

Hence, the direction cosines of the axis of the resultant couple at O' are proportional of L', M' and N'. But this axis is normal to the null plane. Therefore, the equation of this plane through O' is

$$L'(x - x_0) + M'(y - y_0) + N'(z - z_0) = 0$$

$$\Rightarrow \quad (x - x_0)(L - y_0 Z + z_0 Y) + (y - y_0)(M - z_0 X + x_0 Z) + (z - z_0)(N - x_0 Y + y_0 X) = 0$$

$$\Rightarrow \quad x(L - y_0 Z + z_0 Y) + y(M - z_0 X + x_0 Z) + z(N - x_0 Y + y_0 X) = L x_0 + M y_0 + N z_0$$

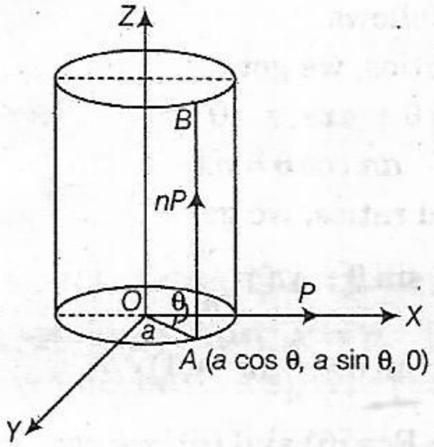
To find the condition for a line through (x_0, y_0, z_0) to be a null line, see the solution of Q. 1 of Short Answer Questions.

Q 7. A force P acts along X-axis and another force nP along a generator of the cylinder $x^2 + y^2 = a^2$, show that the central axis lies on the cylinder $n^2(nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2$.

(2017, 11, 09, 06)

Or Show that the central axis lies on cylinder
$$n^2(nx - z^2) + (1+n^2)^2 y^2 = n^4 a^2.$$ **(2015, 13, 04, 02, 1998, 96, 92)**

Sol. A force P acts along X-axis whose equation is
$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$
Another force np acts along the generator AB of the cylinder $x^2 + y^2 = a^2$ whose axis is Z-axis, so the direction cosines of the generator AB are 0, 0, 1 which is parallel to Z-axis.



Any point A on the cylinder is $x = a\cos\theta$, $y = a\sin\theta$ and $z = 0$.
Now, the equations of the generator AB are
$$\frac{x-a\cos\theta}{0} = \frac{y-a\sin\theta}{0} = \frac{z-0}{1} \quad [\because \text{ direction cosines of } AB \text{ are } 0, 0, 1]$$
Therefore, the forces acting on the body are as follows

(i) The force P acting at the point (0, 0, 0) along X-axis whose direction cosines are 1, 0, 0.

(ii) The force nP acting at the point $(a\cos\theta, a\sin\theta, 0)$ along the generator AB whose direction cosines are (0,0, 1).

The components (X_1, Y_1, Z_1) etc.of these forces parallel to the axes are
$$X_1 = P\cdot 1 = P, \quad X_2 = n\,P\cdot 0 = 0$$
$$Y_1 = P\cdot 0 = 0, \quad Y_2 = nP\cdot 0 = 0$$
$$Z_1 = P\cdot 0 = 0, \quad Z_2 = nP\cdot 1 = nP$$
If the system reduces to a single force $R = (X, Y, Z)$ acting at O and a couple $G = (L, M, N)$, then
$$X = X_1 + X_2 = P, \; Y = Y_1 + Y_2 = 0$$
and
$$Z = Z_1 + Z_2 = nP$$
$$L = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2)$$
$$= (0-0) + (a\sin\theta\, nP - 0) = anP\sin\theta$$
$$M = (z_1 X_1 - x_1 Z_1) + (z_2 X_2 - x_2 Z_2)$$
$$= (0-0) + (0 - a\cos\theta\cdot nP) = -anP\cos\theta$$
and
$$N = (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2)$$
$$= (0-0) + (a\cos\theta\cdot 0 - a\sin\theta\cdot 0) = 0$$

∴ Equation of the cental axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

$$\Rightarrow \quad \frac{anP\sin\theta - ynP + z\cdot 0}{P} = \frac{-an\,P\cos\theta - zP + xnP}{0} = \frac{0 - x\cdot 0 + y\cdot P}{nP}$$

$$\Rightarrow \quad \frac{an\sin\theta - yn}{1} = \frac{-an\cos\theta - z + xn}{0} = \frac{y}{n}$$

The required surface is obtained by eliminating θ from these equations. For this, we proceed as follows

From first and second ratios, we get

$$-an\cos\theta + nx - z = 0$$

$$\Rightarrow \quad an\cos\theta = nx - z \quad \text{...(i)}$$

Also, from first and third ratios, we get

$$an\sin\theta - yn = \frac{y}{n}$$

$$\Rightarrow \quad an\sin\theta = \frac{y}{n}(n^2 + 1) \quad \text{...(ii)}$$

On squaring and adding Eqs. (i) and (ii), we get

$$a^2n^2(\cos^2\theta + \sin^2\theta) = (nx - z)^2 + \frac{y^2}{n^2}(n^2 + 1)^2 \quad \text{...(ii)}$$

$$\therefore \quad n^2(nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2$$

which is the required equation of cylinder. **Hence proved.**

Q 8. Find the null point of the plane $lx + my + nz = 1$ for the system $(X, Y, Z; L, M, N)$. **(2006, 01)**

Sol. Let (f, g, h) be the null point of the given plane

$$lx + my + nz = 1 \quad \text{...(i)}$$

Then,

$$lf + mg + nh = 1 \quad \text{...(ii)}$$

The equation of null plane through (f, g, h) is

$$x(L - gZ + hY) + y(M - hX + fZ) + z(N - fY + gX) = Lf + Mg + Nh \quad \text{...(iii)}$$

Since, Eqs. (i) and (iii) represent the same plane, then comparing Eqs. (i) and (iii), we get

$$\frac{L - gZ + hY}{l} = \frac{M - hX + fZ}{m} = \frac{N - fY + gX}{n} = \frac{Lf + Mg + Nh}{1}$$

Taking first and second ratios, first and third ratios, and second and third ratios, we get

$$lfZ + mgZ - (lX + mY)h + (lM - mL) = 0$$

$$lYf - (nZ + lX)g + nYh + (nL - lN) = 0$$

and

$$-(mY + nZ)f + mXg + nXh + (mN - nM) = 0$$

From Eq. (ii), we get

$$(1-nh)\,Z-(lX+mY)h+(lM-mL)=0$$
$$(1-mg)Y-(nZ+lX)g+(nL-lN)=0$$
$$-(mY+nZ)f+(1-lf)X+(mN-nM)=0$$
$$-(Lm-lM-Z)=(lX+mY+nZ)h,$$
$$(Ln-lN+Y)=(lX+mY+nZ)\,g$$
$$-(nM-mN-X)=(lX+mY+nZ)f$$

Therefore, $\dfrac{f}{X-nM+mN}=\dfrac{g}{Y-lN+nL}=\dfrac{h}{Z-mL+lM}=\dfrac{1}{lX+mY+hZ}$

$$\Rightarrow \quad \frac{f}{X-\begin{vmatrix}M & N\\ m & n\end{vmatrix}}=\frac{g}{Y-\begin{vmatrix}N & L\\ n & l\end{vmatrix}}=\frac{h}{Z-\begin{vmatrix}L & M\\ l & m\end{vmatrix}}=\frac{l}{lX+mY+nZ}$$

Q 9. Forces X, Y, Z act along the three straight lines $y=b$, $z=-c$, $z=c$, $x=-a$ and $x=a$, $y=-b$, respectively. Show that they will have a single resultant (i.e. force), if $\dfrac{a}{X}+\dfrac{b}{Y}+\dfrac{c}{Z}=0$ and that the equations of the line of action are any two of the three $\dfrac{y}{Y}-\dfrac{z}{Z}-\dfrac{a}{X}=0$, $\dfrac{z}{Z}-\dfrac{x}{X}-\dfrac{b}{Y}=0$, $\dfrac{x}{X}-\dfrac{y}{Y}-\dfrac{c}{Z}=0$. **(2000, 1992, 91)**

Sol. Equation of the lines along which forces X, Y, Z act can be written as

$$\frac{x-0}{1}=\frac{y-b}{0}=\frac{z+c}{0};\ \frac{x+a}{0}=\frac{y-0}{1}=\frac{z-c}{0};\ \frac{x-a}{0}=\frac{y+b}{0}=\frac{z-0}{1}.$$

From above, it is obvious that the first force (X) acts at the point $(0, b, -c)$ along the first line whose direction cosines are $(1, 0, 0)$.

The second force (Y) acts at the point $(-a, 0, c)$ along the second line whose direction cosines are $(0, 1, 0)$.

The third force (Z) acts at the point $(a, -b, 0)$ along the third line whose direction cosines are $(0, 0, 1)$.

Hence, if the dyname be $(X, Y, Z; L, M, N)$, then

$X=X\cdot 1+0+0=X; Y=Y\cdot 0+Y\cdot 1+Y\cdot 0=Y; Z=Z\cdot 0+Z\cdot 0+Z\cdot 1=Z,$

$L=\Sigma(y_1Z_1-z_1Y_1)=0+(0-cY)+(-bZ-0)=-(cY+bZ)$

$M=\Sigma(z_1X_1-x_1Z_1)=(-cX-0)+(0)+(0-aZ)=-(cX+aZ)$

$N=\Sigma(x_1Y_1-y_1X_1)=(0-bX)+(-aY-0)+(a\cdot 0-b\cdot 0)=-(bX+aY)$

Now, the system will reduce to a single resultant, if the condition $LX+MY+NZ=0$ is satisfied, i.e. if

$$-(cY+bZ)X-(cX+aZ)Y-(bX+aY)Z=0$$
$$\Rightarrow \quad (cY+bZ)X+(cX+aZ)Y+(bX+aY)Z=0$$

$$\therefore \quad \frac{c}{Z}+\frac{b}{Y}+\frac{a}{X}=0$$

[after dividing by XYZ throughout the above relation]

Also, the equations of the central axis of the system are

$$\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}$$

$$=\frac{LX+MY+NZ}{X^2+Y^2+Z^2}=0$$

$$\Rightarrow \quad \frac{-(cY+bZ)-yZ+zY}{X}=\frac{-(cX+aZ)-zX+xZ}{Y}$$

$$=\frac{-(aY+bX)-xY+yX}{Z}=0$$

$$\Rightarrow \quad \left.\begin{array}{l} cY+bZ+yZ-zY=0 \\ cX+aZ+zX-xZ=0 \\ aY+bX+xY-yX=0 \end{array}\right\} \quad \text{...(i)}$$

These given $\frac{c}{Z}+\frac{b}{Y}+\frac{y}{Y}-\frac{z}{Z}=0$

[after dividing by YZ the first relation of Eq. (i)]

$$\frac{c}{Z}+\frac{a}{X}+\frac{z}{Z}-\frac{x}{X}=0$$

[after dividing by ZX the second relation of Eq. (i)]

and $\frac{a}{X}+\frac{b}{Y}+\frac{x}{X}-\frac{y}{Y}=0$

[after dividing by XY the third relation of Eq. (i)]

But $\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0$ [from the above]

Hence, making use of this relation, the last three equations are converted into the following three equations

$$\left(-\frac{a}{X}\right)+\frac{y}{Y}-\frac{z}{Z}=0, \left(-\frac{b}{Y}\right)+\frac{z}{Z}-\frac{x}{X}=0, \left(-\frac{c}{Z}\right)+\frac{x}{X}-\frac{y}{Y}=0$$

Thus, the relation of the line of action of the single force are any two of the above three relations.