

## *Chapter Four*

# CALCULUS OF RESIDUE

### ⌚ Important Points from the Chapter

- 1. Residue at a Pole** Let  $f(z)$  be an analytic function within a circle  $C$  of radius  $r$  and centre at  $z = a$  except at the centre  $z = a$ , which is a pole of order  $m$ . Then,  $f(z)$  is analytic within the annulus  $0 < |z - a| < r$ , hence it can be expanded within this annulus in a Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n + \sum_{n=1}^{\infty} b_n(z - a)^{-n}$$

where,  $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - a)^{n+1}} dz$  and  $b_n = \frac{1}{2\pi i} \int_C (z - a)^{n-1} f(z) dz$ .

In particular,  $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$

where, the coefficient  $b_1$  is called the **residue** of  $f(z)$  at  $z = a$ .

- 2. Residue at Infinity** Residue of  $f(z)$  at  $z = \infty$  is defined as

$-\frac{1}{2\pi i} \int_C f(z) dz$ , where the integration along  $C$  is taken anti-clockwise direction.

- 3. Computation of Residue at a Finite Pole**

(i) **Residue at a simple pole** If  $z = a$  is a simple pole of  $f(z)$ . Then, the residue at the simple pole  $z = a$  is given by  $\lim_{z \rightarrow a} (z - a) f(z)$ .

Another form is obtained as follows If  $f(z) = \frac{\phi(z)}{\psi(z)}$ , where

$\phi(a) \neq 0$  and  $\psi(z) = (z - a) f(z)$ ,  $f(a) \neq 0$ .

Since,  $z = a$  is a simple pole of  $f(z)$ , therefore  $\psi(a) = 0$  and  $\psi'(a) \neq 0$ .

Then, the residue of  $f(z)$  at a simple pole  $z = a$  is given by  $\frac{\phi(a)}{\psi'(a)}$ .

- (ii) **Residue at a pole of order  $m$**  Let  $z = a$  be pole of order  $m$  of  $f(z)$ .

Then,  $f(z)$  is of the form  $\frac{\phi(z)}{(z - a)^m}$ , where  $\phi(z)$  is analytic.

Hence, the residue of  $f(z)$  at the pole of order  $m$  is  $\frac{\phi^{m-1}(a)}{(m-1)!}$ , where

$z = a$ , is the pole of order  $m$ .

(2016, 14, 07)

- (a) **Residue at a pole  $z = a$  of any finite order** (general method)  
 If  $z = a$  is a pole of order  $m$  (which may be equal to 1), then the residue at the pole  $z = a$  is

$b_1 = \text{Coefficient of } \frac{1}{t}$  in the Laurent's expansion of  $f(a + t)$ , where  $t$  is sufficiently small.

- (b) **Residue of  $f(z)$  at  $\infty$**  Residue of  $f(z)$  at  $z = \infty$  is  $\lim_{z \rightarrow \infty} \{-z f(z)\}$ , provided the limit exists.

$$= -[\text{Coefficient of } \frac{1}{z} \text{ in the expansion of } f(z) \text{ for values of } z \text{ in the neighbourhood of } \infty]$$

4. **Cauchy's Residue Theorem** Let  $f(z)$  be an analytic function, except at a finite number of poles within a closed contour  $C$  and continuous on the boundary  $C$ , then

$$\int_C f(z) dz = 2\pi i (\text{sum of the residues at the poles within } C) \\ = 2\pi i \Sigma R$$

If an analytic function has singularities at a finite number of points (including that at infinity), then the sum of the residues of  $f(z)$  at all these poles along with infinity is zero. (2011, 06)

5. **Evaluation of Real Definite Integrals by Contour Integration**  
 For this we take a suitable closed contour  $C$  and find the residues of the function  $f(z)$  at all its poles which lie within  $C$ . Then, by using Cauchy's theorem of residues, we have

$$\int_C f(z) dz = 2\pi i (\text{sum of residues of } f(z) \text{ at the poles within } C)$$

■ Note

(i) If  $(z - a) f(z) \rightarrow a$  as  $z \rightarrow a$  and  $C$  is a small circle  $|z - a| = r$ , then we have  $\theta_2 - \theta_1 = 2\pi$ , so that  $\int_C f(z) dz = 2\pi i A$ .

(ii) If  $(z - a) f(z) \rightarrow 0$  as  $z \rightarrow a$ , then we have  $\int_C f(z) dz \rightarrow 0$  as  $z \rightarrow 0$ .

6. **Integration Round the Unit Circle** We consider the integral of the type  $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$ , where the integral is a rational function of  $\sin \theta$  and  $\cos \theta$ .

If we take  $z = e^{i\theta}$ , so that  $dz = e^{i\theta} id\theta \Rightarrow \frac{dz}{iz} = d\theta$

and  $\frac{1}{2} \left( z + \frac{1}{z} \right) = \cos \theta, \frac{1}{2i} \left( z - \frac{1}{z} \right) = \sin \theta$ .

$$\text{Then, } \int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta = \frac{1}{i} \int_C f \left[ \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right] \frac{dz}{z} \\ = \int_C F(z) dz \text{ (say)}$$

where,  $C$  is the unit circle  $|z| = 1$ .

Clearly,  $F(z)$  is a rational function of  $z$ . Hence, by residue theorem, we have

$$\int_C F(z) dz = 2\pi i \Sigma \text{Res}$$

where, ' $\Sigma \text{Res}'$  is the sum of residues of  $F(z)$  at its poles inside  $C$ .

**7. Evaluation of  $\int_{-\infty}^{\infty} f(x) dx$**  (When no Poles on the Real Axis) If the function  $f(z)$  is such that it has no poles on the real axis and possibility has pole in the upper half of the  $Z$ -plane, we can evaluate  $\int_{-\infty}^{\infty} f(x) dx$ .

Consider the real integral  $\int_C f(z) dz$ .

Taken round a closed contour  $C$  consisting of a semi-circle of radius  $R$  large enough to include all the poles of  $f(z)$  and the part of the real axis from  $x = -R$  to  $x = R$ . The only singularities of  $f(z)$  in the upper half are poles.

By Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of the residues of } f(z) \text{ at the poles within } C] \\ \Rightarrow \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz &= 2\pi i \\ \Rightarrow \int_{-R}^R f(x) dx &= - \int_{\Gamma} f(z) dz + 2\pi i \end{aligned} \quad \dots(i)$$

Taking limits on both the sides of Eq. (i), when  $R \rightarrow \infty$ , we get

$$\begin{aligned} \therefore \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx &= - \lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz + 2\pi i \\ \Rightarrow \int_{-\infty}^{\infty} f(x) dx &= - \lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz + 2\pi i \end{aligned} \quad \dots(ii)$$

All the poles of  $f(z)$  can be found by putting the denominator of  $f(z)$  equal to zero. Then, using the method for calculating the residue at the pole, we compute the residues only at those poles which lie inside  $C$  and their sum is taken.

Now, if  $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz$  exists, then the right hand side in Eq. (ii) is perfectly known. Hence,  $\int_{-\infty}^{\infty} f(x) dx$  is determined.

**8. Jordan's Inequality** If  $0 < \theta < \frac{\pi}{2}$ , then  $\frac{2}{\pi} < \frac{\sin \theta}{\theta} < 1$ .

**9. Jordan's Lemma** If  $\phi(z) \rightarrow 0$  uniformly as  $z \rightarrow \infty$  and  $f(z)$  is meromorphic in the upper half plane, then  $\lim_{R \rightarrow \infty} \int_{\Gamma} e^{imz} \phi(z) dz = 0$ ,

$m > 0$ , where, ' $\Gamma$ ' denotes the semi-circle  $|z| = R$ ,  $I(z) > 0$ .

**10. Poles lie on the Real Axis** Let the function  $f(z)$  has poles within the semi-circle  $\Gamma$  as well as on the real line. We exclude the poles on the real line by enclosing them with semi circles of small radii. This method is known as **indenting** at a point.

**11. Integral of Many Valued Functions** We consider the integrals of the type

$$\int_0^\infty x^{a-1} f(x) dx \text{ or } \int_0^\infty \frac{\log x}{\phi(x)} dx$$

where,  $a$  is not an integer and  $a > 0$ . Also,  $\phi(x)$  is not an even function of  $x$ . These integrals are not single valued, to evaluate such integrals we consider only those contours whose interiors do not contain any branch point. For these integrals, we generally use double circle contour intended at the centre.

### Very Short Answer Questions

**Q 1.** Find the sum of residues of function  $f(z) = \frac{z^2}{z^2 + a^2}$  at its poles. (2019, 16, 03)

$$\text{Sol. We have, } f(z) = \frac{z^2}{z^2 + a^2} = \frac{z^2}{(z + ia)(z - ia)}$$

Clearly,  $z = ia, -ia$  are the simple poles.

$$\text{Therefore, Res}(z = -ia) = \lim_{z \rightarrow -ia} [(z + ia) f(z)]$$

$$= \lim_{z \rightarrow -ia} \left[ (z + ia) \frac{z^2}{(z + ia)(z - ia)} \right] \\ = \frac{(-ia)^2}{(-ia - ia)} = -\frac{a^2}{-2ia} = \frac{1}{2} ai$$

$$\text{Res}(z = ia) = \lim_{z \rightarrow ia} [(z - ia) f(z)]$$

$$= \lim_{z \rightarrow ia} \left[ (z - ia) \frac{z^2}{(z + ia)(z - ia)} \right] \\ = \frac{(ia)^2}{(ia + ia)} = -\frac{a^2}{2ia} = \frac{1}{2} ai$$

$$\therefore \text{Sum of residues} = \text{Res}(z = -ia) + \text{Res}(z = ia) = -\frac{1}{2} ai + \frac{1}{2} ai = 0.$$

**Q 2.** Determine the pole and residues of the function  $\operatorname{cosec} z$ . (2018)

$$\text{Sol. Here, } f(z) = \operatorname{cosec} z = \frac{1}{\sin z}$$

Then, the poles of  $f(z)$  are given by  $\sin z = 0 = \sin 0$   
 $\therefore z = 0$  is a simple pole of  $f(z)$ .

Taking  $f(z) = \frac{\phi(z)}{\psi(z)}$  then  $\phi(z) = 1, \psi(z) = \sin z$

$$\text{Res}(z=0) = \lim_{z \rightarrow 0} \frac{\phi(z)}{\psi'(z)} = \lim_{z \rightarrow 0} \frac{1}{\cos z} = \frac{1}{\cos 0} = 1$$

$$\therefore \text{Res}(z=0) = 1.$$

**Q 3.** If  $f(z)$  has a pole of order 3 at  $z = a$ , then find the residue at it. (2004)

**Sol.** Let  $f(z) = \frac{\phi(z)}{(z-a)^3}$

Clearly,  $f(z)$  has a pole of order 3 at  $z = a$ .

$$\therefore \text{Residue at } z = a = \frac{1}{(3-1)!} \left[ \frac{d^2}{dz^2} \phi(z) \right]_{z=a} = \frac{1}{2!} [\phi''(z)]_{z=a} = \frac{1}{2} \phi''(a)$$

**Q 4.** Determine the residue of  $f(z) = \frac{z+3}{z^2 - 2z}$  at the respective poles. (2006)

**Sol.** Let  $f(z) = \frac{z+3}{z^2 - 2z}$

The poles of  $f(z)$  are given by  $z^2 - 2z = 0$

$$\Rightarrow z(z-2) = 0 \Rightarrow z = 0, 2$$

Therefore,  $z = 0, 2$  are the simple poles.

$$\begin{aligned} \therefore \text{Res}(z=0) &= \lim_{z \rightarrow 0} [(z-0) f(z)] \\ &= \lim_{z \rightarrow 0} \left[ z \left( \frac{z+3}{z^2 - 2z} \right) \right] = \lim_{z \rightarrow 0} \left( \frac{z+3}{z-2} \right) = -\frac{3}{2} \end{aligned}$$

$$\text{and } \text{Res}(z=2) = \lim_{z \rightarrow 2} [(z-2) f(z)]$$

$$= \lim_{z \rightarrow 2} \left[ (z-2) \frac{z+3}{z^2 - 2z} \right] = \lim_{z \rightarrow 2} \left[ \frac{z+3}{z} \right] = \frac{5}{2}$$

**Q 5.** Find the residue of  $\frac{z^3}{(z-1)^4 (z-2)(z-3)}$  at a pole of order 4.

**Sol.** Let  $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

Clearly,  $z = 1$  is a pole of order 4 of  $f(z)$ .

To find the residue at  $z = 1$ , we will put  $z = 1 + t$  in  $f(z)$ , then the coefficient of  $\frac{1}{t}$  will be the residue at  $z = 1$ .

Now,  $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

On putting  $z-1=t$ , we get

$$\begin{aligned} f(z) &= \frac{(1+t)^3}{t^4(t-1)(t-2)} \\ &= \frac{1}{2t^4} \left[ (1+t)^3 (1-t)^{-1} \left(1 - \frac{t}{2}\right)^{-1} \right] \\ &= \frac{1}{2t^4} \left[ (1+3t+3t^2+t^3) (1+t+t^2+t^3+\dots) \right. \\ &\quad \left. \left(1 + \frac{1}{2}t + \frac{1}{4}t^2 + \frac{1}{8}t^3 + \dots\right) \right] \\ &= \frac{1}{2t^4} [(1+3t+3t^2+t^3) \left(1 + \frac{3}{2}t + \frac{7}{4}t^2 + \frac{15}{8}t^3 + \dots\right)] \end{aligned}$$

$$\therefore \text{Coefficient of } \frac{1}{t} = \frac{1}{2} \left[ 1 + \frac{9}{2} + \frac{21}{4} + \frac{15}{8} \right] = \frac{101}{16}$$

Hence, the residue at  $z=1$  is  $\frac{101}{16}$ .

**Q 6.** Find the residue of  $f(z) = z^3 \cos\left(\frac{1}{z-2}\right)$  at  $z=2$ . (2017)

**Sol.** We have,  $f(z) = z^3 \cos\left(\frac{1}{z-2}\right)$

On putting  $z-2=t$ , we get

$$f(t+2) = (t+2)^3 \cos \frac{1}{t} = (t^3 + 8t^2 + 24t + 16) \left(1 - \frac{1}{2!t^2} + \frac{1}{4!t^4} + \dots\right)$$

$$\text{Coefficient of } \frac{1}{t} = -\frac{12}{2!} + \frac{1}{4!} = -6 + \frac{1}{24} = -\frac{143}{24}$$

Hence, the residue at  $z=2$  is  $\left(-\frac{143}{24}\right)$ .

### Short Answer Questions

**Q 1.** Define residue of a function  $f(z)$  at a point  $z=a$ . If  $f(z)$  is analytic within and on a simple closed contour  $C$ , except at a finite number of poles  $z_1, z_2, \dots, z_n$  within  $C$ , and if  $R_1, R_2, \dots, R_n$  be the residues of  $f(z)$  at these poles, then

prove that  $\int_C f(z) dz = 2\pi i (R_1 + R_2 + \dots + R_n)$ . (2011, 06)

**Or** State and prove Cauchy's Residue theorem. (2008, 05)

**Sol. Part I. Residue at a Pole** Let  $f(z)$  be an analytic function within a circle  $C$  of radius  $r$  and centre at  $z = a$  except at the centre  $z = a$ , which is a pole of order  $m$ . Then,  $f(z)$  is analytic within the annulus  $0 < |z - a| < r$ , hence it can be expanded within this annulus in a Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} b_n (z - a)^{-n}$$

where,  $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - a)^{n+1}} dz$  and  $b_n = \frac{1}{2\pi i} \int_C (z - a)^{n-1} f(z) dz$ .

In particular,  $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$

where, the coefficient  $b_1$  is called the residue of  $f(z)$  at  $z = a$ .

**Part II. Cauchy's Residue Theorem** Let  $f(z)$  be an analytic function, except at a finite number of poles within a closed contour  $C$  and continuous on the boundary  $C$ , then

$$\int_C f(z) dz = 2\pi i$$

(sum of residues at the poles within  $C$ ) =  $2\pi i \sum R$

If an analytic function has singularities at a finite number of points (including that at infinity), then the sum of the residues of  $f(z)$  at all these poles along with infinity is zero.

Let  $z_1, z_2, \dots, z_n$  be the poles of order  $m_1, m_2, \dots, m_n$  respectively of  $f(z)$  within a closed contour  $C$ . Let  $\gamma_1, \gamma_2, \dots, \gamma_n$  be the circles with the centres  $z_1, z_2, \dots, z_n$  respectively and each of radius  $r$ , so small that all the circles lie entirely within  $C$  and do not overlap. Then,  $f(z)$  is analytic within the region enclosed by the curve  $C$  and these circles.

Now, by Cauchy's theorem for multi-connected regions, we have

$$\int_C f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \dots + \int_{\gamma_n} f(z) dz \quad \dots(i)$$

$$\text{But } \text{Res}(z = z_1) = \frac{1}{2\pi i} \int_{\gamma_1} f(z) dz \quad [\text{by definition of residue}]$$

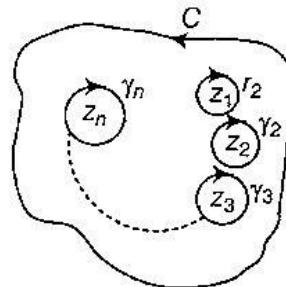
$$\Rightarrow \int_{\gamma_1} f(z) dz = 2\pi i \text{Res}(z = z_1)$$

$$\text{Similarly, } \int_{\gamma_2} f(z) dz = 2\pi i \text{Res}(z = z_2)$$

$$\int_{\gamma_3} f(z) dz = 2\pi i \text{Res}(z = z_3)$$

$$\vdots \quad \vdots \quad \vdots$$

$$\int_{\gamma_n} f(z) dz = 2\pi i \text{Res}(z = z_n)$$



On putting these values in Eq. (i), we get

$$\int_C f(z) dz = 2\pi i \text{Res}(z = z_1) + 2\pi i \text{Res}(z = z_2) + \dots + 2\pi i \text{Res}(z = z_n)$$

$$= 2\pi i [\text{Res}(z = z_1) + \text{Res}(z = z_2) + \dots + \text{Res}(z = z_n)] = 2\pi i \sum R$$

**Q 2.** Find the residue of  $\frac{z^2 - 2z}{(z+1)^2(z^2+4)}$  at all its poles in the finite plane.

$$\text{Sol. Let } f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)} = \frac{z^2 - 2z}{(z+1)^2(z+2i)(z-2i)}$$

The poles of  $f(z)$  are given by  $(z+1)^2(z+2i)(z-2i) = 0$

Since,  $z = -1$  (pole of order 2) and  $z = 2i, -2i$  (simple poles)

$$\begin{aligned} \text{Now, Res}(z=2i) &= \lim_{z \rightarrow 2i} \left[ (z-2i) \cdot \frac{z^2 - 2z}{(z+1)^2(z+2i)(z-2i)} \right] \\ &= \lim_{z \rightarrow 2i} \left[ \frac{z^2 - 2z}{(z+1)^2(z+2i)} \right] = \frac{(2i)^2 - 2(2i)}{(2i+1)^2(2i+2i)} \\ &= \frac{-4 - 4i}{(2i+1)^2 \cdot 4i} = \frac{(i-1)}{(2i+1)^2} = \frac{(i-1)}{(-3+4i)} \\ &= \frac{(i-1)(-3-4i)}{25} = \frac{7+i}{25} \end{aligned}$$

$$\begin{aligned} \therefore \text{Res}(z=-2i) &= \lim_{z \rightarrow -2i} \left[ (z+2i) \cdot \frac{z^2 - 2z}{(z+1)^2(z+2i)(z-2i)} \right] \\ &= \lim_{z \rightarrow -2i} \left[ \frac{(-2i)^2 - 2(-2i)}{(-2i+1)^2(-2i-2i)} \right] = \frac{7-i}{25} \end{aligned}$$

$$\begin{aligned} \text{and Res}(z=-1) &= \frac{1}{1!} [\phi'(z)]_{z=-1} \\ &= \left[ \frac{d}{dz} \phi(z) \right]_{z=-1} = \left[ \frac{d}{dz} \left( \frac{z^2 - 2z}{z^2 + 4} \right) \right]_{z=-1} \\ &= \left[ \frac{2z^2 + 8z - 8}{(z^2 + 4)^2} \right]_{z=-1} = -\frac{14}{25} \end{aligned}$$

**Q 3.** Evaluate the residue of  $\frac{z^3}{(z-1)(z-2)(z-3)}$  at  $z=1, 2, 3$  and infinity and show that their sum is zero.

$$\text{Sol. Let } f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$

Clearly,  $z=1, 2, 3$  are simple poles of  $f(z)$ .

$$\begin{aligned} \therefore \text{Res}(z=1) &= \lim_{z \rightarrow 1} \left[ (z-1) \frac{z^3}{(z-1)(z-2)(z-3)} \right] \\ &= \lim_{z \rightarrow 1} \left[ \frac{z^3}{(z-2)(z-3)} \right] = \frac{1}{2} \end{aligned}$$

Similarly, residue at  $z = 2$  is  $-8$  and residue at  $z = 3$  is  $\frac{27}{2}$ .

$$\begin{aligned} \text{Now, } f(z) &= \left(1 - \frac{1}{z}\right)^{-1} \left(1 - \frac{2}{z}\right)^{-1} \left(1 - \frac{3}{z}\right)^{-1} \\ &= \left(1 + \frac{1}{z} + \dots\right) \left(1 + \frac{2}{z} + \dots\right) \left(1 + \frac{3}{z} + \dots\right) \\ &= 1 + \frac{6}{z} + \text{Terms containing higher powers of } z \text{ in denominator} \end{aligned}$$

$$\therefore \text{Residue at infinity} = - \left[ \text{Coefficient of } \frac{1}{z} \text{ in } f(z) \right] = -6$$

Hence, sum of residues of  $f(z)$  at  $z = 1, 2, 3$  and infinity

$$= \frac{1}{2} + (-8) + \frac{27}{2} - 6 = 0$$

**Q 4.** By the method of Calculus of residues, evaluate  $\int_0^\infty \frac{dx}{1+x^2}$ .

(2013, 11, 05)

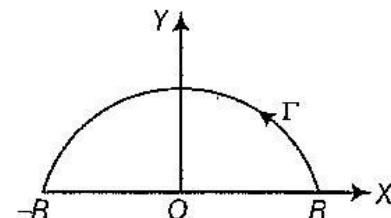
$$\text{Sol. Let } \int_C f(z) dz = \int_C \frac{dz}{1+z^2} \quad \dots(i)$$

where,  $C$  is the contour consisting of a large semi-circle  $\Gamma$  of radius  $R$  together with real axis from  $-R$  to  $R$ .

So,  $z = i, -i$  are the simple poles of  $f(z)$ .

Out of these, only  $z = i$  lies inside  $C$ .

$$\begin{aligned} \therefore \text{Residue at } z = i &= \lim_{z \rightarrow i} (z - i) f(z) \\ &= \lim_{z \rightarrow i} (z - i) \frac{1}{(z^2 + 1)} = \frac{1}{2i} \end{aligned}$$



By Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= \int_{-R}^R \frac{dx}{1+x^2} + \int_{\Gamma} \frac{dz}{1+z^2} \\ &= 2\pi i (\text{sum of the residues within } C) \quad \dots(ii) \end{aligned}$$

$$\text{Since, } \lim_{z \rightarrow \infty} z f(z) = \lim_{z \rightarrow \infty} \frac{z}{1+z^2} = 0$$

$$\therefore \lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{1+z^2} = 0$$

From Eq. (ii), we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \frac{1}{2i} \Rightarrow 2 \int_0^{\infty} \frac{dx}{1+x^2} = \pi$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

**Q 5.** If  $f(z)$  has a pole of order  $m$  at  $z = a$ , then show that residue at  $a$  is the limit

$$\left[ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right] \text{ as } z \rightarrow a. \quad (2016, 14, 07)$$

Or If  $f(z)$  has a pole of order  $m$  at  $z = a$ , then show that residue

$$\text{at } a \text{ is } \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]. \quad (2018)$$

**Sol.** Since,  $f(z)$  has a pole of order  $m$  at  $z = a$ , so that  $f(z)$  may be expressed as

$$f(z) = \frac{\phi(z)}{(z-a)^m} \quad \dots(i)$$

where,  $\phi(z)$  is analytic and  $\phi(a) \neq 0$ . Residue of  $f(z)$  at  $z = a$  is  $b_1$ , where

$$\begin{aligned} b_1 &= \frac{1}{2\pi i} \int_C f(z) dz = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{(z-a)^m} dz \\ &= \frac{1}{(m-1)!} \frac{(m-1)!}{2\pi i} \int_C \frac{\phi(z)}{(z-a)^{m-1+1}} dz \\ &= \frac{1}{(m-1)!} \phi^{m-1}(a) \quad [\text{by Cauchy's integral formula}] \end{aligned}$$

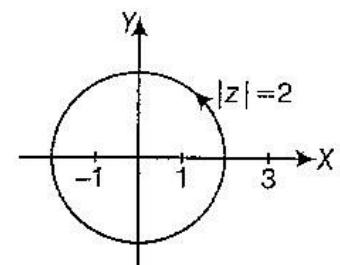
$$\Rightarrow \begin{aligned} b_1 &= \text{Res}(z=a) \\ &= \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \quad [\text{from Eq. (i)}] \end{aligned}$$

**Q 6.** Evaluate  $\int_C \frac{3z^2 + z + 1}{(z^2 - 1)(z + 3)} dz$ , where  $C$  is the circle  $|z| = 2$ .

**Sol.** The poles of the integrand are given by

$$(z^2 - 1)(z + 3) = 0 \Rightarrow z = -1, 1, 3$$

Out of these three poles only two  $z = -1$  and  $z = 1$  lie inside the circle  $|z| = 2$ .



$$\begin{aligned} \therefore \text{Res}(z=-1) &= \lim_{z \rightarrow -1} \left[ (z+1) \frac{(3z^2 + z + 1)}{(z^2 - 1)(z+3)} \right] \\ &= \lim_{z \rightarrow -1} \left[ \frac{3z^2 + z + 1}{(z-1)(z+3)} \right] = -\frac{3}{4} \end{aligned}$$

$$\text{and } \text{Res}(z=1) = \lim_{z \rightarrow 1} \left[ (z-1) \frac{3z^2 + z + 1}{(z^2 - 1)(z+3)} \right] = \lim_{z \rightarrow 1} \left[ \frac{3z^2 + z + 1}{(z+1)(z+3)} \right] = \frac{5}{8}$$

Hence, by Cauchy Residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{Res}(z=-1) + \text{Res}(z=1)] \\ &= 2\pi i \left[ -\frac{3}{4} + \frac{5}{8} \right] = -\frac{\pi i}{4} \end{aligned}$$

**Q 7.** Find the residue of  $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$  at its poles

and hence evaluate  $\int_C f(z) dz$ , where  $C$  is the circle  $|z| = \frac{5}{2}$ .

**Sol.** Poles of  $f(z)$  are given by

$$(z-1)^4(z-2)(z-3) = 0$$

$\Rightarrow z = 1$  (pole of order four),  $z = 2, 3$  (simple poles)

Here,  $\text{Res}(z=2)$

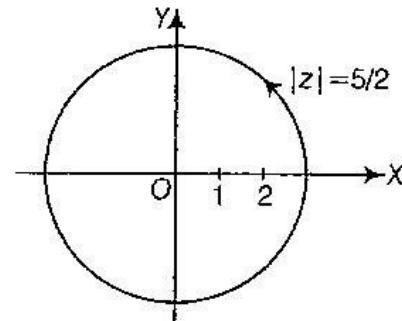
$$\begin{aligned} &= \lim_{z \rightarrow 2} \left[ (z-2) \frac{z^3}{(z-1)^4(z-2)(z-3)} \right] \\ &= \lim_{z \rightarrow 2} \left[ \frac{z^3}{(z-1)^4(z-3)} \right] = 1 \\ \text{Res}(z=3) &= \lim_{z \rightarrow 3} \left[ (z-3) \frac{z^3}{(z-1)^4(z-2)(z-3)} \right] \\ &= \lim_{z \rightarrow 3} \left[ \frac{z^3}{(z-1)^4(z-2)} \right] = \frac{27}{16} \end{aligned}$$

$$\begin{aligned} \text{and } \text{Res}(z=1) &= \frac{1}{(4-1)!} \left[ \frac{d^3}{dz^3} \left\{ (z-1)^4 \frac{z^3}{(z-1)^4(z-2)(z-3)} \right\} \right]_{z=1} \\ &= \frac{1}{6} \left[ \frac{d^3}{dz^3} \left( \frac{z^3}{(z-2)(z-3)} \right) \right]_{z=1} = \frac{1}{6} \left[ \frac{d^3}{dz^3} \left( z+5 + \frac{19z-30}{z^2-5z+6} \right) \right]_{z=1} \\ &= \frac{1}{6} \left[ \frac{d^3}{dz^3} \left\{ z+5 + \frac{19}{z-3} + 8 \left( \frac{1}{(z-3)} - \frac{1}{(z-2)} \right) \right\} \right]_{z=1} \\ &= \frac{1}{6} \left[ \frac{d^2}{dz^2} \left\{ 1 - \frac{27}{(z-3)^2} + \frac{8}{(z-2)^2} \right\} \right]_{z=1} \\ &= \frac{1}{6} \left[ \frac{d}{dz} \left\{ \frac{54}{(z-3)^3} - \frac{16}{(z-2)^3} \right\} \right]_{z=1} \\ &= \frac{1}{6} \left[ \frac{-162}{(z-3)^4} + \frac{48}{(z-2)^4} \right]_{z=1} = \frac{1}{6} \left[ \frac{-162}{16} + 48 \right] = 8 - \frac{27}{16} = \frac{101}{16} \end{aligned}$$

Clearly, only the poles  $z = 1$  and  $z = 2$  lie inside the circle  $|z| = \frac{5}{2}$ .

Hence, by Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{Res}(z=1) + \text{Res}(z=2)] \\ &= 2\pi i \left[ \frac{101}{16} - 2 \right] = 2\pi i \left( \frac{69}{16} \right) = \frac{69\pi i}{8} \end{aligned}$$



**Q 8.** Show that  $\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2\pi}{\sqrt{3}}$ .

**Sol.** We have,  $I = \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \int_0^{2\pi} \frac{d\theta}{2 + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}$

Put  $e^{i\theta} = z \Rightarrow ie^{i\theta} d\theta = dz \Rightarrow \frac{dz}{iz} = d\theta$

$$\therefore I = \frac{1}{i} \int_C \frac{dz}{z \left[ 2 + \frac{1}{2} \left( z + \frac{1}{z} \right) \right]} = \frac{2}{i} \int_C \frac{dz}{z^2 + 4z + 1}$$

where,  $C$  is the unit circle  $|z| = 1$ .

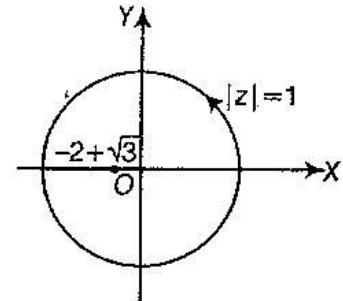
The poles of integrand  $f(z) = \frac{2}{i(z^2 + 4z + 1)}$  are given by  $z^2 + 4z + 1 = 0$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$$

Out of these two poles, only  $z = -2 + \sqrt{3}$  lies inside the circle  $C$ .

$\therefore$  Residue at  $z = -2 + \sqrt{3}$

$$\begin{aligned} &= \lim_{z \rightarrow -2 + \sqrt{3}} \left[ (z + 2 - \sqrt{3}) \frac{2}{i(z^2 + 4z + 1)} \right] \\ &= \lim_{z \rightarrow -2 + \sqrt{3}} \left[ \frac{2}{i(z + 2 + \sqrt{3})} \right] \\ &\quad [\because z^2 + 4z + 1 \equiv (z + 2 - \sqrt{3})(z + 2 + \sqrt{3})] \\ &= \frac{2}{i2\sqrt{3}} = \frac{1}{i\sqrt{3}} \end{aligned}$$



$$\begin{aligned} \therefore \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} &= 2\pi i && [\text{sum of residues of } f(z) \text{ at the poles } C] \\ &= 2\pi i \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}} \end{aligned}$$

Hence proved.

**Q 9.** By the method of contour integration, prove that

$$\int_0^{2\pi} e^{\cos \theta} \cos(\sin \theta - n\theta) d\theta = \frac{2\pi}{n!}, \text{ where } n \text{ is a positive integer.}$$

(2005)

**Sol.** We have,  $I = \int_0^{2\pi} e^{\cos \theta} \cos(n\theta - \sin \theta) d\theta$

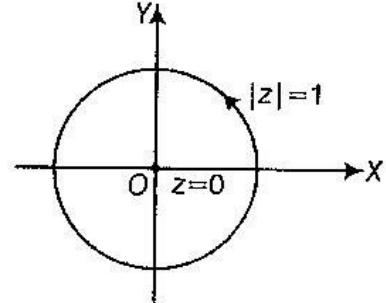
$$= \int_0^{2\pi} e^{\cos \theta} \cos(\sin \theta - n\theta) d\theta \quad [\because \cos(-\theta) = \cos \theta]$$

$$\begin{aligned} \text{Now, consider } I &= \int_0^{2\pi} e^{\cos \theta} e^{i(\sin \theta - n\theta)} d\theta = \int_0^{2\pi} e^{(\cos \theta + i \sin \theta)} \cdot e^{-in\theta} d\theta \\ &= \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta} d\theta = \int_C e^z \cdot z^{-n} \cdot \frac{dz}{iz} \quad [\text{put } z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta] \\ &= \int_C \frac{e^z}{iz^{n+1}} dz = \int_C f(z) dz \text{ (say)} \end{aligned}$$

where,  $C$  is the unit circle.

So,  $f(z)$  has a pole of order  $(n+1)$  at  $z=0$ .

$$\begin{aligned} \therefore \text{Residue at } (z=0) &= \frac{1}{n!} \left[ D^n \left( \frac{e^z}{i} \right) \right]_{z=0} \\ &= \frac{1}{in!} (e^z)_{z=0} = \frac{1}{in!} \end{aligned}$$



$$\therefore I = 2\pi i (\text{sum of residues at poles inside } C)$$

$$= 2\pi i \cdot \frac{1}{in!} = \frac{2\pi}{n!}$$

$$\Rightarrow \int_0^{2\pi} e^{\cos \theta} e^{i(\sin \theta - n\theta)} d\theta = \frac{2\pi}{n!}$$

Now, equating the real parts on both sides, we get

$$\int_0^{2\pi} e^{\cos \theta} \cos(\sin \theta - n\theta) d\theta = \frac{2\pi}{n!} \quad \text{Hence proved.}$$

**Q 10.** Prove that  $\int_{-\pi}^{\pi} \frac{\cos t}{5 + 4 \cos t} dt = -\frac{\pi}{3}$  (2004, 01)

**Sol.** Let  $I = \int_{-\pi}^{\pi} \frac{\cos t}{5 + 4 \cos t} dt = 2 \int_0^{\pi} \frac{\cos t}{5 + 4 \cos t} dt$  [as  $f(-t) = f(t)$ ]

$$I = \int_0^{2\pi} \frac{\cos t}{5 + 4 \cos t} dt \quad \text{[by property]}$$

$$= \text{Real part} \int_0^{2\pi} \frac{e^{it}}{5 + 2(e^{it} + e^{-it})} dt$$

$$\text{Put } z = e^{it} \Rightarrow dz = ie^{it} dt \Rightarrow dt = \frac{dz}{iz}$$

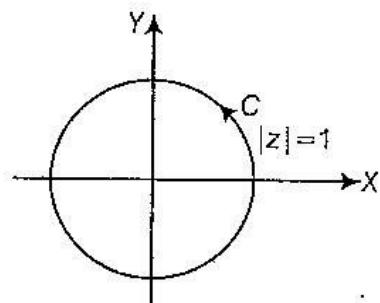
$$\therefore I = \text{Real part} \frac{1}{i} \int_C \frac{z}{2z^2 + 5z + 2} dz$$

$$= \text{Real part} \frac{1}{i} \int_C f(z) dz \quad \dots(i)$$

$$\text{where, } f(z) = \frac{z}{2z^2 + 5z + 2}$$

The poles of  $f(z)$  are given by  $2z^2 + 5z + 2 = 0$

$$\Rightarrow (2z+1)(z+2) = 0 \Rightarrow z = -\frac{1}{2}, -2$$



Out of these two poles only  $z = -\frac{1}{2}$  lies within  $C$ .

$$\begin{aligned}\therefore \text{Res} \left( z = -\frac{1}{2} \right) &= \lim_{z \rightarrow -1/2} \left( z + \frac{1}{2} \right) f(z) = \lim_{z \rightarrow -1/2} \left( z + \frac{1}{2} \right) \frac{z}{(2z+1)(z+2)} \\ &= \lim_{z \rightarrow -1/2} \frac{z}{2(z+2)} = -\frac{1}{6}\end{aligned} \quad \dots(\text{ii})$$

$\therefore$  By Cauchy's residue theorem, we have

$$\begin{aligned}\int_C f(z) dz &= 2\pi i (\text{sum of residues within } C) \\ &= 2\pi i \left( -\frac{1}{6} \right) = -\frac{i\pi}{3}\end{aligned} \quad \dots(\text{iii})$$

Using Eq. (iii) in Eq. (i), we get

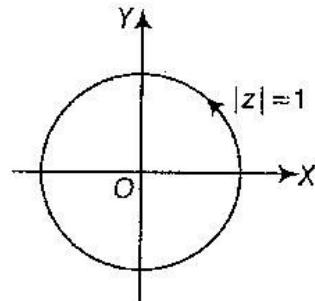
$$\begin{aligned}I &= \text{Real part} \frac{1}{i} \left( -i \frac{\pi}{3} \right) = -\frac{\pi}{3} \\ \therefore \int_{-\pi}^{\pi} \frac{\cos t}{5 + 4 \cos t} dt &= -\frac{\pi}{3} \quad \text{Hence proved.}\end{aligned}$$

**Q 11.** Apply Calculus of residues to prove that

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2p \cos \theta + p^2} d\theta = \frac{\pi p^2}{1 - p^2}, \quad 0 < p < 1. \quad (2006, 02)$$

**Sol.** Let  $I = \int_0^\pi \frac{\cos 2\theta}{1 - 2p \cos \theta + p^2} d\theta$

$$\begin{aligned}&= \frac{1}{2} \int_0^{2\pi} \frac{\cos 2\theta}{1 - 2p \cos \theta + p^2} d\theta \quad [\text{by property}] \\ &= \frac{1}{2} \text{Real part of} \int_0^{2\pi} \frac{e^{i2\theta}}{1 - 2p \cos \theta + p^2} d\theta \\ &= \frac{1}{2} \text{Real part of} \int_C \frac{z^2}{1 - p \left( z + \frac{1}{z} \right) + p^2} \frac{dz}{iz} \\ &\quad [\text{put } z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta] \\ &= \frac{1}{2} \text{Real part of} \int_C \frac{z^2}{i(1 - pz)(z - p)} dz \\ &= \frac{1}{2} \text{Real part of} \int_C f(z) dz \quad (\text{say}) \quad [\text{where, } C \text{ is the unit circle}]\end{aligned}$$



On equating  $(1 - pz)(z - p)$  equal to zero, we obtained,  $z = p, \frac{1}{p}$  as simple poles of  $f(z)$ .

$\therefore z = p$  is the simple pole which lies inside  $C$ .

Residue at  $z = p$  is

$$\lim_{z \rightarrow p} (z - p) f(z) = \lim_{z \rightarrow p} (z - p) \frac{z^2}{i(1 - pz)(z - p)} = \lim_{z \rightarrow p} \frac{z^2}{i(1 - pz)} = \frac{p^2}{i(1 - p^2)}$$

Thus, by Cauchy's residue theorem, we get

$$\int_C f(z) dz = 2\pi i (\text{sum of residues at poles within } C)$$

$$= 2\pi i \cdot \frac{p^2}{i(1 - p^2)} = \frac{2\pi p^2}{1 - p^2}$$

$$\therefore \int_0^\pi \frac{\cos 2\theta}{1 - 2p \cos \theta + p^2} d\theta = \frac{1}{2} \text{Real part of } \int_C f(z) dz$$

$$= \frac{\pi p^2}{1 - p^2}$$

**Q 12.** Using the method of contour integrations, prove that

$$\int_0^\infty \frac{\cos x}{a^2 - x^2} dx = \frac{\pi}{2a} \sin a, a > 0. \quad (2010, 08)$$

**Sol.** Let  $f(z) = \frac{e^{iz}}{a^2 - z^2}$ , then poles of  $f(z)$  are  $a^2 - z^2 = 0$  or  $z = \pm a$ .

Both the poles lie on the real axis.

Consider the integral  $\int_C f(z) dz$ , where

$C$  is closed contour consisting of a large semi-circle of radius  $R$  and real axis from  $x = -R$  to  $R$  and at  $z = a, z = -a$  by small semi-circles of radii be  $r_1, r_2$  respectively.

Since,  $f(z)$  has no poles within  $C$ .

Hence, by Cauchy's residues theorem, we have

$$\int_C f(z) dz = 2\pi i \Sigma \text{Res} = 2\pi i (0)$$

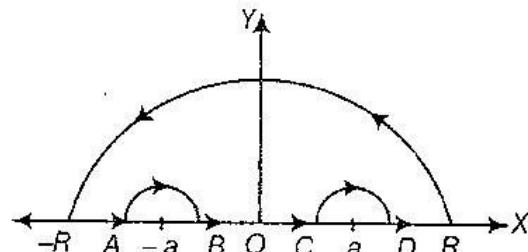
$$\therefore \int_{-R}^{-a+r_1} f(x) dx + \int_{r_1}^a f(z) dz + \int_{-(a-r_1)}^{a-r_2} f(x) dx + \int_{r_2}^a f(z) dz + \int_{a+r_2}^R f(x) dx + \int_{\Gamma} f(z) dz = 0 \quad \dots(i)$$

Since,  $\lim_{|z| \rightarrow \infty} \frac{1}{a^2 - z^2} = 0$ .

Hence, by Jordan's lemma, we get

$$\lim_{R \rightarrow \infty} \int_{\Gamma} \frac{e^{iz}}{a^2 - z^2} dz = 0 \Rightarrow \lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0 \quad \dots(ii)$$

$$\text{Again, } \lim_{z \rightarrow -a} (z + a) f(z) = \lim_{z \rightarrow -a} \frac{e^{iz}}{a - z} = \frac{e^{-ia}}{2a}$$



We have,  $\lim_{r_1 \rightarrow 0} \int_{r_1} f(z) dz = -i(\pi - 0) \frac{e^{-ia}}{2a} = \lim_{r_1 \rightarrow 0} \int_{r_1} f(z) dz = \frac{-i\pi}{2a} e^{-ia}$  ... (iii)

Since,  $\lim_{z \rightarrow a} (z - a) f(z) = \lim_{z \rightarrow a} (z - a) \frac{e^{iz}}{a^2 - z^2} = \frac{e^{ia}}{-2a}$

$$\therefore \lim_{r_2 \rightarrow 0} \int_{r_2} f(z) dz = -i(\pi - 0) \left( \frac{e^{ia}}{-2a} \right) = \frac{i\pi}{2a} e^{ia}$$
 ... (iv)

In Eqs. (iii) and (iv), we have chosen negative sign in the result because  $r_1$  and  $r_2$  are described in clockwise direction.

Using Eqs. (ii), (iii), (iv) in Eq. (i) and making  $R \rightarrow \infty$ ,  $r_1 \rightarrow 0$ ,  $r_2 \rightarrow 0$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} f(x) dx - \frac{i\pi}{2a} e^{-ia} + \int_{-a}^a f(x) dx + \frac{i\pi}{2a} e^{ia} + \int_a^{\infty} f(x) dx + 0 = 0 \\ \Rightarrow & \int_{-\infty}^{\infty} f(x) dx = -\frac{i\pi}{2a} 2i \sin a \quad [\because e^{ia} - e^{-ia} = 2i \sin a] \\ \Rightarrow & \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} dx = \frac{\pi}{a} \sin a \\ \Rightarrow & \int_{-\infty}^{\infty} \left( \frac{\cos x + i \sin x}{a^2 - x^2} \right) dx = \frac{\pi}{a} \sin a \\ \Rightarrow & \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{a^2 - x^2} dx = \frac{\pi}{a} \sin a \end{aligned}$$

On comparing the real parts from both sides, we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi}{a} \sin a \\ \Rightarrow & \int_0^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi}{2a} \sin a \quad \text{Hence proved.} \end{aligned}$$

**Q 13.** By the method of contour integration, prove that

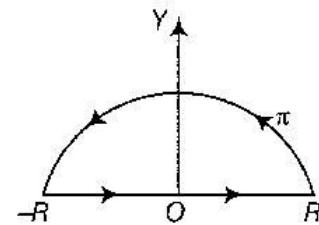
$$\int_{-\infty}^{\infty} \frac{\sin mx}{a^2 + x^2} dx = 0. \quad (2004, 1999, 96, 93, 90)$$

**Sol.** Consider the integral  $\int_C f(z) dz$ , where  $f(z) = \frac{e^{imz}}{a^2 + z^2}$ ,

round a contour  $C$  consisting a large semi-circle of radius  $R$  along with part of the real axis from  $x = -R$  to  $x = R$ .

Hence, by Cauchy's residue theorem, we have

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = 2\pi i \sum \text{Res}$$



where,  $\Sigma$  is sum of residue of  $f(z)$  at its poles lies within  $C$ .

$$\int_{-R}^R \frac{e^{jmx}}{a^2 + x^2} dx + \int_{\Gamma} \frac{e^{jmz}}{a^2 + z^2} dz = 2\pi i \Sigma \text{Res} \quad \dots(i)$$

Poles of  $f(z)$  are given by  $z^2 + a^2 = 0 \Rightarrow z = \pm ai$

Out of these two poles  $f(z)$  has only one simple pole  $z = ai$  lies inside  $C$ .

$$\begin{aligned} \therefore \text{Res}(z = ai) &= \lim_{z \rightarrow ai} (z - ai) f(z) = \lim_{z \rightarrow ai} \frac{(z - ai) e^{jmz}}{(z + ai)(z - ai)} \\ \Rightarrow \text{Res}(z = ai) &= \frac{1}{2ia} e^{-ma} \end{aligned} \quad \dots(ii)$$

$$\text{Now, } \lim_{z \rightarrow \infty} \frac{1}{z^2 + a^2} = 0$$

Therefore, by Jordan's lemma, we get

$$\lim_{R \rightarrow \infty} \int_{\Gamma} \frac{e^{jmz}}{z^2 + a^2} dz = \lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0 \quad \dots(iii)$$

Using Eqs. (ii) and (iii) in Eq. (i) and making  $R \rightarrow \infty$ , we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{jmx}}{a^2 + x^2} dx &= 2\pi i \frac{e^{-ma}}{2ia} = \frac{\pi}{a} e^{-ma} \\ \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{a^2 + x^2} dx &= \frac{\pi}{a} e^{-ma} \\ \int_{-\infty}^{\infty} \frac{\cos mx}{a^2 + x^2} dx + i \int_{-\infty}^{\infty} \frac{\sin mx}{a^2 + x^2} dx &= \frac{\pi}{a} e^{-ma} \end{aligned}$$

On comparing the imaginary parts from both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin mx}{a^2 + x^2} dx = 0 \quad \text{Hence proved.}$$

**Q 14.** Using the method of contour integration, prove that

$$\int_0^{\infty} \frac{\cos mx}{x^4 + a^4} dx = \frac{\pi}{2a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right). \quad (2008)$$

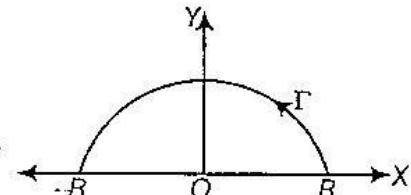
**Sol.** Let  $\int_C \frac{e^{jmz}}{z^4 + a^4} dz = \int_C f(z) dz$

where,  $C$  is the contour consisting of a large semi-circle of radius  $R$  containing all the poles of the integrand in the upper half plane and the part of real axis from  $-R$  to  $R$ .

By Cauchy's residue theorem, we have

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = 2\pi i \Sigma \text{Res}$$

By Jordan's lemma, we have



$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$$

When  $R \rightarrow \infty$ , then we have

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \text{Res} \quad \dots(i)$$

Poles of  $f(z)$  are given by  $z^4 + a^4 = 0$

$$\Rightarrow z = ae^{i(2n+1)\pi/4}, n = 0, 1, 2, 3, \dots$$

Thus,  $z = ae^{i\pi/4}, ae^{3i\pi/4}, ae^{5i\pi/4}, ae^{7i\pi/4}$  are the simple poles of  $f(z)$ . Out of these, the first two lie within  $C$ .

Let us denote any of these by  $\alpha$ .

$$\begin{aligned} \therefore \text{Residue at } (z = \alpha) &= \lim_{z \rightarrow \alpha} \left[ (z - \alpha) \frac{e^{imz}}{z^4 + a^4} \right] \quad \left[ \begin{array}{l} 0 \\ 0 \end{array} \text{ form} \right] \\ &= \lim_{z \rightarrow \alpha} \left[ \frac{(z - \alpha)e^{imz} im + e^{imz}}{4z^3} \right] = \frac{e^{ima}}{4\alpha^3} \quad [\text{using L'Hospital's rule}] \end{aligned}$$

Sum of the residues at poles inside  $C$

$$\begin{aligned} &= \frac{1}{4} \left[ \frac{e^{jmae^{i\pi/4}} e^{j\pi/4}}{a^3 e^{j3\pi/4}} + \frac{e^{jmae^{3i\pi/4}}}{a^3 e^{9i\pi/4}} \right] \quad \left[ \because e^{j\pi/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1+i}{\sqrt{2}} \right] \\ &= \frac{1}{4a^3} \left[ \frac{e^{jma(1+i)/\sqrt{2}}}{-e^{-\pi i/4}} + \frac{e^{jma(-1+i)/\sqrt{2}}}{e^{j\pi/4}} \right] \quad \left[ \because e^{3i\pi/4} = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = \frac{-1+i}{\sqrt{2}} \right] \\ &= \frac{e^{-ma/\sqrt{2}}}{4a^3} \left[ \frac{e^{jma/\sqrt{2}}}{-e^{-i\pi/4}} + \frac{e^{-ima/\sqrt{2}}}{e^{j\pi/4}} \right] \quad [\because e^{j3\pi/4} = e^{j\pi} \cdot e^{-i\pi/4} = (-1)e^{-i\pi/4} = -e^{j\pi/4}] \\ &= -\frac{e^{-ma/\sqrt{2}}}{4a^3} \left[ \exp i \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right) - \exp \left\{ -i \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right) \right\} \right] \\ &\qquad\qquad\qquad \left[ \because e^{9i\pi/4} = e^{2i\pi} \cdot e^{j\pi/4} = (1)e^{j\pi/4} = e^{j\pi/4} \right] \\ &= -\frac{e^{-ma/\sqrt{2}}}{4a^3} 2i \sin \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right) \quad \left[ \because e^{j\theta} - e^{-j\theta} = 2i \sin \theta \right] \end{aligned}$$

From Eq. (i), we get

$$\int_{-\infty}^{\infty} \frac{e^{jmx}}{x^4 + a^4} dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right)$$

Now, comparing the real part on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4 + a^4} dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right)$$

$$\therefore \int_0^{\infty} \frac{\cos mx}{x^4 + a^4} dx = \frac{\pi}{2a^3} e^{-ma/2} \sin \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right)$$

**Hence proved.**

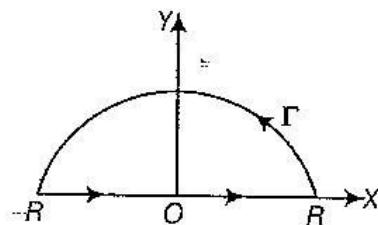
**Q15.** Apply Calculus of residues to evaluate

$$\int_0^\infty \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ma}; a > 0, m > 0. \quad (2012, 1995, 94, 92, 90)$$

**Sol.** Consider the integral

$$\int_C \frac{ze^{imz}}{(z^2 + a^2)} dz = \int_C f(z) dz$$

where,  $C$  is the circle consisting of a large semi-circle of radius  $R$  containing all the poles of the integrand in the upper half plane and the part of real axis from  $-R$  to  $R$ .



By Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= \int_{-R}^R \frac{xe^{imx}}{x^2 + a^2} dx + \int_{\Gamma} \frac{ze^{imz}}{z^2 + a^2} dz \\ &= 2\pi i (\text{sum of the residues within } C) \end{aligned}$$

Since,  $\lim_{z \rightarrow \infty} \frac{z}{(z^2 + a^2)} = 0$

Therefore, we have  $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$  [by Jordan's Lemma]

$$\therefore \lim_{R \rightarrow \infty} \int_{-R}^R \frac{xe^{imx}}{x^2 + a^2} dx = 2\pi i (\text{sum of the residues within } C)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2} dx = 2\pi i (\text{sum of the residues within } C) \quad \dots(i)$$

The poles of  $f(z)$  are given by

$$z^2 + a^2 = 0 \Rightarrow (z + ia)(z - ia) = 0 \Rightarrow z = \pm ia.$$

Out of these poles  $z = ia$  lies inside the contour  $C$ .

$$\begin{aligned} \therefore \text{Res}(z = ia) &= \lim_{z \rightarrow ia} [(z - ia) f(z)] \\ &= \lim_{z \rightarrow ia} \left[ (z - ia) \frac{ze^{imz}}{(z^2 + a^2)} \right] = \lim_{z \rightarrow ia} \left[ \frac{ze^{imz}}{(z + ia)} \right] = \frac{e^{-ma}}{2} \end{aligned}$$

From Eq. (i), we have

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2} dx = 2\pi i \frac{e^{-ma}}{2} = \pi i e^{-ma}$$

Comparing the imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \pi e^{-ma}$$

$$\therefore \int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ma}$$

**Q 16.** Prove that  $\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma}$ ,  $m \geq 0$ ,  $a > 0$ .

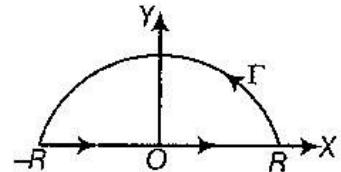
(2015, 09, 07, 01, 1999, 96, 93, 90)

Or With the help of residue theorem, prove that

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma}; m \geq 0, a > 0. \quad (2018)$$

**Sol.** Consider the integral

$$\int_C \frac{e^{imz}}{z^2 + a^2} dz = \int_C f(z) dz$$



where,  $C$  is the contour consisting of a large semi-circle  $\Gamma$  of radius  $R$  containing all the poles of the integrand in the upper half plane and the part of real axis from  $-R$  to  $R$ .

By Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= \int_{-R}^R \frac{e^{imx}}{x^2 + a^2} dx + \int_{\Gamma} \frac{e^{imz}}{z^2 + a^2} dz \\ &= 2\pi i \cdot (\text{sum of the residues within } C) \end{aligned}$$

Since,  $\lim_{z \rightarrow \infty} \left( \frac{1}{z^2 + a^2} \right) = 0$ , therefore we have  $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$

[by Jordan's Lemma]

$$\therefore \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{imx}}{(x^2 + a^2)} dx = 2\pi i \cdot (\text{sum of the residues within } C)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2 + a^2)} dx = 2\pi i \cdot (\text{sum of the residues within } C) \quad \dots(i)$$

Since,  $z = \pm ia$  are the simple poles of  $f(z)$  and the pole  $z = ia$  lies inside  $C$ .

$$\therefore \text{Res}(z = ia) = \lim_{z \rightarrow ia} \left[ (z - ia) \frac{e^{imz}}{(z^2 + a^2)} \right] = \lim_{z \rightarrow ia} \left[ \frac{e^{imz}}{(z + ia)} \right] = \frac{e^{-ma}}{2ia}$$

From Eq. (i), we have

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} dx = 2\pi i \frac{e^{-ma}}{2ia} = \frac{\pi}{a} e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx + i \int_{-\infty}^{\infty} \frac{\sin mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$

On comparing the real parts from both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$

$$\therefore \int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma}$$

Hence proved.

**Q 17.** By the method of contour integration, prove that

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma}. \text{ Hence, deduce that}$$

$$\int_0^\infty \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ma}. \quad (2011)$$

**Sol. Part I** See the solution of Q. 16.

**Part II** Since,  $\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma}$

On differentiating w.r.t.  $m$ , we get

$$\begin{aligned} & \frac{d}{dm} \int_0^\infty \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} \frac{d}{dm} (e^{-ma}) \\ \Rightarrow & \int_0^\infty \frac{\partial}{\partial m} \left( \frac{\cos mx}{x^2 + a^2} \right) dx = \frac{\pi}{2a} (-ae^{-ma}) \\ \Rightarrow & \int_0^\infty \frac{-x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2a} (-ae^{-ma}) \\ \therefore & \int_0^\infty \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ma} \end{aligned}$$

Hence proved.

**Q 18.** Prove that  $\int_0^\infty \frac{\cos mx}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}$ , where  $m > 0$ . (2004)

**Sol.** Consider the integral  $\int_C \frac{e^{imz}}{z^2 + 1} dz = \int_C f(z) dz$

where,  $C$  is the contour consisting of a large semi-circle of radius  $R$  containing all the poles of the integrand in the upper half plane and the part of real axis from  $-R$  to  $R$ . By Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= \int_{-R}^R \frac{e^{imx}}{x^2 + 1} dx + \int_{\Gamma} \frac{e^{imz}}{z^2 + 1} dz \\ &= 2\pi i (\text{sum of the residues within } C) \end{aligned}$$

$$\text{Since, } \lim_{z \rightarrow \infty} \left( \frac{1}{z^2 + 1} \right) = 0$$

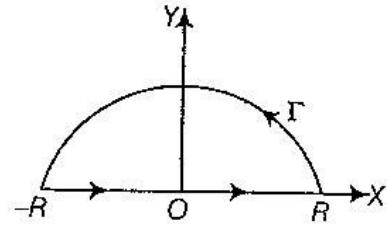
Therefore, we have  $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$  [by Jordan's Lemma]

$$\therefore \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{imx}}{(x^2 + 1)} dx = 2\pi i (\text{sum of the residue within } C)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2 + 1)} dx = 2\pi i (\text{sum of the residues within } C) \quad \dots(i)$$

Now,  $z = \pm i$  are the simple poles of  $f(z)$ .  
and the pole  $z = i$  lies inside  $C$ .

$$\begin{aligned}\therefore \text{Res}(z=i) &= \left[ \lim_{z \rightarrow i} (z-i) \frac{e^{imz}}{(z^2+1)} \right] \\ &= \lim_{z \rightarrow i} \left[ \frac{e^{imz}}{(z+i)} \right] = \frac{e^{-m}}{2i}\end{aligned}$$



From Eq. (i), we have

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{x^2+1} dx = 2\pi i \frac{e^{-m}}{2i} = \pi e^{-m}$$

On comparing the real parts from both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1} dx = \pi e^{-m} \quad \therefore \quad \int_0^{\infty} \frac{\cos mx}{x^2+1} dx = \frac{\pi}{2} e^{-m} \text{ Hence proved.}$$

**Q 19.** Prove that  $\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \pi$  (2012)

**Sol.** Consider  $\int_C f(z) dz = \int_C \frac{e^{inx}}{z(1-z^2)} dz$

where,  $C$  is the contour consisting of a semi-circle of radius  $R$  in the upper half plane intended at  $z = -1, 0, 1$  as clear from the figure.

Since,  $f(z)$  is analytic within and on  $C$ , therefore by Cauchy's residue theorem, we have  $\int_C f(z) dz = \int_{\Gamma} f(z) dz + \int_{-R}^{-1+r_1} f(x) dx + \int_{r_1}^{1-r_3} f(z) dz + \int_{-(1-r_1)}^{-r_2} f(x) dx + \int_{r_2}^{r_3} f(z) dz + \int_{1+r_3}^R f(x) dx = 0$  ... (i)

Since,  $\lim_{z \rightarrow \infty} \frac{1}{z(1-z^2)} = 0$ , therefore we have

$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0 \quad [\text{by Jordan's lemma}]$$

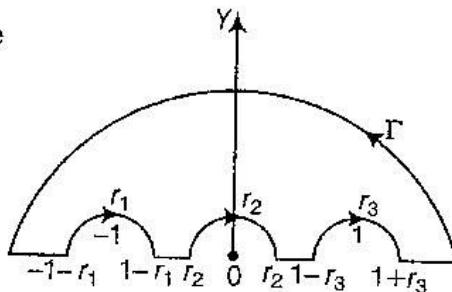
Since,  $\lim_{z \rightarrow -1} (z+1) f(z) = \frac{1}{2}$ , therefore we have

$$\lim_{r_1 \rightarrow 0} \int_{r_1}^{-1} f(z) dz = \frac{1}{2} i(0 - \pi) = -\frac{i\pi}{2}.$$

Since,  $\lim_{z \rightarrow 0} z f(z) = 1$ , therefore we have

$$\lim_{r_2 \rightarrow 0} \int_{r_2}^0 f(z) dz = -i\pi$$

Also, since  $\lim_{z \rightarrow 1} (z-1) f(z) = \frac{1}{2}$ , therefore, we have



$$\lim_{r_3 \rightarrow 0} \int_{r_3} f(z) dz = -\frac{i\pi}{2}$$

Hence, when  $r_1 \rightarrow 0$ ,  $r_2 \rightarrow 0$ ,  $r_3 \rightarrow 0$  and  $R \rightarrow \infty$ , we have from Eq. (i), we get

$$\int_{-\infty}^{-1} f(x) dx - \frac{1}{2}\pi i + \int_{-1}^0 f(x) dx - i\pi + \int_0^1 f(x) dx - \frac{i\pi}{2} + \int_1^\infty f(x) dx = 0$$

$$\Rightarrow \int_{-\infty}^\infty f(x) dx = 2\pi i \Rightarrow \int_{-\infty}^\infty \frac{e^{ix}}{x(1-x^2)} dx = 2\pi i$$

On comparing the imaginary parts from both sides, we get

$$\int_{-\infty}^\infty \frac{\sin \pi x}{x(1-x^2)} dx = 2\pi$$

$$\therefore \int_0^\infty \frac{\sin \pi x}{x(1-x^2)} dx = \pi$$

Hence proved.

## Long Answer Questions

**Q 1.** By the method of contour integration, prove that

$$(a) \int_0^\pi \frac{ad\theta}{a^2 + \sin^2 \theta} = \frac{\pi}{\sqrt{1+a^2}} \quad (a > 0) \quad (2019)$$

$$(b) \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12} \quad (2019)$$

$$Sol. (a) Let I = \int_0^\pi \frac{ad\theta}{a^2 + \sin^2 \theta} = \int_0^\pi \frac{2ad\theta}{2a^2 + 1 - \cos 2\theta} \quad ... (i)$$

$$\text{Put } 2\theta = \phi \text{ in Eq. (i), we get, } I = \int_0^{2\pi} \frac{ad\phi}{2a^2 + 1 - \cos \phi}$$

Now, using transformation  $z = e^{i\phi}$ , we get

$$I = \int_C \frac{a}{2a^2 + 1 - \frac{1}{2} \left( z + \frac{1}{z} \right)} \frac{dz}{iz} \left[ \because \cos \phi = \frac{1}{2} \left( z + \frac{1}{z} \right) \right]$$

$$I = \int_C \frac{2ai}{z^2 - 2(2a^2 + 1)z + 1} dz = \int_C g(z) dz = 2\pi i R^+ \quad ... (ii)$$

Here,  $C : |z| = 1$  is circle

$$g(z) = \frac{2ai}{z^2 - 2(2a^2 + 1)z + 1} \text{ and in } g(z), \text{ sum of residues of poles is } \Sigma R^+.$$

Now, roots of poles equation of  $g(z)$  is

$$z^2 - 2(2a^2 + 1)z + 1 = 0 \Rightarrow z^2 - 2(2a^2 + 1)z + 1 = 0$$

$$z = \frac{+2(2a^2 + 1) \pm \sqrt{4(2a^2 + 1)^2 - 4 \times 1}}{2} = \frac{2(2a^2 + 1) \pm 2\sqrt{4a^4 + 4a^2 + 1 - 1}}{2}$$

$$\therefore z = (1 + 2a^2) + 2a\sqrt{(1 + a^2)} = \alpha \text{ (let)}$$

$$\text{and } z = (1 + 2a^2) - 2a\sqrt{(1 + a^2)} = \beta \text{ (let)}$$

Then,  $z = \alpha$  and  $z = \beta$  is simple pole, and  $\alpha\beta = 1$

$$\text{Since, } |\alpha| = |(1 + 2a^2) + 2a\sqrt{1 + a^2}| > 1 \therefore |\beta| = |(1 + 2a^2) - 2a\sqrt{1 + a^2}| < 1$$

Therefore, only one pole  $z = \beta$  is inside of  $C$ .

$$\begin{aligned} \text{Residue of } g(z) \text{ on } z = \beta &= \lim_{z \rightarrow \beta} (z - \beta)g(z) = \lim_{z \rightarrow \beta} (z - \beta) \frac{2ai}{(z - \alpha)(z - \beta)} \\ &= \frac{2ai}{\beta - \alpha} = \frac{2ai}{-4a\sqrt{1 + a^2}} = \frac{i}{-2\sqrt{1 + a^2}} = \Sigma R^+ \end{aligned} \quad \dots (\text{iii})$$

Now, using Cauchy's residues and from Eqs. (ii) and (iii), we get

$$\int_0^\pi \frac{ad\theta}{a^2 + \sin^2 \theta} = \frac{\pi}{\sqrt{(1 + a^2)}}$$

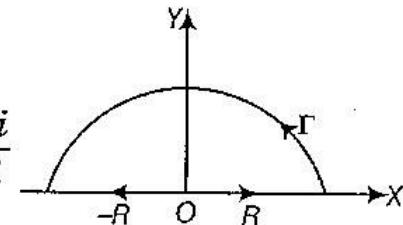
(b) Take integration  $\int_C f(z) dz$ , where

$$f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}. \text{ Poles of function } f(z) = z^4 + 10z^2 + 9 = (z^2 + 1)(z^2 + 9)$$

are  $z = \pm i$  and  $z = \pm 3i$  and all poles are simple. Since pole of  $f(z)$  are not on real axis. So, we take segment of circumference  $C$  in the mid between the upper semiplane of large semi-circle  $\gamma : |z| = R$  and real axis  $-R$  and  $R$ . Therefore, in  $C$ ,  $z = i$  and  $z = 3i$  are only two pole of  $f(z)$ .

Residues on simple pole at  $z = i$  is

$$\begin{aligned} \lim_{z \rightarrow i} (z - i)f(z) &= \lim_{z \rightarrow i} (z - i) \frac{z^2 - z + 2}{(z + i)(z - i)(z^2 + 9)} \\ &= \frac{(i)^2 - i + 2}{(i + i)(i^2 + 9)} = \frac{-1 - i + 2}{2i(-1 + 9)} = \frac{1 - i}{16i} \end{aligned}$$



And, residues on simple pole at  $z = 3i$  is

$$\begin{aligned} \lim_{z \rightarrow 3i} (z - 3i)f(z) &= \lim_{z \rightarrow 3i} (z - 3i) \frac{z^2 - z + 2}{(z^2 + 1)(z + 3i)(z - 3i)} \\ &= \frac{(3i)^2 - (3i) + 2}{((3i)^2 + 1)(3i + 3i)} = \frac{-9 - 3i + 2}{(-9 + 1)(6i)} = \frac{7 + 3i}{48i} \end{aligned}$$

By using Cauchy's residue theorem  $\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = 2\pi i \Sigma R^+$

$$\Rightarrow \int_{-R}^R \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx + \int_{\Gamma} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$$

$$= 2\pi i \left[ \frac{1-i}{16i} + \frac{7+3i}{48i} \right] = 2\pi i \left[ \frac{10}{48i} \right] \quad \dots(i)$$

$$\text{Now, } \lim_{z \rightarrow \infty} zf(z) = \lim_{z \rightarrow \infty} \frac{z(z^2 - z + 2)}{z^4 + 10z^2 + 9} = \lim_{z \rightarrow \infty} \frac{\frac{1}{z} - \frac{1}{z^2} + \frac{2}{z^3}}{1 + \frac{10}{z^2} + \frac{9}{z^4}} = 0$$

Therefore,  $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$

Hence, take  $\lim R \rightarrow \infty$  and from Eq. (i), we get  $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

$$= 2\pi i \left[ \frac{5}{24i} \right] = \frac{5\pi}{12} \quad \text{Hence proved.}$$

**Q 2.** If  $\alpha, \beta$  and  $\gamma$  are real and  $\alpha^2 - \beta^2 - \gamma^2 > 0$ , then prove

$$\text{that } \int_0^{2\pi} \frac{d\theta}{(\alpha + \beta \cos \theta + \gamma \sin \theta)} = \frac{2\pi}{\sqrt{(\alpha^2 - \beta^2 - \gamma^2)}}. \quad (2000)$$

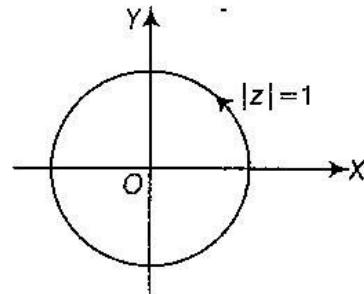
**Sol.** Let  $I = \int_0^{2\pi} \frac{d\theta}{\alpha + \beta \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) + \frac{\gamma}{2i} (e^{i\theta} - e^{-i\theta})} \quad \left[ \text{put } z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz} \right]$

$$\begin{aligned} I &= \int_C \frac{1}{\alpha + \frac{\beta}{2} \left( z + \frac{1}{z} \right) + \frac{\gamma}{2i} \left( z - \frac{1}{z} \right)} \cdot \frac{dz}{iz} \\ &\quad [\text{where, } C \text{ is a unit circle } |z| = 1] \\ &= \frac{2}{i} \int_C \frac{dz}{(\beta - i\gamma) z^2 + 2\alpha z + (\beta + i\gamma)} \\ &= -\frac{2i}{(\beta - i\gamma)} \int_C \frac{dz}{z^2 + \frac{2\alpha}{\beta - i\gamma} z + \frac{\beta + i\gamma}{\beta - i\gamma}} \\ &= -\frac{2i}{\beta - i\gamma} \int_C f(z) dz \end{aligned} \quad \dots(i)$$

$$\text{where, } f(z) = \frac{1}{z^2 + \frac{2\alpha}{\beta^2 + \gamma^2} z + \frac{(\beta + i\gamma)^2}{\beta^2 + \gamma^2}}$$

The poles of  $f(z)$  are given by  $z^2 + \frac{2\alpha}{\beta^2 + \gamma^2} z + \frac{(\beta + i\gamma)^2}{\beta^2 + \gamma^2} = 0$

$$z = \frac{-2\alpha(\beta + i\gamma)}{\beta^2 + \gamma^2} \pm \sqrt{\frac{4\alpha^2(\beta + i\gamma)^2}{(\beta^2 + \gamma^2)^2} - \frac{4(\beta + i\gamma)^2}{\beta^2 + \gamma^2}}$$



$$\begin{aligned}
 &= \frac{-\alpha(\beta + i\gamma) \pm (\beta + i\gamma)\sqrt{\alpha^2 - \beta^2 - \gamma^2}}{(\beta^2 + \gamma^2)} \\
 &= \frac{-\alpha \pm \sqrt{\alpha^2 - \beta^2 - \gamma^2}}{(\beta - i\gamma)}
 \end{aligned}$$

Let  $z_1$  and  $z_2$  be the two poles, then

$$z_1 = \frac{-\alpha + \sqrt{\alpha^2 - \beta^2 - \gamma^2}}{(\beta - i\gamma)} \text{ and } z_2 = \frac{-\alpha - \sqrt{\alpha^2 - \beta^2 - \gamma^2}}{(\beta - i\gamma)}$$

Out of these two poles only  $z = z_1$  lies within  $C$  because  $\alpha^2 - \beta^2 - \gamma^2 > 0$ .

$$\begin{aligned}
 \therefore \operatorname{Res}(z = z_1) &= \lim_{z \rightarrow z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)} \\
 &= \frac{1}{z_1 - z_2} = \frac{(\beta - i\gamma)}{2\sqrt{\alpha^2 - \beta^2 - \gamma^2}}
 \end{aligned} \quad \dots \text{(ii)}$$

By Cauchy's residue theorem, we have

$$\begin{aligned}
 \int_C f(z) dz &= 2\pi i \quad (\text{sum of residues lie within } C) \\
 &= 2\pi i \frac{(\beta - i\gamma)}{2\sqrt{\alpha^2 - \beta^2 - \gamma^2}} \quad [\text{from Eq. (ii)}]
 \end{aligned}$$

From Eq. (i), we get

$$I = -\frac{2i}{(\beta - i\gamma)} \frac{2\pi i (\beta - i\gamma)}{2\sqrt{\alpha^2 - \beta^2 - \gamma^2}} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}} \quad \text{Hence proved.}$$

**Q 3.** By the method of contour integration, prove that

$$(i) \int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}, m > 0. \quad (2018, 13, 10, 07)$$

$$(ii) \int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2. \quad (2013, 09)$$

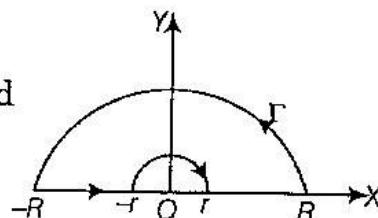
**Sol.**

(i) Consider the integral

$$\int_C f(z) dz = \int_C \frac{e^{izm}}{z} dz$$

where,  $C$  is the contour consisting of

- (a) the upper half of the circle  $|z| = R$ .
- (b) real axis from  $r$  to  $R$ , where  $r$  is small and  $R$  is large.
- (c) real axis from  $-R$  to  $-r$



(d) upper half of the circle  $\gamma, |z| = r$ .

Clearly, the function  $f(z)$  has no singularity inside  $C$ , therefore by Cauchy's residue theorem, we have

$$\int_C f(z) dz = \int_r^R f(x) dx + \int_{\Gamma} f(z) dz + \int_{-R}^{-r} f(x) dx + \int_{\gamma} f(z) dz = 0 \quad \dots(i)$$

By Jordan's lemma, we have

$$\lim_{R \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$$

Also,

$$\lim_{z \rightarrow 0} z f(z) = 1$$

$$\therefore \lim_{r \rightarrow 0} \int_{\gamma} f(z) dz = i(0 - \pi) = -i\pi$$

Hence, when  $r \rightarrow 0$ , then  $R \rightarrow \infty$

From Eq. (i), we get

$$\begin{aligned} & \int_0^{\infty} f(x) dx + \int_{-\infty}^0 f(x) dx - i\pi = 0 \\ \Rightarrow & \int_{-\infty}^{\infty} f(x) dx = i\pi \Rightarrow \int_{-\infty}^{\infty} \frac{e^{jmx}}{x} dx = i\pi \end{aligned}$$

Now, comparing the imaginary parts on both sides, we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi \\ \therefore & \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2} \qquad \text{Hence proved.} \end{aligned}$$

(ii) Consider  $\int_C \frac{\log(i+z)}{1+z^2} dz = \int_C f(z) dz$

where,  $C$  is the contour consisting of a large semi-circle of radius  $R$  containing all the poles of the integrand in the upper half plane and the part of real axis from  $-R$  to  $R$ .

By Residue theorem, we have

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = 2\pi i \sum \text{Res} \quad \dots(i)$$

$$\text{Now, } \lim_{z \rightarrow \infty} zf(z) = \lim_{z \rightarrow \infty} \frac{z \log(i+z)}{(i+z)(z-i)} = \lim_{z \rightarrow \infty} \frac{z}{z-i} \cdot \lim_{z \rightarrow \infty} \frac{\log(i+z)}{1+z} = 0$$

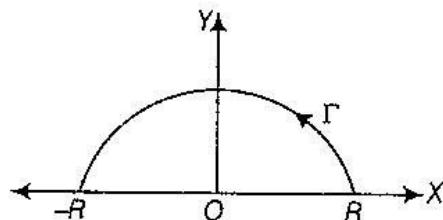
When  $R \rightarrow \infty$ , then from Eq. (i), we get

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \text{Res} \quad \dots(ii)$$

So,  $z = \pm i$  are the simple poles of  $f(z)$  and only  $z = i$  lies inside  $C$ .

$\therefore$  Residue at  $(z = i) = \lim_{z \rightarrow i} (z - i) f(z)$

$$= \lim_{z \rightarrow i} \frac{\log(i+z)}{z+i} = \frac{\log 2i}{2i} = \frac{\log 2 + i\pi/2}{2i}$$



∴ From Eq. (ii), we have

$$\int_{-\infty}^{\infty} \frac{\log(i+x)}{1+x^2} dx = \pi \left( \log 2 + \frac{i\pi}{2} \right)$$

On equating the real parts from both sides, we get

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2} \log(x^2+1)}{x^2+1} dx = \pi \log 2$$

$$\therefore \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2$$

**Hence proved.**

**Q 4.** Using the method of residues, evaluate the following.

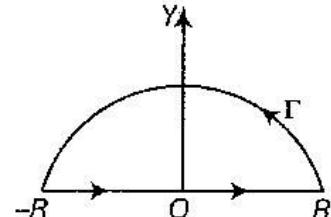
$$(i) \int_0^{\infty} \frac{dx}{x^4 + a^4}, a > 0 \quad (2016, 05, 01)$$

$$(ii) \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8} \quad (2016, 03, 01)$$

**Sol.**

$$(i) \text{ Consider } \int_C f(z) dz = \int_C \frac{dz}{z^4 + a^4}$$

where,  $C$  is the contour consisting of a large semi-circle of radius  $R$  together with real axis from  $-R$  to  $R$ .



Poles of  $f(z)$  are given by

$$\begin{aligned} z^4 + a^4 &= 0 \\ \Rightarrow z &= e^{(2n+1)\pi i/4}, \text{ where } n = 0, 1, 2, 3, \dots \end{aligned}$$

∴  $z = ae^{i\pi/4}, ae^{3i\pi/4}, ae^{5i\pi/4}, ae^{7i\pi/4}$  are the simple poles of  $f(z)$ .

Out of these only  $z = ae^{i\pi/4}, ae^{3i\pi/4}$  lie inside  $C$ .

If  $\alpha$  denotes any of these poles, then

$$\begin{aligned} \text{Residue at } (z = \alpha) &= \lim_{z \rightarrow \alpha} (z - \alpha) f(z) = \lim_{z \rightarrow \alpha} \frac{z - \alpha}{z^4 + a^4} \quad \left[ \frac{0}{0} \text{ form} \right] \\ &= \lim_{z \rightarrow \alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4} = \frac{-\alpha}{4a^4} \quad [\because \alpha^4 = -a^4] \end{aligned}$$

Now, sum of residues at poles inside  $C$

$$= \frac{-1}{4a^4} (ae^{i\pi/4} + ae^{3i\pi/4}) = \frac{-1}{4a^3} \frac{2i}{\sqrt{2}} = \frac{-i\sqrt{2}}{4a^3} \quad \dots(i)$$

$$\begin{aligned} \Rightarrow \int_C f(z) dz &= \int_{-R}^R \frac{dx}{x^4 + a^4} + \int_{\Gamma} \frac{dz}{z^4 + a^4} \\ &= 2\pi i (\text{sum of these residues within } C) \end{aligned}$$

Since,  $\lim_{R \rightarrow \infty} z f(z) = 0$ , therefore

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{z^4 + a^4} = 0 \\ \therefore & \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^4 + a^4} = 2\pi i \text{ (sum of the residues within } C) \\ \Rightarrow & \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i \text{ (sum of the residues within } C) \end{aligned}$$

Now, from Eq. (i), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i \left( \frac{-i\sqrt{2}}{4a^3} \right) = \frac{\pi\sqrt{2}}{2a^3} \\ \Rightarrow & \int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3} \qquad \text{Hence proved.} \end{aligned}$$

(ii) We have,  $\int_C \frac{(\log x)^2}{1+x^2} dx = \int_C \frac{(\log z)^2}{1+z^2} dz = \int_C f(z) dz$

where,  $C$  is the contour as given in the figure.

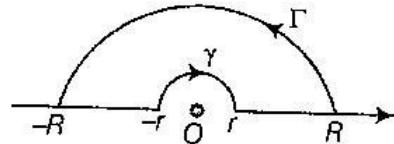
$\therefore$  By Residue theorem, we have

$$\begin{aligned} \int_C f(z) dz &= \int_r^R f(x) dx + \int_{\Gamma} f(z) dz + \int_R^r f(xe^{i\pi}) e^{i\pi} dx \\ &\quad + \int_{\gamma} f(z) dz = 2\pi i \Sigma \text{Res} \end{aligned} \quad \dots(i)$$

Now, we have  $\left| \int_{\Gamma} f(z) dz \right|$

$$\begin{aligned} &\leq \int_0^{\pi} \left| \frac{(\log R e^{i\theta})^2}{1+R^2 e^{i2\theta}} R ie^{i\theta} \right| d\theta \quad [\text{put } z = Re^{i\theta} \Rightarrow dz = Rie^{i\theta} d\theta] \\ &\leq \int_0^{\pi} \frac{|(\log R + i\theta)^2| R d\theta}{R^2 - 1} \leq \int_0^{\pi} \frac{(\log R)^2 - \theta^2 + 2\theta i \log R}{R^2 - 1} R d\theta \\ &= \frac{R}{R^2 - 1} \left[ \pi (\log R)^2 - \frac{\pi^3}{3} + \pi^2 i \log R \right] \\ &= \frac{R^2}{R^2 - 1} \left[ \pi \frac{(\log R)^2}{R} - \frac{\pi^3}{3R} + \pi^2 i \frac{\log R}{R} \right] \rightarrow 0 \text{ as } R \rightarrow \infty \end{aligned}$$

$$\left[ \because \frac{\log R}{R} \rightarrow 0, \text{ as } R \rightarrow \infty \text{ and } \lim_{R \rightarrow \infty} \frac{(\log R)^2}{R} = \lim_{R \rightarrow \infty} \frac{2(\log R)(1/R)}{1} = 0 \right]$$



Similarly, we get

$$\begin{aligned}
 |\int_{\gamma} f(z) dz| &\leq \int_{\pi}^0 \frac{(\log r e^{i\theta})^2}{r^2 e^{i2\theta} + 1} d\theta \\
 &\quad \left[ \because \lim_{r \rightarrow 0} r(\log r)^2 = \lim_{r \rightarrow 0} \frac{(\log r)^2}{1/r} \text{ and } \frac{\infty}{\infty} \text{ form} \right] \\
 &= \lim_{r \rightarrow 0} \frac{2(\log r) 1/r}{-1/r^2} = \lim_{r \rightarrow 0} \frac{2 \log r}{-(1/r)} \quad \left[ \frac{\infty}{\infty} \text{ form} \right] \\
 &= \lim_{r \rightarrow 0} \frac{2(1/r)}{1/r^2} = \lim_{r \rightarrow 0} 2r = 0
 \end{aligned}$$

When  $r \rightarrow 0, R \rightarrow \infty$ , then from Eq. (i),

$$\int_0^{\infty} f(x) - \int_{\infty}^0 f(xe^{i\pi}) dx = 2\pi i \sum \text{Res} \quad \dots (\text{ii})$$

Now,  $z = \pm i$  are the simple poles of  $f(z)$  of which only  $z = i$  lies within  $C$ .

$$\begin{aligned}
 \therefore \lim_{z \rightarrow i} (z - i) f(z) &= \lim_{z \rightarrow i} \frac{(\log z)^2}{z + i} = \frac{(\log i)^2}{2i} = \frac{(\log e^{i\pi/2})^2}{2i} \quad [\because i = e^{i\pi/2}] \\
 &= \frac{1}{2i} \left( \frac{i\pi}{2} \right)^2 = -\frac{\pi^2}{8i}
 \end{aligned}$$

From Eq. (ii), we have

$$\begin{aligned}
 \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx + \int_0^{\infty} \frac{(\log xe^{i\pi})^2}{1+x^2 e^{i2\pi}} dx &= -\frac{\pi^3}{4} \\
 \Rightarrow \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx + \int_0^{\infty} \frac{(\log x + i\pi)^2}{1+x^2} dx &= -\frac{\pi^3}{4}
 \end{aligned}$$

On comparing the real parts from both sides, we get

$$\begin{aligned}
 2 \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx - \pi^2 \int_0^{\infty} \frac{dx}{1+x^2} &= -\frac{\pi^3}{4} \\
 \Rightarrow 2 \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx - \pi^2 [\tan^{-1} x]_0^{\infty} &= -\frac{\pi^3}{4} \\
 \Rightarrow 2 \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx &= -\frac{\pi^3}{4} + \frac{\pi^3}{2} \\
 \therefore \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx &= \frac{\pi^3}{8} \quad \text{Hence proved.}
 \end{aligned}$$