Chapter Five

DIFFERENTIABILITY

⏻ Important Points from the Chapter

1. **Differentiability of a Function** The value of the $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ or $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ if exists, is called right hand or progressive derivative of $f(x)$ at $x = a$ and denotes by $Rf'(a)$.

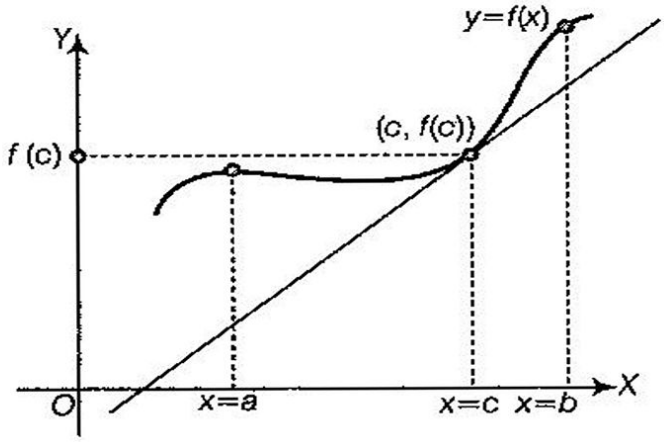
   Similarly, the value of the $\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ or $\lim_{h \to a^-} \frac{f(x) - f(a)}{x - a}$ if exists, is called left hand or progressive derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$. **(2010, 02, 01)**

2. **Differentiability of a Function on an Interval** Let f be a function defined on an interval $[a, b]$. Then, f is said to be differentiable on $[a, b]$, if

   (i) f is differentiable at each point of (a, b).

   (ii) f is differentiable from the right at a and from left at b.

   Geometrical Interpretation $f'(c)$ is the tangent of angle which the tangent at a point $(c, f(c))$ to the curve $y = f(x)$ makes with X-axis.



3. **Sign of Derivative at a Point** If the function $f : [a, b] \to R$ is derivable at the point $c \in (a, b)$ and $f'(c) \neq 0$, then

   (i) f is increasing at c, if $f'(c) > 0$.

   (ii) f is decreasing at c, if $f'(c) < 0$. **(2004, 1997)**

4. **Some Important Theorems on Derivatives**

   (i) **Darboux's Theorem** If a function $f : [a, b] \to R$ is derivable in $[a, b]$; and $f'(a)$, $f'(b)$ have opposite signs, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$. **(2011, 03, 1998, 96)**

(ii) **Intermediate Value Theorem** If f is derivable in $[a, b]$ and $f'(a) \neq f'(b)$. Then, f assumes every value lying between $f'(a)$ and $f'(b)$ in $[a, b]$.

(iii) **Rolle's Theorem** If a function $f : [a, b] \rightarrow R$ is
(a) continuous on the closed interval $[a, b]$.
(b) derivable in the open interval (a, b).
(c) $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$. **(2013, 11, 09, 05, 01, 1999)**

(iv) **Lagrange's Mean Value Theorem** (LMVT) Let the function $f : [a, b] \rightarrow R$ be
(a) continuous on $[a, b]$.
(b) derivable on (a, b), then there exists atleast one point $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. **(2014, 12, 10, 07, 1996)**

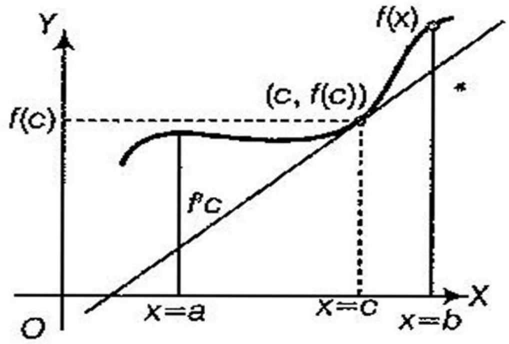
(v) **Cauchy's Mean Value Theorem** If two functions f and g are
(a) continuous on $[a, b]$. (b) derivable in (a, b).
(c) $g'(x) \neq 0$ for any $x \in (a, b)$, then there exists at least one point c in (a, b) such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$. **(2006, 02)**

Very Short Answer Questions

Q 1. Define differentiability of the function $f(x)$ on an interval and give its geometrically interpreted.

Sol. **Part I Differentiability** Let f be a function defined on an interval $[a, b]$. Then, f is said to be differentiable on $[a, b]$, if
(i) f is differentiable at each point of (a, b).
(ii) f is differentiable from the right at a and from left at b.

Part II Geometrical Interpretation $f'(c)$ is the tangent of the angle which the tangent at a point $(c, f(c))$ to the curve $y = f(x)$ makes with X-axis.



Q 2. Discuss the derivability of the following function at $x = 0$.

$$f(x) = \frac{1}{x^2}, \forall\, x \in [-1, 1] \text{ except } x = 0,\ f(0) = 0. \qquad \textbf{(2009)}$$

Sol. $$f'_{0^-}(x) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{\frac{1}{(0-h)^2} - 0}{-h} = \lim_{h \to 0} -\frac{1}{h^3} = -\infty$$

$$f'_{0^+}(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{(0+h)^2} - 0}{h} = \lim_{h \to 0} \frac{1}{h^3} = +\infty$$

$\because \quad f'_{0^-}(x) \neq f'_{0^+}(x)$

Hence, $f(x)$ is not derivable at $x = 0$.

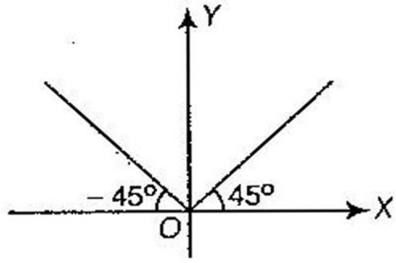
Q 3. Given an example of a function which is continuous but not differentiable. **(2014, 06)**

Sol. Consider the function $f(x) = |x|, \forall\, x \in [-1, 1]$

This function is continuous at $x = 0$, because for an arbitrary $\varepsilon > 0$, there exists $\delta\ (= \varepsilon)$ such that

$$|f(x) - f(0)| = |x| < \varepsilon$$

$$\Rightarrow \quad |x - 0| < \delta = \varepsilon$$



However, f is not derivable at $x = 0$.

To prove this not differentiable

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Now, $$f'(0+0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x - 0} = 1$$

and $$f'(0-0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{-x - 0}{x - 0} = -1$$

$\because \quad f'(0^-) \neq f'(0^+)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Q 4. Show that a function is differentiable at a point, then it is continuous at that point. **(2013)**

Sol. Let us consider a function $f(x) = x^2 \cos\frac{1}{x}, x \neq 0, f(0) = 0$.

First, we test the differentiability of $f(x)$ at $x = 0$.

$$f'_{0^-}(x) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \cos\frac{1}{0-h} - 0}{-h} = \lim_{h \to 0}\left(-h \cos\frac{1}{h}\right)$$

$$= \lim_{h \to 0} (-h) \cdot \lim_{h \to 0} \left(\cos \frac{1}{h} \right)$$

$$= 0 \times (\text{A finite quantity persist between } -1 \text{ to } +1) = 0$$

$$f'_{0^-}(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \cos \frac{1}{0+h} - 0}{h} = \lim_{h \to 0} \left(h \cos \frac{1}{h} \right)$$

$$= \lim_{h \to 0} (h) \cdot \lim_{h \to 0} \left(\cos \frac{1}{h} \right)$$

$$= 0 \times (\text{A finite quantity persist between } -1 \text{ to } +1) = 0$$

$\because \quad f'_{0^-} = f'_{0^+}$

Hence, $f(x)$ is derivable at $x = 0$ and therefore it is continuous at $x = 0$, because every differentiable function is continuous. **Hence proved.**

***Q* 5. Explain whether the Lagrange's mean value theorem is applicable for the function defined by**
$f(x) = |x|, \forall x \in [-2, 1]$ or not. **(2016)**

Sol. Since, $f(x)$ is continuous on $[-2, 1]$ but not derivable at $x = 0$ and such on $[-2, 1]$, hypothesis is not valid.

Clearly, $f(x) = \begin{cases} -x, & \text{if } x \in [-2, 0] \\ x, & \text{if } x \in [0, 1] \end{cases}$

and $f(a) = f(-2) = 2$ and $f(b) = f(1) = 1$

Then, by Lagrange's mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 2}{1 + 2} = -\frac{1}{3}$$

which shows that there is no point c in $(-2, -1)$ such that $f'(c) = -\frac{1}{3}$.

Hence, the conclusion is also not valid and Lagrange's mean value theorem is not applicable.

***Q* 6. Find the value of *c* in the Cauchy's mean value theorem for the following pair of functions.**
$f(x) = \sqrt{x}$ and $g(x) = 2x + 1$ in $[1, 4]$ **(2017)**

Sol. We have, $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2} \frac{1}{\sqrt{x}}$

$\Rightarrow \quad f'(c) = \frac{1}{2} \frac{1}{\sqrt{c}}$

and $\quad g(x) = 2x + 1 \Rightarrow g'(x) = 2 \Rightarrow g'(c) = 2$

Now, by Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)} \text{ gives}$$

$$\frac{f(4)-f(1)}{g(4)-g(1)}=\frac{f'(c)}{g'(c)} \Rightarrow \frac{2-1}{9-3}=\frac{\frac{1}{2}\frac{1}{\sqrt{c}}}{2}$$

$$\Rightarrow \qquad \frac{1}{6}=\frac{1}{4\sqrt{c}} \Rightarrow \sqrt{c}=\frac{3}{2}$$

$$\therefore \qquad c=\sqrt{\frac{3}{2}}$$

Short Answer Questions

Q 1. **Examine the continuity and differentiability of the function**

$$f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & \text{for } x\neq 0\\ 0, & \text{for } x=0\end{cases}.$$

(2013)

Sol. First, we examine the differentiability of the function at $x=0$.

$$\therefore \qquad f'_{0^-}(x)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{\dfrac{0-h}{1+e^{1/(0-h)}}-0}{-h}=\lim_{h\to 0}\frac{1}{1+e^{-1/h}}=1 \qquad [\because \lim_{h\to 0}e^{-1/h}=0]$$

and
$$f'_{0^+}(x)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\dfrac{0+h}{1+e^{1/(0+h)}}-0}{h}$$

$$=\lim_{h\to 0}\frac{1}{1+e^{1/h}}=0 \qquad [\because \lim_{h\to 0}e^{1/h}=\infty]$$

$$\because \qquad f'_{0^-}(x)\neq f'_{0^+}(x)$$

So, $f(x)$ is not differentiable at $x=0$.

Now, we examine the continuity of this function at $x=0$.

$$f_{0^-}(x)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\frac{(0-h)}{1+e^{1/0-h}}$$

$$=\lim_{h\to 0}\frac{-h}{1+e^{-1/h}}=\frac{0}{1+0}=0$$

$$f_{0^+}(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{(0+h)}{1+e^{1/0+h}}$$

$$= \lim_{h \to 0} \frac{h}{1 + e^{1/h}} = \lim_{h \to 0} \frac{he^{-1/h}}{e^{-1/h} + 1} = 0$$

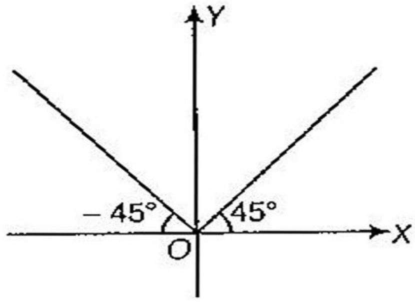
$\because \quad f_{0^-}(x) = f(0) = f_{0^+}(x)$

Hence, $f(x)$ is continuous at $x = 0$.

Q 2. Show that the function $f : R \to R$ defined by $f(x) = |x|$ is both left and right differentiable at O but not differentiable at O. **(2017)**

Sol. Given function is $f(x) = |x|$, $\forall\, x \in R$, which is continuous at $x = 0$.

Now, we show the differentiability of $f(x) = |x|$ at $x = 0$.



$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ +x, & \text{if } x \geq 0 \end{cases}$$

$$f'(0 + 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = 1$$

$f(x)$ is right differentiable at $x = 0$

and $\quad f'(0 - 0) = \lim_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \dfrac{-x - 0}{x - 0} = -1$

$f(x)$ is left differentiable at $x = 0$.

$\because \quad f'(0^-) \neq f'(0^+)$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

Q 3. Discuss the differentiability of the function f at the point $x = 1$ and $x = 2$, where $f(x) = |x - 1| + |x - 2|$; $\forall\, x \in R$. Draw its graph in the interval [0, 3]. **(2012)**

Sol. Given function may be written as

$$f(x) = \begin{cases} -2x + 3, & \text{if } x < 1 \\ 1, & \text{if } 1 \leq x < 2 \\ 2x - 3, & \text{if } x \geq 2 \end{cases}$$

At $x = 1$,

$$f'_{1^-}(x) = \lim_{h \to 0} \frac{f(1 - h) - f(1)}{-h} = \lim_{h \to 0} \frac{-2(1 - h) + 3 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{2h + 3 - 3}{-h} = -2$$

and $\quad f'_{1^+}(x) = \lim_{h \to 0} \dfrac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \dfrac{1 - 1}{h} = 0$

$\because \quad f'_{1^-}(x) \neq f'_{1^+}(x)$

So, $f(x)$ is not differentiable at $x = 1$.

At $x = 2$,

$$f'_{2^-}(x) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = 0$$

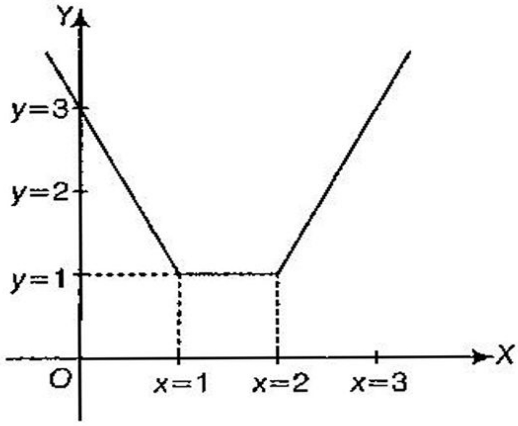
$$f'_{2^+}(x) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{2(2+h) - 3 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$$

$\because$ $f'_{2^-}(x) \neq f'_{2^+}(x)$

So, $f(x)$ is not differentiable at $x = 2$.

The graph of $f(x)$ in the interval [0, 3] is



***Q* 4. Prove that the necessary but not sufficient condition that a function be derivable at a point is that it is continuous at that point.** **(2014, 13, 05, 04, 01)**

Sol. Let $f : [a, b] \to R$ be derivable at any point $x = c$.

Then, $f'(c) = \lim_{x \to 0} \frac{f(x) - f(c)}{x - c}$, $x \neq c$ exists, because

$$f(x) = \left[\frac{f(x) - f(c)}{x - c}\right](x - c) + f(c)$$

We have,

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left[\frac{f(x) - f(c)}{x - c}\right] \cdot \lim_{x \to c}(x - c) + f(c)$$

$$= f'(c) \cdot 0 + f(c) = f(c)$$

and as such $f(x)$ is continuous at $x = c$

Thus, the derivability implies continuity.

In order to prove that the condition is not sufficient we given the counter example.

Let $f(x) = |x|, \forall x \in [-1, 1]$

Thus, $f(x)$ is continuous at $x = 0$, because for an arbitrary $\varepsilon > 0$, there exists $\delta (= \varepsilon)$ such that $|f(x) - f(0)| = |x| < \varepsilon$

$\therefore$ $|x - 0| < \delta = \varepsilon$

However, f is not derivable at $x=0$. To prove, we first note that

$$f(x)=\begin{cases}-x, & \text{if } -1\le x<0\\ x, & \text{if } 0\le x\le 1\end{cases}$$

Now, $$f'_{0^-}(x)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{-x-0}{x-0}=-1$$

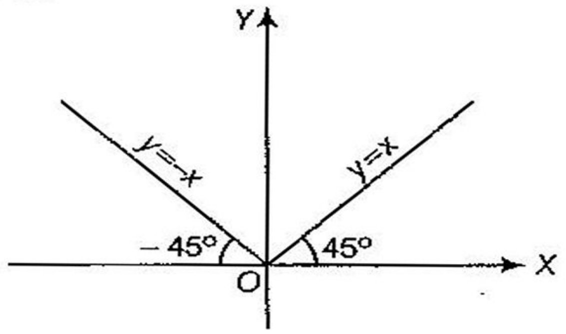
and $$f'_{0^+}(x)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x-0}{x-0}=1$$

$\because$ $f'_{0^-}(x)\neq f'_{0^+}(x)$

So, $f(x)$ is not derivable at $x=0$.

Hence, the continuity is not a sufficient condition for the derivability of function at that point.

The graph of function $f(x)=|x|$ is shown in the figure below for $x\in R$



Thus, the graph is an unbroken curve and therefore gradient of the curve $y=|x|$ at (0, 0) is not unique. **Hence proved.**

***Q 5.* Show that the function** $$f(x)=\begin{cases}\dfrac{x(e^{1/x}-e^{-1/x})}{(e^{1/x}+e^{-1/x})}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$$

is continuous at $x=0$, but not differentiable at $x=0$.

(2010, 02)

Sol. **Differentiability at $x=0$**

$$f'_{0^-}(x)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{(0-h)\dfrac{e^{\frac{1}{0-h}}-e^{-\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+e^{-\frac{1}{0-h}}}-0}{-h}$$

$$=\lim_{h\to 0}\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}=\lim_{h\to 0}\frac{e^{-2/h}-1}{e^{-2/h}+1}=\frac{0-1}{0+1}=-1 \qquad [\because e^{-\infty}=0]$$

$$f'_{0^+}(x)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h)\dfrac{e^{\frac{1}{0+h}}-e^{-\frac{1}{0+h}}}{e^{\frac{1}{0+h}}+e^{-\frac{1}{0+h}}}-0}{h}$$

$$= \lim_{h \to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1-0}{1+0} = 1$$

$\because \quad f'_{0^-}(x) \neq f'_{0^+}(x)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Continuity at $x = 0$

$$f_{0^-}(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{(0-h)\left(e^{1/(0-h)} - e^{-1/(0-h)}\right)}{e^{\frac{1}{0-h}} + e^{-\frac{1}{0-h}}}$$

$$= \lim_{h \to 0} \frac{-h(e^{-1/h} - e^{1/h})}{(e^{-1/h} + e^{1/h})} = \lim_{h \to 0} \frac{-h(e^{-2/h} - 1)}{(e^{-2/h} + 1)} = \frac{0(0-1)}{(0+1)} = 0$$

$$f_{0^+}(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{(0+h)\left(e^{1/(0+h)} - e^{-1/(0+h)}\right)}{\left(e^{1/(0+h)} - e^{-1/(0+h)}\right)}$$

$$= \lim_{h \to 0} \frac{h(e^{1/h} - e^{-1/h})}{(e^{1/h} + e^{-1/h})} = \lim_{h \to 0} \frac{h(1 - e^{-2/h})}{(1 + e^{-2/h})} = \frac{0(1-0)}{(1+0)} = 0$$

$\because \; f_{0^-}(x) = f(0) = f_{0^+}(x)$

Hence, $f(x)$ is continuous at $x = 0$.

***Q* 6.** **Define $f(x) = \begin{cases} 1 + x, \text{ if } 0 \le x < 1 \\ 3 - x, \text{ if } 1 \le x \le 2 \end{cases}$. Prove that f is continuous at $x = 1$ but not differentiable at $x = 1$.** **(2008)**

Sol. Here, $f(1) = 2$

Continuity at $x = 1$

$$f_{1^-}(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [1 + (1-h)] = 2$$

$$f_{1^+}(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [3 - (1+h)] = 2$$

$\because \quad f_{1^-}(x) = f(1) = f_{1^+}(x)$

So, $f(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$

$$f'_{1^-}(x) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1 + (1-h)) - 2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$f'_{1^+}(x) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(3 - (1+h)) - 2}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

$\because \quad f'_{1^-}(x) \neq f'_{1^+}(x)$

So, $f(x)$ is not differentiable at $x = 1$.

Q 7. State and prove Darboux's theorem. **(1998, 96)**

Sol. **Statement** If a function $f : [a, b] \to R$ is derivable in $[a, b]$ and $f'(a)$, $f'(b)$ have opposite signs, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof Let us suppose that $f'(a) > 0$ and $f'(b) < 0$

Now, $$f'(a) = f'(a + 0) > 0$$

$\Rightarrow f$ is increasing function on the right at 'a', such that

$$f(x) > f(a), \forall\, x \in (a, a + \delta_1)$$

Again, $$f'(b) = f'(b - 0) < 0$$

$\Rightarrow f$ is decreasing function on the left at 'b', such that

$$f(x) > f(b), \forall\, x \in (b - \delta_2, b)$$

Therefore, f is not supremum at $x = a$ or $x = b$.

Since, f is derivable in $[a, b]$, it is continuous in $[a, b]$. Therefore, it is bounded in $[a, b]$.

Let $M =$ supremum of f in $[a, b]$.

Since, a continuous function in $[a, b]$ attains its supremum for $c \in (a, b)$ such that $f(c) = M$.

We will prove that $f'(c) = 0$.

If $f'(c) > 0$, then f is increasing at c.

Hence, there exists a$\delta > 0$ such that $f(x) > f(c), \forall\, x \in (c, c + \delta)$

which is impossible as f has supremum at c.

If $f'(c) < 0$, then f is decreasing at c.

Hence, there exists a$\delta > 0$ such that $f(x) > f(c), \forall\, x \in (c - \delta, c)$.

Again, this is impossible as f has supremum at c.

$\therefore$ $$f'(c) = 0$$ **Hence proved.**

Q 8. State and prove Rolle's theorem. **(2013, 11, 09, 05, 01)**

Sol. **Statement** If a function $f : [a, b] \to R$ is

(i) continuous on $[a, b]$.

(ii) differentiable on (a, b)

(iii) $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof Since, $f(x)$ is continuous on $[a, b]$, it is bounded and attains its supremum and infimum at some points of $[a, b]$.

Let $M =$ supremum of in $[a, b]$

$m =$ infimum of in $[a, b]$

Now, either $M = m$ or $M \neq m$

If $M = m$, then $f(x)$ is constant on $[a, b]$.

Hence, $f'(x) = 0, \forall\ x \in [a, b]$

Thus, the theorem holds in this case.

In other case, if $M \neq m$, then at least one of M and m, if not both, must be different from the equal value $f(a)$ and $f(b)$.

Suppose $M \neq f(a) = f(b)$

Then, there exists $c \in [a, b]$ such that $f(c) = M$.

Also, $f'(c)$ exists because of condition (ii) holds.

Now, we will show that $f'(c) = 0$

If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) = M\ ,\ \forall\ x \in (c, c + \delta)$$

But $f(x) \leq M, \forall\ x \in [a, b]$

where, M being the supremum of f.

Thus, we arrived at a contradiction, so $f'(c) \not> 0$.

If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(x) > f(c) = M, \forall\ x \in (c - \delta, c)$

Again, this is not possible, so $f'(c) \not< 0$

Hence, $f'(c) = 0$. **Hence proved.**

Q 9. Verify Rolle's theorem in case of function $f(x) = e^x \sin x$ in $[0, \pi]$. **(2012)**

Sol. Given function $f(x) = e^x \sin x$ is obviously a continuous function as well as differentiable in the interval $[0, \pi]$.

Now, $f(0) = e^0 = \sin 0 = 0$

$$f(\pi) = e^{\pi} \sin \pi = 0$$

$\therefore$ $$f(0) = f(\pi)$$

Thus, all three conditions holds of Rolle's theorem. Therefore, there must be at least one value of x in the interval $(0, \pi)$ at which $f'(x) = 0$.

Now, $$f'(x) = 0 \Rightarrow e^x \sin x + e^x \cos x = 0$$

$\Rightarrow$ $$e^x(\sin x + \cos x) = 0$$

$\Rightarrow$ $$\sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 0$$

$\Rightarrow$ $$\sqrt{2}e^x \cos\left(x - \frac{\pi}{4}\right) = 0 \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 0$$

$\Rightarrow$ $$\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{2} \Rightarrow x - \frac{\pi}{4} = \frac{\pi}{2}$$

$\Rightarrow$ $$x = \frac{3\pi}{4}$$

$\because$ $$\frac{3\pi}{4} \in (0, \pi)$$

Hence, Rolle's theorem verified.

Q 10. Verify Rolle's theorem for the function
$$f(x) = (x+1)(x-2)(x-3), \forall\ x \in [2, 3].$$ **(2010)**

Sol. Given function $f(x)$ is a polynomial, so $f(x)$ is continuous and differentiable for all real value of x.

Also, $f(x) = 0 \Rightarrow x^3 - 6x^2 - 11x - 6 = 0$

$\Rightarrow$ $(x+1)(x-2)(x-3) = 0$

$\Rightarrow$ $x = -1, 2, 3$

Thus, $f(-1) = f(2) = f(3) = 0$

In the interval $[-1, 3]$, $f(x)$ holds the all three conditions of Rolle's theorem. Therefore, there must be at least one value of x in the interval $(-1, 3)$ for which $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3x^2 - 12x - 11 = 0$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{144 + 132}}{6} = 2 \pm \frac{\sqrt{69}}{6}$$ [by Sridharacharya formula]

$$\because \quad \left(2 - \frac{\sqrt{69}}{6}\right) \in (-1, 3)$$

Hence, $f(x)$ verified the Rolle's theorem.

Q 11. Verify Rolle's theorem for the following function.
$$f(x) = (x-2)^2 (x-4)^3, \forall\ x \in [2, 4]$$ **(2007)**

Sol. We have, $f(x) = (x-2)^2 (x-4)^3, \forall\ x \in [2, 4]$

The given function is a polynomial in x, so it is continuous and differentiable in [2, 4].

Also, $f(2) = 0 = f(4)$

Thus, all the three conditions of Rolle's theorem holds, therefore there must be at least one value of x in (2, 4) for which $f'(x) = 0$.

Now, $f'(x) = 2(x-2)(x-4)^3 + 3(x-2)^2(x-4)^2 = 0$

$\Rightarrow$ $(x-2)(x-4)^2 \cdot [2(x-4) + 3(x-2)] = 0$

$\Rightarrow$ $(x-2)(x-4)^2 (5x-14) = 0$

$\Rightarrow$ $x = 2, 4, 2\frac{4}{5}$

Since, $2 < 2\frac{4}{5} < 4$ or $2\frac{4}{5} \in (2, 4)$

Hence, Rolle's theorem satisfied.

Q 12. Discuss the applicability of the Rolle's theorem on the following function.
$$f(x) = |x-1|, \forall\ x = (0, 2)$$

Sol. Given function $f(x) = |x-1|, \forall\ x \in (0, 2)$.

Then, $f(0) = 1$ and $f(2) = 1$

Given function can be written as

$$f(x) = \begin{cases} -(x-1), & \text{if } x < 1 \\ (x-1), & \text{if } x \geq 1 \end{cases}$$

Differentiability at $x = 1$

$$f'_{1^-}(x) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{-((1-h)-1)-1}{-h} = \lim_{h \to 0} \frac{h-1}{-h}$$

$$= \lim_{h \to 0} \frac{1-h}{h} = \lim_{h \to 0}\left(\frac{1}{h} - 1\right) = \infty - 1 = +\infty$$

$$f'_{1^+}(x) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)-1-1}{h} = \lim_{h \to 0} \frac{h-1}{h}$$

$$= \lim_{h \to 0}\left(1 - \frac{1}{h}\right) = 1 - \infty = -\infty$$

$\because \quad f'_{1^-}(x) \neq f'_{1^+}(x)$

So, $f(x)$ is not differentiable at $x = 1 \in (0, 2)$.

Hence, Rolle's theorem is not applicable.

Q 13. State and prove Lagrange's mean value theorem and give the geometrical interpretation. **(2014, 12, 10, 07, 1996)**

Sol. **Statement** Let the function $f : [a, b] \to R$ be

(i) continuous on $[a, b]$.

(ii) derivable on (a, b), then there exists at least one point

$c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Proof Let us consider an auxiliary function $h : [a, b] \to R$ defined by

$$h(x) = f(x) + Ax$$

where, A is to be determined such that $h(a) = h(b)$,

i.e. $\quad f(a) + Aa = f(b) + Ab \Rightarrow -A = \dfrac{f(b) - f(a)}{b - a}$

Now, the function $h(x)$ is the sum of two continuous and differentiable functions, therefore it is also

(i) continuous on $[a, b]$.

(ii) differentiable on (a, b).

(iii) $h(a) = h(b)$.

Thus, $h(x)$ satisfies the conditions of Rolle's theorem.

Therefore, there exists a point $c \in (a, b)$ such that

$$h'(c) = 0 \Rightarrow f'(c) + A = 0$$

Because $\quad h'(x) = f'(x) + A$

Hence, $\quad f'(c) = \dfrac{f(b) - f(a)}{b - a}.$ **Hence proved.**

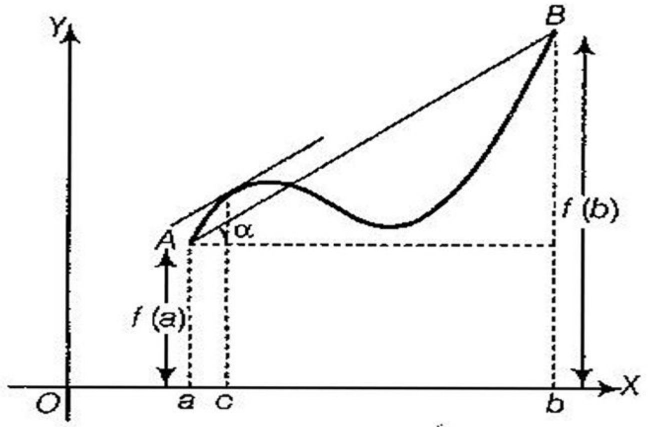
Geometrical Interpretation The graph of this function $f : [a, b] \to R$, from the point $A : (a, f(a))$ to the point $B : (b, f(b))$ has neither breaks nor sharp corners anywhere between A and B.

If the chord AB makes an angle α with the X-axis, then

$$\tan \alpha = \frac{f(b) - f(a)}{b - a} = f'(c)$$

[by Lagrange's mean value theorem]

where, $a < c < b$



Thus, according to the Lagrange's mean value theorem, the graph of the function f has a tangent parallel to the chord joining the points on the graph with abscissae a and b, for at least on point c in (a, b).

Q 14. Verify Lagrange's mean value theorem for the function $f(x) = (x + 1)(x - 2)(x - 3), \forall\ x \in [0, 1]$. **(2014, 11, 09, 06, 01)**

Sol. Given function $f(x)$ is a polynomial in x, hence it is

(i) continuous in $[0, 1]$.

(ii) differentiable in $(0, 1)$.

Therefore, Lagrange's mean value theorem is applicable.

Now, $f(0) = 6$ and $f(1) = 4$

$$\therefore \quad \frac{f(1) - f(0)}{1 - 0} = \frac{4 - 6}{1 - 0} = -2$$

Form any $x \in (0, 1)$, we have

$$f'(x) = (x - 2)(x - 3) + (x + 1)(x - 3) + (x + 1)(x - 2)$$

$$= 3x^2 - 8x + 1$$

$$\therefore \quad \frac{f(1) - f(0)}{1 - 0} = f'(c), \forall\ c \in (0, 1)$$

$$\Rightarrow \quad -2 = 3c^2 - 8c + 1$$

$$\Rightarrow \qquad 3c^2 - 8c + 3 = 0$$

$$\Rightarrow \qquad c = \frac{+8 \pm \sqrt{64-36}}{2 \cdot 3} = \frac{4 \pm \sqrt{7}}{3}$$

$\therefore\ c = 4 - \dfrac{\sqrt{7}}{3} \in (0, 1)$ such that $f'(c) = 0$

Hence, Lagrange's mean value theorem verified.

Q 15. Verify Lagrange's theorem for the function
$f(x) = (x-1)(x-2), \forall\ 0 \le x \le 1.$ **(2013)**

Sol. Given function $f(x)$ is a polynomial in x, hence it is

(i) continuous in [0, 1].

(ii) differentiable in (0, 1).

Therefore, Lagrange's mean value theorem is applicable.

Now, $f(0) = 2$ and $f(1) = 0$

$$\therefore \qquad \frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0} = \frac{0-2}{1-0} = -2$$

For any $x \in (0, 1)$, we have $f'(x) = 2x - 3$

$$\therefore \qquad \frac{f(1)-f(0)}{1-0} = f'(c), \forall\ c \in (0, 1)$$

$$\Rightarrow \qquad -2 = 2c - 3 \Rightarrow c = \frac{1}{2}$$

So, $c = \dfrac{1}{2} \in (0, 1)$ such that $f'(c) = 0$.

Hence, Lagrange's mean value theorem verified for the function.

Q 16. Verify Lagrange's mean value theorem for the function
$f(x) = x(x-1)(x-2), \forall\ x \in \left[0, \frac{1}{2}\right].$ **(2005)**

Sol. Given function is a polynomial in x, so it is continuous in [0,1/2] and differentiable in (0, 1/2). Therefore, $f(x)$ holds all the conditions of Lagrange's mean value theorem.

Now, $\quad f(0) = 0$ and $f\left(\dfrac{1}{2}\right) = \dfrac{3}{8}$

$$\therefore \qquad \frac{f(b)-f(a)}{b-a} = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{4}$$

Now, $\qquad f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$

$$\Rightarrow \qquad f'(x) = 3x^2 - 6x + 2$$

$$\therefore \qquad f'(c) = 3c^2 - 6c + 2, \forall\, c \in \left(0, \frac{1}{2}\right)$$

Now, according to Lagrange's mean value theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \qquad \frac{3}{4} = 3c^2 - 6c + 2 \Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow \qquad c = 1 \pm \frac{1}{24}\sqrt{336}$$

$$\because 1 - \frac{1}{24}\sqrt{336} \in \left(0, \frac{1}{2}\right) \text{ such that } f\left(1 - \frac{1}{24}\sqrt{336}\right) = 0$$

Hence, Lagrange's mean value theorem verified.

Long Answer Questions

***Q 1.* Prove that a necessary but not sufficient condition that a function be derivable at a point is that it is continuous at that point, also discuss the continuity and differentiability of the function $f(x) = |x - 1| + |x - 2|$, $\forall\, x \in R$ at the point $x = 1$ and $x = 2$. Draw its graph in the interval [0, 3].** **(2016, 09, 06, 01)**

Sol. **Part I** See the solution of Q. 4 of Short Answer Questions.

Part II Given function is $f(x) = |x - 1| + |x - 2|, \forall\, x \in R$

It can be written as $f(x) = \begin{cases} -2x + 3, & \text{if } x < 1 \\ 1, & \text{if } 1 \le x < 2 \\ 2x - 3, & \text{if } x \ge 2 \end{cases}$

Continuity at $x = 1$

$$f_{1^-}(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} [-2(1 - h) + 3] = \lim_{h \to 0} (2h + 1) = 1$$

$$f_{1^+}(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} [1] = 1$$

$$\because \qquad f_{1^-}(x) = f(1) = f_{1^+}(x)$$

So, $f(x)$ is continuous at $x = 1$.

Continuity at $x = 2$

$$f_{2^-}(x) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} (1) = 1$$

$$f_{2^+}(x) = \lim_{h \to 0} (2 + h) = \lim_{h \to 0} [2(2 + h) - 3] = \lim_{h \to 0} (2h + 1) = 1$$

$$\because \qquad f_{2^-}(x) = f(2) = f_{2^+}(x)$$

So, $f(x)$ is continuous at $x = 2$.

Differentiability at $x = 1$

$$f'_{1^-}(x) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{-2(1-h)+3-1}{-h} = \lim_{h \to 0} \frac{2h}{-h} = -2$$

$$f'_{1^+}(x) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\because \quad f'_{1^-}(x) \neq f'_{1^+}(x)$$

So, $f(x)$ is not differentiable at $x = 1$.

Differentiability at $x = 2$

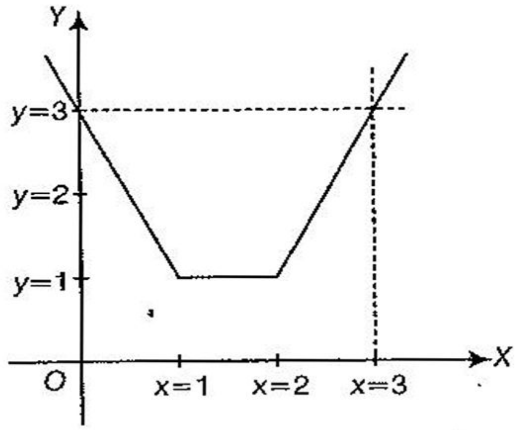
$$f'_{2^-}(x) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f'_{2^+}(x) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{2(2+h)-3-1}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$$

$$\therefore \quad f'_{2^-}(x) \neq f'_{2^+}(x)$$

So, $f(x)$ is not differentiable at $x = 2$.

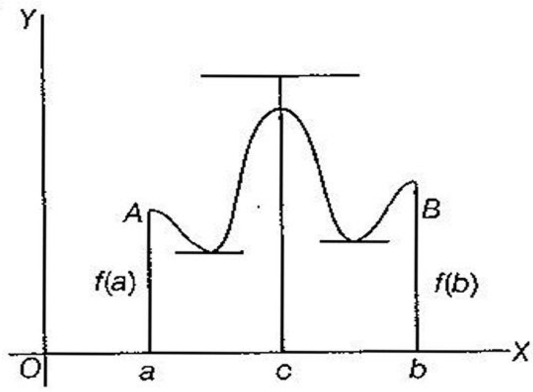
Part III Geometrical Interpretation The graph in the interval [0, 3] is



Q 2. State and prove Rolle's mean value theorem. Give its geometrical interpretation, also verify it for the function $f(x) = x(x+3)e^{-x/2}$ in [− 3, 0]. **(2015, 07, 04)**

Sol. **Part I** See the solution of Q. 8 of Short Answer Questions.

Part II Geometrical Interpretation of Rolle's Theorem Geometrically Rolle's theorem says that the curve representing the graph of the function f, under the slated conditions, has a tangent parallel to X-axis for at least one point between a and b, as shown in the figure. The given function is



$$f(x) = x(x+3)e^{-x/2} = (x^2+3x)e^{-x/2}$$

Then,

$$f'(x) = -\frac{1}{2}(x^2+3x)e^{-x/2} + (2x+3)e^{-x/2} = -\frac{1}{2}e^{-x/2}(x^2-x-6)$$

Here, $f'(x)$ exists for every value of $x \in [-3, 0]$. Thus, $f(x)$ is differentiable in $(-3, 0)$ and therefore it is also continuous in $[-3, 0]$.

Also, $$f(-3) = 0 = f(0).$$

Thus, all three conditions of Rolle's theorem holds, so $f'(x) = 0$ at least one value of $x \in (-3, 0)$.

Now, for $f'(x) = 0$, $-\frac{1}{2}e^{-x/2}(x^2 - x - 6) = 0 \Rightarrow x^2 - x - 6 = 0$ $\left[\because -\frac{1}{2}e^{-x/2} \neq 0\right]$

$$\Rightarrow \quad (x-3)(x+2) = 0 \Rightarrow x = -2, 3$$

Since, $-3 < -2 < 0$, therefore Rolle's theorem is verified.

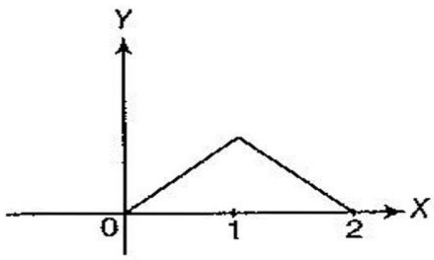
Q 3. State and prove Rolle's theorem. Give its geometrical interpretation, also explain why $f(x) = 1 - |x - 1|$ does not satisfy Rolle's theorem in [0, 2]. **(2017)**

Sol. **Part I** See the solution of Q. 2.

Part II Given function is

$$f(x) = 1 - |x - 1| = \begin{cases} x, & \text{if } x < 1 \\ 2 - x, & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 1, & x < 1 \\ -1, & x \geq 1 \end{cases}$$



The given function is not differentiable at $x = 1$, so $f(x)$ is not satisfies the condition of Rolle's theorem.

Q 4. State and prove Cauchy's mean value theorem and show that Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem. **(2006, 02)**

Sol. **Statement** Let the function $f : (a, b) \to R$ and $g : (a, b) \to R$ both be

(i) continuous in the closed interval $[a, b]$.

(ii) differentiable in the open interval (a, b).

(iii) $g'(x) \neq 0, \forall x \in [a, b]$, then there exists $c \in [a, b]$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof Let $g(b) = g(a)$, then g satisfies all the conditions of Rolle's theorem and hence there exists a point $\lambda \in [a, b]$ such that $g'(\lambda) = 0$, which is a contradiction.

Hence, $g(b) \neq g(a)$

Now, let us consider an auxiliary equation

$$h(x) = f(x) + Ag(x), \forall x \in [a, b]$$

where, A is a constant to be determine such that $h(a) = h(b)$.

Thus, $f(a) + Ag(a) = f(b) + Ag(b) \Rightarrow A = -\dfrac{f(b) - f(a)}{g(b) - g(a)}$ ...(i)

Since, $h(x)$ is the function of $f(x)$ and $g(x)$, then $h(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and also $h(a) = h(b)$.

So, $h(x)$ holds all three conditions of Rolle's theorem, so there exists a point $c \in (a, b)$ such that $h'(c) = 0$.

$$\therefore \quad f'(c) + Ag'(c) = 0 \Rightarrow A = -\frac{f'(c)}{g'(c)} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

Hence, Cauchy's theorem proved.

If we put $g(x) = x$, then $h(x) = f(x) + Ax$

Then, from the condition $h(a) = h(b)$

$$A = -\frac{f(b) - f(a)}{b - a} \quad \text{...(iii)}$$

and $\quad h'(x) = f'(x) + A.$

Then, $\quad h'(c) = f'(c) + A, \forall c \in (a, b)$

$\Rightarrow \quad h'(c) = 0 \Rightarrow f'(c) + A = 0$

$\Rightarrow \quad A = -f'(c) \quad \text{...(iv)}$

From Eqs. (iii) and (iv), we get $\dfrac{f(b) - f(a)}{b - a} = f'(c), \forall c \in (a, b)$

which prove the Lagrange's mean value theorem.

Hence, Lagrange's mean value theorem is the particular case of Cauchy's mean value theorem. **Hence proved.**