Chapter Two

VECTORS AND TENSORS

⏻ Important Points from the Chapter

1. **Tensor** A tensor is a system of quantities or functions whose components obey certain laws of transformation of coordinates from one system to the other system.

   e.g. The stress tensor, the moment of inertia tensor, the deformation tensor etc. **(2006, 02)**

   ■ **Note**

   (i) The rank of order of a tensor is defined as the total number of real indices of the component of a tensor.

   (ii) A tensor of order zero is a scalar.

   (iii) A tensor of order one is a vector.

2. **Contravariant Vector** Let A^i $(i = 1, 2, \ldots, n)$ and $\overline{A}^i$ be n quantities in x^j and $\overline{x}^i$-coordinate system which are connected by the relation

   $$\overline{A}^i = A^j \frac{\partial \overline{x}^i}{\partial x^j}, \quad (i, j = 1, 2, \ldots, n)$$

   Then, the quantities A^i and $\overline{A}^i$ are known as the component of contravariant vector x^j and $\overline{x}^i$-coordinate system.

   ■ **Note.** The equations of transformation of components of a contravariant vector possess the group property.

3. **Scalar Invariant or Invariant** A system of quantities which are unchanged by the law of transformation of coordinates from one system to another system, is known as invariant.

4. **Gradient** Let A be scalar function of x^j coordinate. Then, the gradient of A denoted by grad (A) or ∇A, is defined in its ordinary partial derivatives,

   i.e. $\qquad \nabla A = \text{grad } (A) = \dfrac{\partial A}{\partial x^j}.$

5. **Covariant Vector** Let A_i $(i = 1, 2, \ldots, n)$ and $\overline{A}_i$ $(i = 1, 2, \ldots, n)$ be n quantities in x^j and $\overline{x}^i$-coordinates system respectively and they are connected by the relation $\overline{A}_i = A_j \dfrac{\partial x_j}{\partial \overline{x}^i}, (i, j = 1, 2, \ldots, n).$

   Then, the quantities A_i and $\overline{A}_i$ are said to be the component of covariant vector in x^j and $\overline{x}^i$-coordinates system. **(2007, 04)**

   ■ **Note** The equations of transformation of components of a covariant vector possess the group property.

6. **Contravariant Tensor of Second Order** Let A^{ij}, $\overline{A}^{ij}$ be n^2 quantities in x^j and $\overline{x}^i$-coordinate system which are connected by the relation

$$\overline{A}^{ij} = A^{ab} \frac{\partial \overline{x}^i}{\partial x^a} \cdot \frac{\partial \overline{x}^j}{\partial x^b}, \ (i, j, a, b = 1, 2, \ldots, n)$$

Then, the quantities A_{ij} and $\overline{A}_{ij}$ are known as components of contravariant of second order in x^j and $\overline{x}^i$-coordinate system.

7. **Covariant Tensor of Second Order** Let A_{ij} and $\overline{A}_{ij}$ be n^2 quantities in x^j and $\overline{x}^i$-coordinate system which are connected by the relation

$$\overline{A}_{ij} = A_{ab} \frac{\partial x^a}{\partial \overline{x}^i} \cdot \frac{\partial x^b}{\partial \overline{x}^j}, \ (i, j, a, b = 1, 2, \ldots, n).$$ Then, the quantities A_{ij} and $\overline{A}_{ij}$ are known as components of covariant tensor of second order in x^j and $\overline{x}^i$-coordinate system.

8. **Mixed Tensor of Second Order** Let A^i_j and $\overline{A}^i_j$ be n^2 quantities in x^j and $\overline{x}^i$-coordinate system which are connected by the relation

$$\overline{A}^i_j = A^a_b \frac{\partial \overline{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \overline{x}^j}, \ (i, j; a, b = 1, 2, \ldots, n)$$

Then, the quantities A^i_j and $\overline{A}^i_j$ are known as components of mixed tensor of second order in x^j and $\overline{x}^i$-coordinate system. **(2016)**

- **Note** The equations of transformation of components of a covariant tensor or a mixed tensor of second order possess the group property.

9. **Symmetric Tensor** The contravariant (or covariant) tensor of the second order whose components are A^{ij} (or A_{ij}), is known as symmetric, if $A^{ij} = A^{ji}$ (or $A_{ij} = A_{ji}$), $\forall \ i, j$. **(2008)**

10. **Skew-symmetric Tensor** The contravariant tensor of second order whose components are A^{ij} (or A_{ij}) is said to be skew-symmetric, if

$$A^{ij} = - A^{ji} \text{ (or } A_{ij} = -A_{ji}), \forall \ i, j.$$

11. A symmetric covariant (or contravariant) tensor of second order has $\frac{n(n+1)}{2}$, while for a skew-symmetric covariant (or contravariant) tensor, it has $\frac{n(n-1)}{2}$ independent components in space V_n. **(2008, 07)**

12. **Tensor Field** If a tensor is defined at all points of the space V_n itself, then it forms a tensor field.

- **Note** If a tensor is symmetric in one of the coordinate system, then it is symmetric in every other coordinate system.

⊕ Very Short Answer Questions

Q 1. Define a tensor with example and write the law of transformation for A_{ij}.   **(2006, 02)**

Sol. **Part I Tensor** A tensor is a system of quantities or functions whose components obey certain laws of transformation of coordinates from one system to the other system.

e.g. The stress tensor, the moment of inertia tensor, the deformation tensor etc.

Part II The law of transformation for mixed tensor A_k^{ij} or order 3 or a tensor of type (2, 1), if its components $\overline{A}_k^{ij}$ in any other coordinate system $\overline{x}^i$ is given by

$$\overline{A}_k^{ij} = A_c^{ab} \frac{\partial \overline{x}^i}{\partial x^a} \cdot \frac{\partial \overline{x}^j}{\partial x^b} \cdot \frac{\partial x^c}{\partial \overline{x}^k}$$

where, i, j, a, b, c varies from 1 to n.

Q 2. Define mixed tensor of second order and show that Kronecker delta δ_j^i are components of a mixed tensor of second order.   **(2016, 1998, 96)**

Sol. **Part I Mixed Tensor of Second Order** Let A_{ij} and $\overline{A}_j^i$ be n^2 quantities in x^i and $\overline{x}^i$-coordinate system which are connected by the relation

$$\overline{A}_j^i = A_b^a \frac{\partial \overline{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \overline{x}^j}, \ (i, j, a, b = 1, 2, \dots, n)$$

Then, the quantities A_j^i and $\overline{A}_j^i$ are known as components of mixed tensor of second order in x^i and $\overline{x}^i$-coordinate system.

Part II Let us consider the Kronecker deltas δ_j^i and $\overline{\delta}_j^i$ in x^i and $\overline{x}^i$ coordinate system, then

$$\overline{\delta}_j^i = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^h} \cdot \frac{\partial x^h}{\partial \overline{x}^j} = \delta_k^h \frac{\partial \overline{x}^i}{\partial x^h} \cdot \frac{\partial x^k}{\partial \overline{x}^j} \qquad [\because h = k] \ \dots(i)$$

On comparing Eq. (i) with definition of mixed tensor of second order, we get

$$\overline{A}_j^i = A_k^h \cdot \frac{\partial \overline{x}^i}{\partial x^h} \cdot \frac{\partial x^k}{\partial \overline{x}^j}$$

Hence, δ_j^i is a mixed tensor of second order.

Q 3. Find the independent components of symmetric and a skew-symmetric covariant tensor of second order.   **(2010, 07)**

Sol. **Part I** See the solution of Q. 3 of Short Answer Questions.

Part II Let A_{ij} be the components of a skew-symmetric covariant tensor of second order.

Then, $\qquad A_{ij} = -A_{ji}, \forall\ i, j.$

On taking $i = j$, we get

$$A_{ii} = -A_{ii} \Rightarrow 2A_{ii} = 0 \Rightarrow A_{ii} = 0$$

$$\Rightarrow \qquad A_{11} = A_{22} = \dots = A_{nn} = 0$$

Thus, number of independent components corresponding to repeated index is zero and number of independent components corresponding to distinct indices due to skew-symmetric property is $\dfrac{n^2 - n}{2}$.

Hence, total number of independent components

$$= 0 + \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

Q. 4. Prove that a symmetric covariant tensor of second order has 10 independent components in four dimensional space. **(2018)**

Sol. We know that a symmetric covariant (or contravariant) tensor of second order has $\dfrac{n(n+1)}{2}$ independent components in n-dimensional space V_n.

∴ For four dimensional space, $n = 4$, the required independent components

$$= \frac{4(4+1)}{2} = \frac{4 \times 5}{2} = 10$$

⏻ Short Answer Questions

Q. 1. Show that a mixed tensor of second order possess group property. **(2016, 05, 01, 1998)**

Or Prove that the equations of transformation of a mixed tensor of second order possess group property. **(2009)**

Sol. Let A^i_j be the component of a mixed tensor of second order in x^i-coordinate system.

Consider the coordinate transformation

$$x^i \to \bar{x}^i \to \bar{\bar{x}}^i$$
$$A^i_j \to \bar{A}^i_j \to \bar{\bar{A}}^i_j$$

In case of $x^i \to \bar{x}^i$, we have

$$\bar{A}^l = A^m_n \frac{\partial \bar{x}^l}{\partial x^m} \cdot \frac{\partial x^n}{\partial \bar{x}^k} \qquad \dots(i)$$

In case of $\bar{x}^i \to \bar{\bar{x}}^i$, we have

$$\bar{\bar{A}}^i_j = A^l_k \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^l} \cdot \frac{\partial \bar{x}^k}{\partial \bar{\bar{x}}^j} \qquad \dots(ii)$$

On combining Eqs. (i) and (ii), we get

$$\overline{\overline{A}}_{j}^{i} = \left(A_{n}^{m} \cdot \frac{\partial \overline{x}^{i}}{\partial x^{m}} \cdot \frac{\partial x^{n}}{\partial \overline{x}^{k}} \right) \frac{\partial \overline{\overline{x}}^{i}}{\partial \overline{x}^{l}} \cdot \frac{\partial \overline{x}^{k}}{\partial \overline{\overline{x}}^{j}} = A_{n}^{n} \left(\frac{\partial \overline{x}^{l}}{\partial x^{m}} \cdot \frac{\partial x^{n}}{\partial \overline{x}^{k}} \frac{\partial \overline{\overline{x}}^{i}}{\partial \overline{x}^{l}} \cdot \frac{\partial \overline{x}^{k}}{\partial \overline{\overline{x}}^{j}} \right)$$

$$\Rightarrow \qquad \overline{\overline{A}}_{j}^{i} = A_{n}^{m} \frac{\partial \overline{\overline{x}}^{i}}{\partial x^{m}} \cdot \frac{\partial x^{n}}{\partial \overline{\overline{x}}^{j}} \qquad \qquad \text{...(iii)}$$

which proves that, if we make the direct transformation from Eq. (i) to Eq. (iii), we get the same law of transformation. The equation of transformation of the component of mixed tensor of second order possess group property.

Q 2. What is the difference between contravariant and covariant components of a vector in rectangular cartesian system?

(2013)

Sol. **Contravariant Vector** Let A^{i} $(i = 1, 2, \ldots, n)$ and $\overline{A}^{i}$ be n quantities in x^{i} and $\overline{x}^{i}$-coordinate system which are connected by the relation $\overline{A}^{i} = A^{j} \dfrac{\partial \overline{x}^{i}}{\partial x^{j}}$, $(i, j = 1, 2, \ldots, n)$. Then, the quantities A^{i} and $\overline{A}^{i}$ are known as the component of contravariant vector x^{i} and $\overline{x}^{i}$-coordinate system.

Covariant Vector Let A_{i} $(i = 1, 2, \ldots, n)$ and $\overline{A}_{i}$ $(i = 1, 2, \ldots, n)$ be n quantities in x^{i} and $\overline{x}^{i}$-coordinate system respectively and they are connected by the relation. $\overline{A}_{i} = A_{j} \dfrac{\partial x_{j}}{\partial \overline{x}^{i}}$, $(i, j = 1, 2, \ldots, n)$.

Then, the quantities A_{i} and $\overline{A}_{i}$ are said to be the component of covariant vector in x^{i} and $\overline{x}^{i}$-coordinate system.

Q 3. Define a symmetric tensor of second order and find its number of independent components in V_{n}. Also, show that if a tensor is symmetric in one system of coordinates, it is symmetric in every other coordinate system. (2008)

Or Prow that a symmetric covariant tensor of second order has $\dfrac{n(n + 1)}{2}$ independent components in n-dimensional space V_{n}.

(2014, 11)

Sol. **Part I Symmetric Tensor of Second Order** The contravariant (or covariant) tensor of the second order whose components are A^{ij} (or A_{ij}), is known as symmetric, if $A^{ij} = A^{ji}$ (or $A_{ij} = A_{ji}$), $\forall$ i, j.

Part II Let A_{ij} be components of a symmetric covariant tensor of second order, then $A_{ij} = A_{ji}$, $\forall$ i, j

Hence, it has n^{2} components given by number of independent components corresponding to a repeated index is 'n'.

Number of components corresponding to distinct index is $n^{2} - n$.

Due to symmetry property, this number is reduced to $\dfrac{n^{2} - n}{2}$.

Therefore, total number of independent components

$$= n + \frac{n^2 - n}{n} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

Part III Let A_{ij} be the components of a symmetric covariant tensor of second order in x^i-coordinate system.

Then, $\qquad A_{ij} = A_{ji}, \forall\, i, j$ $\qquad\qquad\qquad$ ...(i)

Again, let $\overline{A}_{ij}$ be the components of the same tensor, i.e. covariant tensor in $\overline{x}^i$-coordinate system.

Then, law of transformation is given by

$$\overline{A}_{ij} = A_{hk} \frac{\partial x^h}{\partial \overline{x}^i} \cdot \frac{\partial x^k}{\partial \overline{x}^j}$$

$$\Rightarrow \qquad \overline{A}_{ij} = A_{kh} \frac{\partial x^k}{\partial \overline{x}^j} \cdot \frac{\partial x^h}{\partial \overline{x}^i} \qquad [\because A_{hk} = A_{kh}, \forall\, h, k]$$

$$\Rightarrow \qquad \overline{A}_{ij} = \overline{A}_{ji}, \forall\, i, j$$

which shows that it is also symmetric in $\overline{x}^i$-coordinate system.

Q 4. **Find the independent components of a skew-symmetric covariant tensor of order three.** (2013, 03)

Sol. Let A_{ijk} be components of a skew-symmetric covariant tensor of order three.

Then, $\qquad \left. \begin{array}{l} A_{ijk} = - A_{jik} \\ A_{ijk} = - A_{ikj} \\ A_{ijk} = - A_{kij} \end{array} \right\}$ $\qquad\qquad\qquad$ ...(i)

Taking $i = j = k$ in Eq. (i), we get

$$A_{iii} = - A_{iii} \Rightarrow A_{iii} = 0, \forall\, i$$

Thus, the number of independent components of A_{ijk} (for $i = j = k$) is zero.

Taking $i = j$ in Eq. (i), we get

$$A_{iik} = - A_{iik} \Rightarrow A_{iik} = 0$$

In this case, the number of independent components of A_{ijk} (for $i = j$) is zero.

Thus, the components of A_{ijk} having any two indices equal, has the value zero. Hence, number of independent components of A_{ijk}

$$= {}^nC_3 = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

Q 5. A tensor A_{ij}^k is skew-symmetric in suffixes i and j. Find the number of its independent components and also that at remain skew-symmetric metric in every coordinate system in same indices whose i, j, k take the value 1 to n. (2012)

Sol. **Part I** Let A_{ij}^k be the components of a skew-symmetric in i and j, then

$$A_{ij}^k = - A_{ji}^k, \forall\, i, j$$

Hence, it has n^3-components.

Taking $i = j$, we get $A_{ij}^h = - A_{ij}^h \Rightarrow 2A_{ij}^h = 0 \Rightarrow A_{ij}^h = 0$

Hence, the number of components corresponding to distinct index
$$= n^3 - n^2$$

Due to skew-symmetry property, this number is reduced to $\dfrac{n^3 - n^2}{2}$.

Hence, total number of independent components of A_{ij}^k
$$= \frac{n^3 - n^2}{2} = \frac{n^2(n-1)}{2}$$

Part II Let $\overline{A}_{ij}^k$ be the components of mixed tensor A_{ij}^k in $\overline{x}^i$-coordinate system. Then, law of transformation is

$$\overline{A}_{ij}^k = A_{bc}^a \frac{\partial x^b}{\partial \overline{x}^i} \cdot \frac{\partial x^c}{\partial \overline{x}^j} \cdot \frac{\partial \overline{x}^k}{\partial x^a} \text{ or } A_{ab}^a \frac{\partial x^c}{\partial \overline{x}^j} \cdot \frac{\partial x^b}{\partial \overline{x}^i} \cdot \frac{\partial \overline{x}^k}{\partial x^a}$$

$\Rightarrow \qquad A_{ij} = A_{ji}, \forall \, i, j$ ...(i)

Let $\overline{A}_{ij}$ be the components of the same tensor in $\overline{x}^i$-coordinate system. Then, law of transformation is given by

$$\overline{A}_{ij} = A_{hk} \cdot \frac{\partial x^h}{\partial \overline{x}^i} \frac{\partial x^k}{\partial \overline{x}^j} \Rightarrow \overline{A}_{ij} = A_{kh} \frac{\partial x^k}{\partial \overline{x}^j} \frac{\partial x^h}{\partial \overline{x}^i} \qquad [\because A_{hk} = A_{kh}]$$

$\Rightarrow \qquad \overline{A}_{ij} = \overline{A}_{ji}$

which shows that it is also symmetric in $\overline{x}^i$-coordinate system.

☼ Long Answer Questions

Q 1. Define the covariant vector and show that the equation of transformation of components of covariant vector possess the group property (or transitive property).   (2007, 04)

Sol. Part I Covariant Vector Let A_i $(i = 1, 2, \ldots, n)$ and $\overline{A}_i$ $(i = 1, 2, \ldots, n)$ be n quantities in x^i and $\overline{x}^i$-coordinates system respectively and they are connected by the relation $\overline{A}_i = A_j \dfrac{\partial x_j}{\partial \overline{x}^i}$, $(i, j = 1, 2, \ldots, n)$.

Then, the quantities A_i and $\overline{A}_i$ are said to be the component of covariant vector in x^i and $\overline{x}^i$-coordinates system.

Part II Let A_i and $\overline{A}_i$ be the components of a covariant vector in x^i and $\overline{x}^i$-coordinate system, respectively. If coordinates of x^i is transformed to $\overline{x}^i$, then

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j, \, (i, j = 1, 2, \ldots, n) \qquad \text{...(i)}$$

$\Rightarrow \qquad \dfrac{\partial \overline{x}^i}{\partial x^k} \overline{A}_i = \dfrac{\partial \overline{x}^i}{\partial x^k} \cdot \dfrac{\partial x^j}{\partial \overline{x}^i} A_j = \delta_k^i A_j = A_k$

$\Rightarrow \qquad A_k = \dfrac{\partial \overline{x}^i}{\partial x^k} \overline{A}_i \qquad\qquad\qquad \text{...(ii)}$

We take transformations $x^j \to \bar{x}^i \to \bar{\bar{x}}^i$

Then, $\qquad A_i \to \bar{A}_i \to \bar{\bar{A}}_i$

The relation (ii) is the law of transformation for the components of a covariant vector when the coordinates $\bar{\bar{x}}^i$ are transformed to the coordinates x^j.

From Eqs. (i) and (ii), it is clear that the relation between two sets of components is reciprocal.

Let $\bar{\bar{A}}_i$ be the components of same vector in third coordinate system $\bar{\bar{x}}^i$.

Then, $\qquad \bar{\bar{A}}_i = \dfrac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^i} \, \bar{A}_j \ (i, j = 1, 2, \ldots, n)$ ...(iii)

On putting the value of $\bar{A}_j$ in Eq. (iii), we get

$$\bar{\bar{A}}_i = \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^i} \times \frac{\partial x^k}{\partial \bar{x}^j} A_k \Rightarrow \bar{\bar{A}}_j = \frac{\partial x^k}{\partial \bar{\bar{x}}^i} A_k$$

This relation is of the same form as the law of transformation for the components of a covariant vector, when the coordinates x^j are transformed to the coordinates $\bar{\bar{x}}^i$.

Thus, the equations of transformation of components of a covariant vector possess the group property (or transitive property).

Q 2. Prove that the law of transformation of a contravariant tensor of second order possess the group property (or transitive property).

(2007, 02)

Sol. Let A^{ij}, $\bar{A}^{ij}$ and $\bar{\bar{A}}^{ij}$ be the components of a contravariant tensor of second order in x^j, $\bar{x}^i$ and $\bar{\bar{x}}^i$-coordinate system, respectively.

Consider the coordinate transformation

$$x^j \to \bar{x}^i \to \bar{\bar{x}}^i$$

Then, $\qquad A^{ij} \to \bar{A}^{ij} \to \bar{\bar{A}}^{ij}$

In case $x^j \to \bar{x}^i$, we have $\bar{A}^{lk} = A^{mn} \dfrac{\partial \bar{x}^l}{\partial x^m} \cdot \dfrac{\partial \bar{x}^k}{\partial x^n}$ ...(i)

In case $\bar{x}^i \to \bar{\bar{x}}^i$, we have $\bar{\bar{A}}^{ij} = \bar{A}^{lk} \dfrac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^l} \cdot \dfrac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^k}$ ...(ii)

On combining Eqs. (i) and (ii), we get

$$\bar{\bar{A}}^{ij} = \left(A^{mn} \frac{\partial \bar{x}^l}{\partial x^m} \cdot \frac{\partial \bar{x}^k}{\partial x^n} \right) \left(\frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^l} \cdot \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^k} \right) = A^{mn} \left(\frac{\partial \bar{x}^l}{\partial x^m} \cdot \frac{\partial \bar{x}^k}{\partial x^n} \cdot \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^l} \cdot \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^k} \right)$$

$$\Rightarrow \quad \bar{\bar{A}}^{ij} = A^{mn} \cdot \frac{\partial \bar{\bar{x}}^i}{\partial x^m} \cdot \frac{\partial \bar{\bar{x}}^j}{\partial x^n}$$ ...(iii)

which proves that, if we make transformation from Eq. (i) to Eq. (iii), we get the same law of transformation.