Chapter Four

SEQUENCES

Important Points from the Chapter

1. **Sequences** Let $< X, d >$ be a metric space and N be the set of natural numbers. Then, a sequences in X is a mapping $f : N \to X$ such that $f(n) = a_n \in X$, $\forall\ n \in N$, where $a_n = f(n)$ is called the nth term of the sequence f.
2. **Subsequences** Let $f : N \to X$ be a mapping and defined as $f(n) = a_n$, $\forall\ n \in N$, i.e. $< a_n >$ be a sequence in X. **(2007, 1990)**
3. **Range of a Sequence** Let $f: N \to X$ be a sequence defined by $f(n) = a_n$. Then, the set $A = \{a_1, a_2, \dots, a_n, \dots\}$ is said to be the range of the sequence $< a_n >$.
4. **Bounded Sequence** A sequenced $< a_n >$ is said to be a bounded sequence iff is a range set $A = \{a_1, a_2, \dots, a_n, \dots\}$ is a bounded set X, i.e. A has a finite diameter.
5. **Convergence of a Sequences** Let $< X, d >$ be a metric space. A sequence $< a_n >$ in X is said to converge to a point $a \in X$ iff for each $\varepsilon > 0$, there exists a positive integer δ such that $d(a_n, a) < \varepsilon$, $\forall\ n \geq \delta$.

   In other words, a sequence $< a_n >$ in X is said to converge to a point $a \in X$ iff for each open sphere $S_\varepsilon(a)$, centred on a, there exists a positive integer δ such that $a_n \in S_\varepsilon(a)$, $\forall\ n \geq \delta$.

   Here, a is called the **limit of the sequence** and we write
   $$\lim_{n \to \infty} a_n = a \Rightarrow a_n \to a \text{ as } n \to \infty$$
6. **Cauchy Sequence** A sequence $< a_n >$ in a metric space X is said to be Cauchy sequence iff for each $\varepsilon > 0$, there exists positive integer δ such that $m, n \geq \delta \Rightarrow d(a_m, a_n) < \varepsilon$. For the sufficiently large values of n, the terms of a Cauchy sequence are getting closer and closer to each other.
7. **Complete Metric Space** A metric space $< X, d >$ is said to be a complete metric space iff every Cauchy sequence in X converges to a point in X.
8. **Nested Sequence** Let $< X, d >$ be a metric space. A sequence $< F_n >$ of subsets of X is said to be monotonic decreasing or nested sequence iff
   $$F_1 \supset F_2 \supset F_3 \supset \dots \supset F_n \supset F_{n+1} \supset \dots$$
9. **Cantor's Intersection Theorem** Let $< X, d >$ be a metric space and $< F_n >$ be a nested sequence of non-empty closed subset of X such that diam $F_n \to 0$ and $n \to \infty$. Then, X is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point. **(2007, 01, 1999)**

10. **Cluster Points** Let $< a_n >$ be a sequence in a metric space X. A point $a \in A$ is said to be a cluster point of the sequence $< a_n >$ iff for every $\varepsilon > 0$ and any positive integer δ, there exists a positive integer $n \geq \delta$ such that

$$d(a, a_n) < \varepsilon \text{ or } a_n \in S_\varepsilon(a).$$

Very Short Answer Questions

Q 1. Prove that every convergent sequence in a metric space (X, d) is a bounded sequence. **(2016)**

Sol. Let $< a_n >$ be a convergent sequence in the metric space (X, d) and $\lim_{n \to \infty} a_n = a$. Then, for $\varepsilon = 1$, there exists a positive integer $n_0 \in N$ such that

$$d(a_n, a) < 1, \forall n \geq n_0$$

Let $k = \max\{1, d(a_1, a), d(a_2, a), \ldots, d(a_{n-1}, a)\}$.

Clearly, k is some positive real number and $d(a_n, a) \leq k, \forall n \in N$.

Thus, the range set $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is bounded.

Hence, the sequence $< a_n >$ is bounded.

Q 2. If Y is a complete subspace of a metric space $< X, d >$, then prove that Y is closed in X.

Sol. Let $x \in X$ be a limit point of Y and there is a sequence $< x_n >$ in Y, none of which equals x, such that $x_n \to x$

Since, $< x_n >$ is a convergent sequence in Y, then it is a Cauchy sequence in Y.

Thus, X is the limit of the Cauchy sequence $< X_n > \subset Y$.

$\Rightarrow Y$ is a complete metric space.

Therefore $X \in Y$.

Thus, Y contains all its limit points showing that Y is closed.

Q 3. Prove that in a metric space, every Cauchy sequence is bounded but its converse is not true. **(2010, 08)**

Sol. Let $< a_n >$ be a Cauchy sequence in a metric space X and taking $\varepsilon = 1$. Then, for $\varepsilon = 1$, there exists a positive integer δ such that

$$m, n \geq \delta \Rightarrow d(a_m, a_n) < 1$$

Let $m \geq \varepsilon$ be some fixed positive integer, then a_m is finite.

Therefore, $\quad n \geq \delta \Rightarrow d(a_n, a_\delta) < 1 \quad \ldots(i)$

Thus, Eq. (i) holds, except for $a_1, a_2, \ldots, a_{\delta-1}$

Let $r = \max\{d(a_1, a_\delta), d(a_2, a_\delta), \ldots, d(a_{\delta-1}, a_\delta), 1\}$.

Since, the maximum of a finite set of real numbers cannot be infinite.

Therefore, r is positive finite real number.

$\therefore \qquad d(a_n, a_\delta) \le r, \forall n \in N$

Hence, the range set of the sequence $< a_n >$ is bounded, i.e. the Cauchy sequence $< a_n >$ is bounded.

To show the converse, we consider a sequence

$< a_n > = <1, 0, 1, 0, 1, 0, \ldots >$ in the usual metric space R_u.

This sequence is bounded, in the range set $\{1, 0\}$.

Hence, it is non-convergent.

Q 4. Prove that if a sequence $< a_n >$ in a metric space is not a Cauchy sequence, it can never converge to any point in the metric space. **(2012)**

Sol. Let (R, d) be the usual metric space $d(x, y) = | x - y |$.

Since, the sequence $< a_n > = < -1 >^n = < -1, 1, -1, 1, \ldots >$ is not a Cauchy sequence. If $\{a_n\}$ is a Cauchy sequence, then $\varepsilon = 1$, there exists a positive integer m such that $| a_n - a_m | < 1, \forall n \ge m$. If m is an even integer, then $a_m = 1$.

We take $n = 2m + 1 > m$ from which $a_n = -1$ and $| a_n - a_m | = | -1 - 1 | = | -2 | = 2 \not< 1$, a contradiction.

If m is an odd integer, then $a_m = -1$.

We can choose $n = 2m > m$ for which $a_n = 1$ and $| a_n - a_m | = | 1 + 1 | = 2 \not< 1$ a contradiction.

So, the sequence $< (-1)^n >$ is not a Cauchy sequence.

Now, $< a_n >$ can never converge to any point in metric space. Since, $\lim\limits_{n \to \infty} a_n = l$.

Then, for $\varepsilon = 1/2$, there exists a positive integer m such that

$$| a_n - l | < 1/2 \text{ for } n \ge m \qquad \ldots\text{(i)}$$

From Eq. (i), $|1 - l| < 1/2$ for $n \ge m$ and n is even and $| -1 - l | < 1/2$ for $n \ge m$ and n is odd or $| 1 + l | < 1/2$ for $n \ge m$ and n is odd.

So, $\qquad 2 = | (1 + l) + (1 - l) |$

$\Rightarrow \qquad 2 \le | 1 + l | + | 1 - l |$

$\Rightarrow \qquad < 1/2 + 1/2 = 1$

$\Rightarrow \qquad 2 < 1$, which is a contradiction.

Hence, $< (-1)^n >$ is not convergent.

Q 5. By an example, show that the limit of convergent sequence need not be a limit point of the range set of the sequence. **(2013)**

Sol. Let a constant sequence $< a_n >_{n=1}^{\infty} = < 1, 1, 1, 1, \ldots >$ is a metric space in R.

Since, $a_n \to 1$ as $n \to \infty$, therefore 1 is limit of sequence $< a_n >_{n=1}^{\infty}$. Then, the range set of sequence a_n is

$$E = \{1, 1, 1, 1, ...\} = \{1\} \subseteq R$$

Since, E is finite set and E has no limit point.

Hence, the limit of convergent sequence need not be a limit point of the range set of the sequence.

Q 6. Let $< X, d >$ be a complete metric space and Y be a subspace of X. Prove that Y is complete iff Y is a closed subset of X. (2009)

Or **Let (X,d) be a complete metric space and Y be a subspace of X. If Y is closed in X prove that Y is also complete. (2018)**

Sol. Let $< x_n >$ be a Cauchy sequence in Y.

Therefore, $< x_n >$ is a Cauchy sequence in X.

Since, $Y \subset X$ and X is a complete metric space.

Then, there exists $X_0 \in X$ such that $X_n \to X_0$ as $n \to \infty$.

$\Rightarrow$ Every neighbourhood of X_0 contains all but finitely many terms of the sequence $< x_n >$.

$\Rightarrow$ Every neighbourhood of X_0 intersects with Y.

$\Rightarrow \qquad X_0 \in \overline{Y}$

$\Rightarrow \qquad X_0 \in Y \qquad [\because Y \text{ is closed iff } Y = \overline{Y}]$

Hence, Y is complete.

Q 7. Prove that every convergent sequence is a Cauchy sequence. By an example, show that converse need not be true. (2015)

Or **Let $< a_n >$ be a convergent sequence in a metric space (X, d). Prove that $< a_n >$ is a Cauchy sequence. By an example show that the converse is not true in general. (2017)**

Sol. Let $< a_n >$ be a sequence in a metric space X which converges to a. Then for every $\varepsilon > 0$, there exists a positive integer $\delta > 0$,

such that $$n \geq \delta \Rightarrow d(a_n, a) < \frac{\varepsilon}{2} \qquad ...(i)$$

If $m \geq \delta$, then from Eq. (i), we have

$$d(a_m, a) < \frac{\varepsilon}{2} \qquad ...(ii)$$

Thus, when $m, n \geq \delta$, then by triangle inequality, we have

$$d(a_m, a_n) \leq d(a_m, a) + d(a, a_n)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[from Eqs. (i) and (ii)]}$$

$$< \varepsilon$$

Hence, $< a_n >$ is a Cauchy sequence.

To show the converse, take $X = (0, 1]$ with usual metric $d(x, y) = |x - y|$, $y \in X$.

Now, consider the sequence $< \frac{1}{n} >$.

Let $\varepsilon > 0$ be given, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| \le \frac{m+n}{mn} < \varepsilon, \text{ if } \frac{nm}{m+n} > \frac{1}{\varepsilon}$$

Let $\delta \in N$ such that $\delta > \frac{mn}{m+n}$. Then, for $\varepsilon > 0$, there exists is a positive integer δ such that $n, m > \delta$

$$\Rightarrow \qquad |a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$$

$\therefore < \frac{1}{n} >$ is a Cauchy sequence.

But the sequence $< \frac{1}{n} >$ does not converge in $X = (0, 1]$.

$$\left[\because \lim_{n \to \infty} \frac{1}{n} = 0 \notin (0, 1]\right]$$

Q 8. Given an example of a Cauchy sequence which is not converse. **(2006)**

Sol. See the solution of Q. 7.

Short Answer Questions

Q 1. Let $< a_n >_{n=1}^{\infty}$ be a Cauchy sequence in a metric space $< X, d >$ and $< a_{n_i} >_{i=1}^{\infty}$ be a subsequence of $(a_n)_{n=1}^{\infty}$. If $a_{n_i} \to a$ as $n_i \to \infty$, find the lim a_n if it exists ($n \to \infty$). **(2005)**

Sol. Let $< a_n >$ be Cauchy sequence in metric space $< X, d >$ and $< a_{n_i} >_{i=1}^{\infty}$ be a subsequence of $< a_n >$, which converges to a.

Since, $< a_n >$ is a Cauchy sequence, for given $\varepsilon > 0$, there exists a positive integer n, such that

$$d(a_n, a_m) < \frac{\varepsilon}{2}, \forall\, m, n \ge n_1 \qquad \text{...(i)}$$

Since, $a_{n_i} \to a$ as $n_i \to \infty$, then there exists positive integer n_2, such that

$$d(a_n, a) < \frac{\varepsilon}{2}, \forall n_i \ge n_2 \qquad \text{...(ii)}$$

Let $n_0 = \max\{n_1, n_2\}$, then by triangle inequality, we get

$$d(a_n, a) \le d(a_n, a_{g(n)}) + d(a_{g(n)}, a)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \forall\, n \ge n_0 \qquad \text{[from Eqs. (i) and (ii)]}$$

Thus, for every $\varepsilon < 0$, there exists a positive integer n_0 such that

$$d(a_n, a) < \varepsilon, \forall\, n \geq n_0$$

Since, $a_n \to a$ as $n \to \infty$, then $\lim\limits_{n \to \infty} a_n = a$

Hence the sequence $< a_n >$ converges to a.

Q 2. Let $E \subset X$, where (X, d) is a metric space. Prove that 'a' is a limit of E iff there is a sequence $(a_n) \subset E$, $a_n \neq a$ such that $a_n \to a$ as $n \to \infty$. **(2015, 13, 09, 05, 02, 1995, 94)**

Sol. Let a be any limit point of E, then every neighbourhood of a contains point of E different from a.

$$V_n = \left\{x \in X : d(x, a) < \frac{1}{n}, \forall\, n \in N\right\}$$

Since, $(V_n)_{n=1}^{\infty}$ is a sequence of neighbourhood of 'a'.

Then, $V_n \cap E / \{a\} \neq \phi$, $\forall n \in N$.

Choosing $a_n \in V_n \cap E / \{a\}$, we get a sequence $(a_n) \subset E$. Since, $a_n \neq a$ and $a_n \to a$ as $n \to \infty$ and for let $\varepsilon > 0$ we choose n_0, such that $\dfrac{1}{n_0} < \varepsilon$.

For $n \geq n_0$, we have $\dfrac{1}{n} \leq \dfrac{1}{n_0} < \varepsilon$.

Since, $a_n \in V_n$, then $d(a_n, a) < \dfrac{1}{n} \leq \dfrac{1}{n_0} < \varepsilon$, $a_n \to a$ as $n \to \infty$

Conversely If $a_n \to a$ as $n \to \infty$

$\Rightarrow$ Every neighbourhood of a contains all but finitely many term.

$\Rightarrow$ Every neighbourhood of a contains all but finitely many points of the range set $A = \{a_1, a_2, \ldots, a_n, \ldots\}$.

$\Rightarrow a$ is a limit point of A. $[\because A \subseteq E]$

Hence, a is a limit point of E.

Q 3. Let (X, d) be a complete metric space and Y be a subspace of X. If Y is a closed set in X, prove that Y is also a complete metric space. In particular, prove that $[a, b]$ is complete in R. **(2006, 05)**

Sol. Let Y be a complete subspace of metric space X. Then, we have to show that Y is closed.

Let X be a limit point of Y. Then, for every positive integer n, the open sphere $S_{1/n}(a)$ must contain a point a_n of Y.

Therefore, the sequence $< a_n >$ converges to a.

$\Rightarrow$ It is a Cauchy sequence in Y.

Since, Y is complete and $a_n \to a$, then we have $a \in Y$.

Hence, all the limit points of Y belongs to Y showing that Y is closed conversely, let Y is closed. Then, we have to show that Y is complete.

Let $< a_n >$ be any Cauchy sequence in Y. Then, it is also a Cauchy sequence in X.

Since, X is complete $< a_n >$ will converge to a point $a \in X$.

Now, we show that $a \in Y$.

If the range set of the sequence $< a_n >$ consists of finite number of distinct points, then $< a_n >$ will be of the form

$$< a_1, a_2, \ldots, a_n, a, a, a, \ldots >$$

where, n is finite and hence $a \in Y$.

If the range set of $< a_n >$ has infinitely many points, then a is the limit point of the range set of $< a_n >$ and so a is also a limit point of Y.

Since, Y is closed, then $a \in Y$, i.e. $< a_n >$ is a convergent sequence in Y.

$\Rightarrow Y$ is complete.

We know that R is complete and $Y = [a, b]$ is a subspace of R.

Also, $Y = [a, b]$ is closed, because $Y' = [a, b] \subset Y$.

Hence, $Y = [a, b]$ is complete in R.

Q 4. Let x_n be a sequence in a metric space X and $x \in X$. Prove that $x_n \to x$ as $n \to \infty$ if and only if every neighbourhood of x contains all, but finite number of terms of the sequence.

(2011, 08)

Sol. Let $a_n \to a$ as $n \to \infty$ and U be a neighbourhood (nbd) of 'a'.

If $$a \in U^0$$

Then, $$\varepsilon > 0 \text{ such that}$$

$$S(a, \varepsilon) \subset U \qquad \ldots\text{(i)}$$

Since, $a_n \to a$ as $n \to \infty$.

Then, there exists a positive integer n_0 such that

$$d(a_n, a) < \varepsilon, \forall n \geq n_0$$

$\Rightarrow$ $$a_n \in S(a, \varepsilon)\ \forall n \geq n_0$$

$\Rightarrow$ $$a_n \in S(a, \varepsilon) \subset u, \forall n \geq n_0 \qquad \text{[from Eq. (i)]}$$

$\Rightarrow$ U contains $a_n, \forall n \geq n_0$

Here, every neighbourhood of a contains all but finitely many terms of $(a_n)_{n=1}^{\infty}$.

Conversely If $\varepsilon > 0$ be given. Then, $V = \{x \in X \mid d(x, a) < \varepsilon\}$

So, V is an open set which contains a.

Therefore, V is a neighbourhood of a.

According to question, every neighbourhood of a contains all finite terms of the sequence,

i.e. for some positive integer n_0 and for $n \geq n_0$ we have $a_n \in V$, which implies that for $n \geq n_0$,

$$d(a_n, a) < \varepsilon$$

$$a_b \to a \text{ as } n \to \infty$$

Q 5. Prove that a subset E of X is closed iff every convergent sequence of points of E has its limit in E. **(2009, 1999)**

Sol. Suppose, $E \subset X$ be a closed set and let (a_n) be a sequence in E such that $a_n \to a$ as $n \to \infty$, then there arise two cases.

***Case* I** $< a_n >$ is a constant sequence, it implies that $a_n = a$ except at most for a finite number of terms, therefore $a \in E$.

***Case* II** If $< a >$ is not constant, i.e. if $< a_n >$ has infinitely many distinct points, then 'a' is the point of the set

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}.$$

$\because A \subset E$

$\therefore$ $$A \subset E \Rightarrow D(A) \subset D(E) \subset E$$

Since, a is a limit point of E.

than $$a \in D(E) \subset E \Rightarrow a \in E$$

Conversely Let a be a limit point of E.

Let (X, d) be a metric space and $E \subset X$. Then, a is a limit point of E iff there is a sequence $(a_n) \subset E$, $a_n \neq a$ such that $a_n \to a$ as $n \to \infty$.

Therefore, there is a sequence $< a_n > \subset E$ such that $a_n \to a$ by the hypothesis of the theorem $a \in E$ and hence E is closed.

Q 6. Prove that if a subsequence of a Cauchy sequence converges, then the whole Cauchy sequence converges to the same limit. **(2011, 1991)**

Sol. Let $< a_n >$ be a Cauchy sequence in a metric space X and $< a_{n_i} >$ be a subsequence of $< a_n >$ such that $a_{n_i} \to a$ as $n_i \to \infty$

Let $\varepsilon > 0$ be even.

Since, $< a_n >$ is a Cauchy sequence, then there exists a positive integer δ, such that

$$n, m \geq \delta_1 \Rightarrow d(a_n, a_m) < \frac{\varepsilon}{2} \qquad \ldots(\text{i})$$

Further, $a_n \to a$ as $n_i \to \infty$

For every $\varepsilon > 0$, there exists a positive integer δ_2, such that

$$n_i \geq \delta_2 \Rightarrow d(a_{n_i}, a) < \frac{\varepsilon}{2} \qquad \ldots(\text{ii})$$

Let $\delta = \max \{\delta_1, \delta_2\}$. Then, for n, $n_i \geq \delta$, applying triangle inequality, we get

$$\begin{aligned} d(a_n, a) &\leq d(a_n, a_{n_i}) + d(a_n, a) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[from Eqs. (i) and (ii)]} \\ &< \varepsilon \end{aligned}$$

Hence, $< a_n >$ converges to a as $n \to \infty$.

Q 7. Let $(a_n) \subset X$ be a sequence in a metric space (X, d). Prove that (a_n) is a Cauchy sequence iff diam as $N \to \infty$, where $E_N = \{a_N, a_{N+1}, a_{N+2}, a_{N+3}, \ldots\}$ and $(N = 1, 2, 3, 4, \ldots)$. **(2004, 01)**

Sol. Let $< a_n >$ be a Cauchy sequence, then for every $\varepsilon > 0$, there exists a positive N_0 such that

$$m, n \geq N_0 \Rightarrow d(a_m, a_n) < \varepsilon \quad \ldots\text{(i)}$$

$\because \quad E_{N_0} = \{a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots\}$

$\therefore \quad a_n \in E_{N_0}$ iff $n \geq N_0$

If $a_m, a_n \in E_{N_0}$, then $d(a_m, a_n) < \varepsilon$

$\Rightarrow \quad \text{diam } E_{N_0} \leq \varepsilon \quad \ldots\text{(ii)}$

Now, for $N \geq N_0$, $E_N \subset E_{N_0}$

$\Rightarrow \quad \text{diam } E_N \leq \text{diam } E_{N_0}$

$\Rightarrow \quad \text{diam } E_N \leq \text{diam } E_{N_0} \leq \varepsilon$ [from Eq. (ii)]

$\Rightarrow \quad \text{diam } E_N \leq \varepsilon$

$\Rightarrow \quad \text{diam } E_N \to 0$, as $N \to \infty$

Conversely Let diam $(E_N) \to 0$ as $N \to \infty$.

Then, for every $\varepsilon > 0$, there exists a positive integer N_0 such that

$$\text{diam } E_0 < \varepsilon \quad \ldots\text{(iii)}$$

$\therefore \quad n, m \geq E_0 \Rightarrow a_n, a_m \in E_0$

$\Rightarrow \quad d(a_n, a_m) < \varepsilon$ [from Eq. (iii)]

Hence, $< a_n >$ is a Cauchy sequence.

Long Answer Questions

Q 1. Prove that a metric space X is complete iff every Cauchy sequence in X provides a convergent subsequence. **(2006)**

Sol. Let X be a complete metric space.

Then, there exists a Cauchy sequence $< a_n >$ in X converges to a point in X and consider $< a_{n_R} >$ be a sequence of $< a_n >$ in X.

Now, $< a_n >$ be a Cauchy sequence in X and let $\varepsilon > 0$ be a given, then there exists $n_1 \in N$ such that

$$d(a_{n_R}, a_n) < \frac{\varepsilon}{2}, \forall n_{n_R} \; n \geq n_1 \quad \ldots\text{(i)}$$

$\Rightarrow \quad a_n \to a$ in X as $n \to \infty$

Then, there exists $n_2 \in N$ such that

$$d(a_n, a) < \frac{\varepsilon}{2}, \forall n \geq n_2 \quad \ldots\text{(ii)}$$

If, $n_0 = \max(n_1, n_2)$. Then, for n, $n_R \geq n_0$, applying triangle inequality, we get

$$d(a_{n_R}, a) \leq d(a_{n_R}, a_n) + d(a_n, a)$$

$$\Rightarrow \quad d(a_{n_R}, a) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad d(a_{n_R}, a) \leq \varepsilon$$

For every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d(a_{n_R}, a) < \varepsilon, \forall n_R \geq n_0$$

$$a_{n_R}, a \to a \text{ in } X \text{ as } n_R \to \infty$$

From the above, every Cauchy sequence in X is a convergent subsequence.

Contradiction Let $< a_n >$ be a Cauchy sequence in a metric space X and $< a_{n_R} >$ be a subsequence of $< a_n >$ such that $a_{n_R} \to a$ as $n_R \to \infty$.

Suppose $\varepsilon > 0$ be a given.

Since, (a_n) is a Cauchy sequence, then there exists $n_1 \in N$ such that

$$d(a_n, a_m) < \frac{\varepsilon}{2}, \forall\, n, m \geq n_1 \qquad \ldots\text{(i)}$$

$$a_{n_R} \to a \text{ as } n \to \infty$$

For every $\varepsilon > 0$, there exists $n_n \in N$ such that

$$d(a_{n_R}, a) < \frac{\varepsilon}{R}\ \forall n_R \geq n_2 \qquad \ldots\text{(ii)}$$

Let $n_0 = \max\{n_1, n_2\}$. Then, for n, $n_r \geq n_0$, applying triangle inequality, we get

$$d(a_n, a) \leq d(a_n, a_{n_R}) + d(a_{n_R}, a)$$

$$\Rightarrow \quad d(a_n, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad d(a_n, a) < \varepsilon$$

$$\Rightarrow \quad a_n \to a \text{ in } X \text{ as } n \to \infty$$

Thus, there exists $\varepsilon > 0$, $n_0 \in N$

$$d(a_n, a) < \varepsilon, \forall\, n \geq n_0$$

$$\Rightarrow \quad a_n \to a \text{ in } X \text{ as } n \to \infty$$

Hence, every Cauchy sequence in X, is convergent and complete metric space.

Q 2. Define a subsequence and show that limit of a convergent sequence is unique. **(2007, 1990)**

Sol. **Part I Subsequence** Let $f : N \to X$ be a mapping and defined as $f(n) = a_n$, $\forall\, n \in N$, i.e. $< a_n >$ be a sequence in X.

Part II Let f be a sequence in X given by

$$f(n) = a_n$$

and let $g : N \to N$ be an increasing sequence in N

i.e. $$n_1 < n_2$$
$$\Rightarrow \quad g(n_1) < g(n_2)$$

Then, $fog : N \to X$ is a sequence in X called subsequence of f.

Thus, $(fog)(n) = f(g(n)) = a_{g(n)},\ n \in N$

Therefore, $< a_{g(n)} >$ is a subsequence of $< a_n >$.

Since, a subsequence of $< a_n >$ depends on g, where g is arbitrary increasing sequence, then a sequence can have several subsequence.

If $< a_n >$ is a sequence in a metric space (X, d), which converges to a point $a \in X$.

Then, we have to show that it cannot converge to any other point of the space. Let $< a_n >$ converges to another point $b \in X$.

Since, $a_n \to a$, for every $\varepsilon > 0$, then there exists $n_1 \in N$, such that
$$d(a_n, a) < \frac{\varepsilon}{2}, \forall n \geq n_1$$

Since, $a_n \to b$ for every $\varepsilon > 0$, then there exists $n_2 \in N$, such that
$$d(a_n, b) < \frac{\varepsilon}{2}, \forall n \geq n_2$$

Let $n_0 = \max\{n_1, n_2\}$. Then, $d(a_n, a) < \frac{\varepsilon}{2}$

and $$d(a_n, b) < \frac{\varepsilon}{2}, \forall n \geq n_0$$

Now, let $n \geq n_0$, then
$$d(a, b) \leq d(a, a_n) + d(a_n, b)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \qquad \text{[by triangle inequality]}$$

Since, ε is arbitrary positive real number, then we have
$$d(a, b) = 0$$
$$\Rightarrow \quad a = b$$

Hence, the limit of a convergent sequence is unique.

Q 3. Prove that (R^n, d_2), $n > 1$, is complete metric space, where $d_2(x, y) = \left(\sum_{i=1}^{n} |X_i - Y_i|^2\right)^{1/2}$, $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n) \in R^n$. **(2010, 03)**

Or **Define R^n, $n \geq 2$ and $d_2(x, y)$ where $x, Y \in R^n$. Prove that $d_2(x, y)$ is complete metric on R^n.** **(2013)**

Sol. Let $(X^k)_{k=1}^{\infty}$ be a Cauchy sequence in R^n.

Then, we may write x^k as follows
$$x^k = (x_1^k, x_2^k, \ldots, x_n^k);\ k = 1, 2, 3, \ldots$$

Let $\varepsilon > 0$ be any given real number.

Now (x^k) is a Cauchy sequence, there exists a positive integer n_0, such that $d(x^k, X^P) < \varepsilon, \forall P, k \geq n_0$

$$\Rightarrow \quad \sum_{i=1}^{n} |X_i^k - X_i^P|^2 < \varepsilon^2 \text{ for } i = 1,2,\ldots,n \ \forall P, k \geq n_0$$

$$\Rightarrow \quad |X_i^k - X_i^P|^2 < \varepsilon^2 \text{ for } i = 1, 2, \ldots, n \text{ and } \forall P, k \geq n_0$$

The sequence $(X_i^k)_{k=1}^{\infty}$ is a Cauchy sequence in R.

Since, R is complete to any real number.

Let $\lim_{k \to \infty} X_i^k = X_i$, $i = 1, 2, \ldots, n$, we have

$$x = (x_1, x_2, \ldots, x_n)$$

Then, $x \in R^n$

So, the sequence $(x^k)_{k=1}^{\infty}$ in R^n converges to $x \in R^n$.

Since, for every $i (1 \leq i \leq n)$, the sequence $(X_i^k)_{k=1}^{\infty}$ in R converges to X_i in R for $\varepsilon > 0$, there exists $n \in N$ such that

$$|X_i^k - X_i| < \frac{\varepsilon}{\sqrt{n}}, \forall k \geq n_i$$

$$\Rightarrow \quad |X_i^k - X_i|^2 < \frac{\varepsilon^2}{n}, \forall k \geq n_i$$

If $N_0 = \max\{n_1, n_2, \ldots, n_k, \ldots, n_n\}$, then we have

$$\sum_{i=1}^{n} |X_i^k - X_i|^2 < \frac{\varepsilon^2}{n} \cdot n$$

$$= \varepsilon^2, \forall k \geq n_0$$

Hence, R^n is a complete metric space.

Q.4. Define a complete metric space. Let (X, d_x) and (Y, d_y) be two complete metric spaces and (Z, d_Z) be the product metric space, i.e. $Z = X \times Y$ and

$$d_Z\{(x, y), (x', y')\} = \sqrt{d_x^2(x, x') + d_y^2(y, y')}$$

where, $(x, y), (x', y') \in X \times Y$.

Prove that (Z, d_Z) is also complete metric space. (201

Sol. **Part I Complete Metric Space** A metric space $< X, d >$ is said be a complete metric space iff every Cauchy sequence in X converges to point in X.

Part II We know that $d(< x, y >, < x', y' >)$ is a metric on $X \times Y$.

Let $< x_n, y_n >$ be Cauchy sequence of elements in $X \times Y$. Then, for every $\varepsilon > 0$, there exists a positive integer $\delta > 0$ such that

$$n, m \geq \delta \Rightarrow d(< x_n, y_n >, < x_m, y_m >) < \varepsilon$$

$$\Rightarrow \quad \sqrt{d_x^2(x_n, x_m) + d_y^2(y_n, y_m)} < \varepsilon$$

$$\Rightarrow \quad d_x^2(x_n, x_m) + d_y^2(y_n, y_m) < \varepsilon^2$$

$\Rightarrow \qquad d_x^2(x_n, x_m) < \varepsilon^2$ and $d_y^2(y_n, y_m) < \varepsilon^2$

$\Rightarrow \qquad d_x(x_n, x_m) < \varepsilon$ and $d_y(y_n, y_m) < \varepsilon$

$\Rightarrow < x_n >$ is a Cauchy sequence in X and $< y_n >$ is a Cauchy sequence in Y.

Since, X and Y both are complete.

So, there exists $x \in X$ such that $X_n \to X$ and there exists $y \in Y$ such that $y_n \to y$ as $n \to \infty$.

Now, $d(< x_n, y_n >, < x, y >) = \sqrt{d_x^2(x_n, X) + d_y^2(y_n, Y)} < \varepsilon$

For the sufficiently large n or the Cauchy sequence,

$< x_n, y_n >$ converges to a point $< x, y > \in X \times Y$

Hence, $Z = X \times Y$ is complete.

Q 5. Prove that $C[a, b]$ is complete metric space with respect to the supremum metric, where $C[a, b]$ is the collection of all continuous real valued function on $[a, b]$. **(2014, 04)**

Sol. We know that, $d(f, g) = \sup\{| f(t) - g(t) | : t \in [a, b]\}$ ...(i)

defined on $C[a, b]$, where $f, g \in C[a, b]$ is a metric.

If $< f_n >$ be Cauchy sequence in $C[a, b]$. Then, for every $\varepsilon > 0$, there exists a positive integer δ such that $n, m \geq \delta$.

$$\Rightarrow \qquad d(f_n, f_m) < \varepsilon \qquad \text{...(ii)}$$

$$\Rightarrow \qquad \sup\{| f_n(t) - f_m(t) |, t \in [a, b]\} < \varepsilon$$

$$\Rightarrow \qquad | f_n(t) - f_m(t) | < \varepsilon, \forall t \in [a, b] \qquad \text{...(iii)}$$

$\Rightarrow < f_n(t) >$ is a Cauchy sequence in R for any fixed $t \in [a, b]$.

But R being complete, this sequence converges.

Let $f_n(t) \to f(t)$ as $n \to \infty$

i.e.
$$\lim_{n \to \infty} f_n(t) = f(t) \qquad \text{...(iv)}$$

Therefore, we can associate to each $t \in [a, b]$ a unique real number $f(t)$. This define (point wise) a function f on $[a, b]$. Now, we will show that $f \in [a, b]$ and $f_n \to f$.

From Eq. (iii), we have

$$| f_n(t) - f_m(t) | < \varepsilon, \forall n, m \geq \delta \text{ and } \forall t \in [a, b] \qquad \text{...(v)}$$

This verifies that the sequence $< f_n >$ of continuous functions converges uniformly to the function f on $[a, b]$ and hence the limit function f is a continuous function on $[a, b]$.

Consequently, $f \in C[a, b]$

Also, from Eq. (v), we have

$$\sup\{| f_n(t) - f(t) | : t \in [a, b]\} < \varepsilon, \forall n \geq \delta$$

$$\Rightarrow \qquad d(f_n, f) < \varepsilon, \forall n \geq \delta$$

$$\Rightarrow \qquad f_n \to f \text{ in } C[a, b]$$

Hence, $C[a, b]$ is complete.

Q 6. State and prove Barie's Category theorem.

(2008, 06, 04, 1995, 94)

***Or* Prove that every complete metric space is of second category.** **(2012)**

Sol. Let $< X, d >$ be a complete metric space and let X is not of second category. Then, X must be of first category. So, X can be expressed as the union of a countable family of nowhere dense sets. We arrange this family of nowhere dense sets as a sequence $< A_n >$.

Since, A_1 is nowhere dense and X is open, then there exists an open sphere $S_1 \subset X$ of radius less than 1, such that

$$S_1 \cap A_1 = \phi \qquad \text{...(i)}$$

Let F_1 be the concentric closed sphere whose radius is one-half of that of S_1.

Since, A_2 is nowhere dense and F_1^0 is an open set, then there exists an open sphere $S_2 \subset F_1^0$ of radius less than $\frac{1}{2}$, such that

$$S_2 \cap A_2 = \phi \qquad \text{...(ii)}$$

Let F_2 be the concentric closed sphere whose radius is one-half of that of S_2.

Since, A_3 is nowhere dense and F_2^0 is an open set, then there exists an open sphere $S_3 \subset F_2^0$ of radius less than $\frac{1}{4} = \frac{1}{2^2}$, such that

$$S_3 \cap A_3 = \phi \qquad \text{...(iii)}$$

Continuing this process, we get a decreasing sequence $< F_n >$ of non-empty closed subsets of X, where F_n is concentric closed sphere whose radius is one-half of that of S_n, i.e. less than $\frac{1}{2^n}$ as the radius of S_n is less than $\frac{1}{2^{n-1}}$.

Consequently, diam $F_n < \frac{2}{2^n} \to 0$ as $n \to \infty$. Thus, $< F_n >$ is a decreasing sequence of non-empty closed subsets of X and diam $F_m \to 0$ as $n \to \infty$.

Since, X is complete, therefore by Cantor's intersection theorem, such that a point

$$x \in X \text{ such that } \bigcap_{n=1}^{\infty} F_n = \{x\}.$$

Since, for every $n \in N$

$$x \in F_n \subset S_n \text{ and } S_n \cap A_n = \phi$$

$\therefore \qquad x \notin A_n$ for any n

Also, $\qquad x \notin \bigcup_{n=1}^{\infty} A_n$

Thus, $x \in X$ and $x \notin \bigcup_{n=1}^{\infty} A_n$

$$\therefore \qquad X \neq \bigcup_{n=1}^{\infty} A_n.$$

Thus, X can not be expressed as a countable union of nowhere dense sets so that X is not first category, which is contrary to our supposition. Hence, our supposition is wrong. Therefore, X is of second category.

Q 7. State and prove Cantor's intersection theorem.

(2016, 13, 10, 07, 01, 1991)

Or **Let X be a complete metric space and $(E_n)_{n=1}^{\infty}$ be a sequence of closed and bounded sets such that $E_1 \supset E_2 \supset E_3 \ldots E_n \supset E_{n+1} \supset \ldots$ and that $\lim_{n \to \infty} \text{diam } E_n = 0$. Prove that $\bigcup_{n=1}^{\infty} E_n$ consists of exactly one point.** **(2018)**

Sol. **Statement** Let (X, d) be a metric space and let $< F_n >$ be a nested sequence of non-empty closed subsets of X such that diam $F_n \to 0$ as $n \to \infty$. Then, X is complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Proof Let X is complete for each n, we choose $a_n \in F_n$.

Since, diam $F_n \to 0$, for every $\varepsilon > 0$, then there exists a positive integer n_0 such that diam $F_{n_0} < \varepsilon$.

Since, $$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots \supseteq F_n \supseteq F_{n+1} \supseteq \ldots$$

We have, $n, m \geq n_0$

$$\Rightarrow \qquad F_n, F_m \subseteq F_{n_0} = a_n, a_m \in F_{n_0} = d(a_m, a_n) < \varepsilon$$

Thus $< a_n >$ is Cauchy sequence in X. Since, X is complete, $a_n \to a$ for some $a \in X$.

We will prove that $a \in \bigcap_{n=1}^{\infty} F_n$.

Let $m \in N$ be arbitrary.

Then, $$n > m \Rightarrow F_n \subseteq F_m \Rightarrow a_n \in F_m$$

Since, $a_n \to a$, every neighbourhood of a contains an infinite number of point of F_m.

Thus, a is a limit point of F_m.

Since, F_m is closed, then $a \in F_m$

Since, m is arbitrary, we have

$$a \in \bigcap_{n=1}^{\infty} F_n$$

Now, suppose that there exists another point $b \in \bigcap_{n=1}^{\infty} F_n$. Then, diam $\left(\bigcap_{n=1}^{\infty} F_n\right) = \delta > 0$.

Since, $\bigcap_{n=1}^{\infty} F_n \subset F_n, \forall n$, then

$$\text{diam}\left(\bigcap_{n=1}^{\infty} F_n\right) \leq \text{diam } F_n, \forall n$$

$\Rightarrow$ $$0 < \delta \leq \text{diam } F_n, \forall n$$

$\therefore$ diam F_n does not converge to zero which contradicts the hypothesis

Hence, $a = b$ and so $\bigcap_{n=1}^{\infty} F_n = \{a\}$.

Conversely Let $\bigcap_{n=1}^{\infty} F_n$ consists of a single point for every nested sequence $< F_n >$ of non-empty closed subset F_n of X such that diam $F_n \to 0$. Then, we have to prove that X is complete.

Let $< a_n >$ be any Cauchy sequence in X.

Construct the subsets S_n of X as follow

$$S_n = \{a_1, a_2, \ldots\}$$
$$S_2 = \{a_2, a_3, \ldots.\}$$
$$\ldots \ldots \ldots \ldots \ldots \ldots$$
$$\ldots \ldots \ldots \ldots \ldots \ldots$$
$$S_n = \{a_n, a_{n+1}, \ldots\}$$

Since, $< a_n >$ is Cauchy sequence, for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(a_m, a_n) < \varepsilon, \forall n, m \geq n_0$$

$\therefore$ $$n \geq n_0 \Rightarrow \text{diam } S_n < \varepsilon$$

Consequently, diam $S_n \to 0$ as $n \to \infty$

Also, $S_1 \supseteq S_2 \supseteq S_3 \supseteq \ldots$ so that

$$\overline{S}_1 \supseteq \overline{S}_2 \supseteq \overline{S}_3 \supseteq \ldots \qquad [\because A \supseteq B \Rightarrow \overline{A} \supseteq \overline{B}$$

We know that,

$$\text{diam } \overline{S} = \text{diam } S$$

Hence, $< \overline{S}_n >$ is a nested sequence of closed subsets of X whose diameter tends to zero.

Then, by hypothesis, there exists a unique point $a \in X$ such that

$$a \in \bigcap_{n=1}^{\infty} \overline{S}_n$$

We claim that the Cauchy sequence $< a_n >$ converges to a.

Since, diam $\overline{S}_n \to 0$, for a given $\varepsilon > 0$, there exists a positive integer n_0, such that diam $\overline{S}_{n_0} < \varepsilon$.

Consequently, $n > n_0 \Rightarrow a_n, a \in \overline{S}_{n_0}$

$\Rightarrow$ $$d(a_n, a) < \varepsilon$$

$\therefore$ $$< a_n > \text{ converges to } a \in X.$$

Hence, X is complete.