Chapter Five

INDETERMINATE FORMS

Important Points from the Chapter

1. **Indeterminate Forms** The forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0$ and 1^∞ are called indeterminate forms and it has no definite value.
2. **L'Hospital's Rule** If $\phi(x)$ and $\psi(x)$ are two functions of x, which can be expanded by Taylor's theorem in the neighbourhood of $x=a$ and $\phi(a)=\psi(a)=0$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}=\lim_{x\to a}\frac{\phi^n(x)}{\psi^n(x)}$$

[if $\phi'(a), \ldots, \phi^{(n-1)}(a)$ and $\psi'(a), \ldots, \psi^{(n-1)}(a)$ are all zero, but $\phi^{(n)}(a)\neq 0$ and $\psi^{(n)}(a)\neq 0$]

which is known as L' Hospital's rule.

In this while applying L' Hospital's rule, we are not to differentiate $\frac{\phi(x)}{\psi(x)}$ by quotient rule of two functions, but we are to differentiate the numerator and denominator separately.

3. **Form** $\left(\frac{\infty}{\infty}\right)$ Suppose $\lim_{x\to a}\phi(x)=\infty$ and $\lim_{x\to a}\psi(x)=\infty$. Then,

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

4. **Form** $(0\times\infty)$ Suppose $\lim_{x\to a}\phi(x)=0$ and $\lim_{x\to a}\psi(x)=\infty$. Then,

$\lim_{x\to a}\phi(x)\cdot\psi(x)$ can be reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking as

$\lim_{x\to a}\frac{\phi(x)}{1/\psi(x)}$ or $\lim_{x\to a}\frac{\psi(x)}{1/\phi(x)}$.

5. **Forms** $(1^\infty, 0^0, \infty^0)$ Suppose $\lim_{x\to a}\{\phi(x)\}^{\psi(x)}$. takes any one of these three forms. Then, let $y=\lim_{x\to a}\{\phi(x)\}^{\psi(x)}$. Taking logarithm on both sides, we get $\log y=\lim_{x\to a}\psi(x)\cdot\log\phi(x)$.

Now, in any of the above three cases, $\log y$ takes the form $0\times\infty$, which can be evaluated by the process of above point 4.

Algebraic Formulae In many cases, the limits are easily obtained by the use of algebraic and trigonometrical expansions, which are as follows

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \infty,\ |x| < 1.$

(ii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty,\ |x| < 1$

(iii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \infty$

(iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \infty$

(v) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots \infty$

(vi) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \infty$

(vii) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \infty$

(viii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$

(ix) $\sin^{-1} x = x + 1^2 \cdot \frac{x^3}{3!} + 3^2 \cdot 1^2 \cdot \frac{x^5}{5!} + 5^2 \cdot 3^2 \cdot 1^2 \cdot \frac{x^7}{7!} + \ldots \infty$

The following values of logarithms to the base e should also be remembered

$$\log 1 = 0,\ \log e = 1,\ \log \infty = \infty,\ \log 0 = -\infty,\ e^{\infty} = \infty, e^{-\infty} = 0, e^0 = 1$$

Very Short Answer Questions

Q 1. Prove that $\lim_{x \to 0} \frac{\tan x}{x} = 1$. **(2017)**

Sol. We have, $\lim_{x \to 0} \frac{\tan x}{x}$ $\left[\text{form } \frac{0}{0}\right]$

On applying L'Hospital's rule, we get

$$\lim_{x \to 0} \frac{\sec^2 x}{1} = \sec^2 0 = 1$$

Q 2. Evaluate $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^x$. **(2010)**

Sol. Let $y = \lim_{x \to 0} \left(1 + \frac{1}{x}\right)^x$ [form ∞^0]

$$\log y = \lim_{x\to 0} x \log\left(1+\frac{1}{x}\right) = \lim_{x\to 0} \frac{\log\left(1+\frac{1}{x}\right)}{1/x} \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$\lim_{x\to 0} = \frac{\left(\frac{1}{1+1/x}\right)(-1/x^2)}{(-1/x^2)} \Rightarrow \log y = \lim_{x\to 0}\left(\frac{1}{1+\frac{1}{x}}\right) = 0$$

$$\Rightarrow \qquad y = e^0 = 1$$

Q 3. **Evaluate** $\lim\limits_{x\to 0} \dfrac{\log x^2}{\cot x^2}$. **(2009)**

Sol. We have, $\lim\limits_{x\to 0} \dfrac{\log x^2}{\cot x^2}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{x^2}\right)(2x)}{-2x \operatorname{cosec}^2 x^2} \qquad \text{[using L' Hospital's rule]}$$

$$= \lim_{x\to 0}\left(\frac{-\sin^2 x^2}{x^2}\right) = \lim_{x\to 0} -\frac{2\sin x^2 \cos x^2 \cdot 2x}{2x}$$

$$= \lim_{x\to 0} -(2\sin x^2 \cos x^2) = 0$$

Q 4. **Evaluate** $\lim\limits_{x\to 0}\left(\dfrac{\tan x - x}{x - \sin x}\right)$. **(2006)**

Sol. We have, $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} \qquad \text{[using L' Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{\sin x} = \lim_{x\to 0} 2\sec^3 x = 2$$

Q 5. **Evaluate** $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$. **(2005)**

Sol. Let $y = \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$

Taking log on both sides, we get

$$\log y = \lim_{x\to 0} \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0} \frac{1}{x^2} \log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots}\right)\left(\frac{2x}{3}+\frac{8x^3}{15}+\ldots\right)}{2x} = \lim_{x \to 0} \frac{\frac{2}{3}+\frac{8x^2}{15}+\ldots}{2\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots\right)} = \frac{1}{3}$$

$\therefore\ y = e^{1/3}$

Q 6. **Evaluate** $\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2}$.

Sol. We have, $\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2} = \lim_{x \to 0} \frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x}$

[using L' Hospital's rule]

$$= \lim_{x \to 0} \frac{x\cos x + \sin x - \sin x + \frac{1}{(1+x)^2}}{2}$$ [using L' Hospital's rule]

$$= \frac{1}{2}$$

Short Answer Questions

Q 1. **Evaluate** $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

Sol. We have $\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \lim_{x \to 0}\left(\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right)\left(\frac{x^2}{x^2}\right)$

$$= \lim_{x \to 0}\left(\frac{\sin^2 x - x^2}{x^4}\right)\left(\frac{x}{\sin x}\right)^2 = \left(\lim_{x \to 0}\frac{\sin^2 x - x^2}{x^4}\right)\left[\lim_{x \to 0}\left(\frac{x}{\sin x}\right)^2\right]$$

$$= \lim_{x \to 0}\left(\frac{\sin^2 x - x^2}{x^4}\right)$$ $\left[\because \lim_{x \to 0}\frac{x}{\sin x} = 1\right]$

$$= \lim_{x \to 0}\frac{2\sin x\cos x - 2x}{4x^3}$$ [using L' Hospital's rule]

$$= \lim_{x \to 0}\frac{\sin 2x - 2x}{4x^3} = \lim_{x \to 0}\frac{2\cos 2x - 2}{12x^2}$$ [using L'Hospital's rule]

$$= \lim_{x \to 0}\frac{-4\sin 2x}{24x}$$ [using L' Hospital's rule]

$$= \lim_{x \to 0}\frac{-8\cos 2x}{24}$$ [using L' Hospital's rule]

$$= -\frac{1}{3}$$

Q 2. **Evaluate** $\lim_{x \to 0} \log_{\tan^2 x}(\tan^2 2x)$.

Sol. $\lim_{x \to 0} \log_{\tan^2 x}(\tan^2 2x) = \lim_{x \to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$ $\left[\because \log_b a = \frac{\log a}{\log b}\right]$

$$= \lim_{x \to 0} \frac{2\log \tan 2x}{2\log \tan x} \quad [\because \log a^b = b \log a]$$

$$= \lim_{x \to 0} \frac{\log \tan 2x}{\log \tan x} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan 2x} \cdot 2\sec^2 2x}{\frac{1}{\tan x} \cdot \sec^2 x} = \lim_{x \to 0} \frac{\sin 2x}{\sin 2x \cos 2x}$$

$$= \lim_{x \to 0} \frac{1}{\cos 2x} = 1$$

Q 3. **Evaluate** $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$.

Sol. We have, $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$ $\left[\text{form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{(2/\sin 2x) \cdot \cos 2x}{(1/\sin x) \cdot \cos x} \quad [\text{using L' Hospital's rule}]$$

$$= \lim_{x \to 0} \frac{2\cot 2x}{\cot x} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{-4\,\text{cosec}^2 2x}{-\text{cosec}^2 x} \quad \left[\text{form } \frac{\infty}{\infty}\right].$$

$$= \lim_{x \to 0} \frac{4\sin^2 x}{\sin^2 2x} = \lim_{x \to 0} \frac{4\sin^2 x}{(2\sin x \cos x)^2} = \lim_{x \to 0} \frac{1}{\cos^2 x} = 1$$

Long Answer Questions

Q 1. **Evaluate** $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$.

Sol. Here, $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$ is of the form $\frac{0}{0}$, because $\lim_{x \to 0} (1+x)^{1/x} = e$.

To evaluate the given limit, first we will obtain an expansion for $(1+x)^{1/x}$ in ascending powers of x.

Let $\quad y = (1+x)^{1/x}$.

Then, $\log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$

$$= 1 + z, \text{ where } z = -\frac{x}{2} + \frac{x^2}{3} \ldots$$

$$\therefore \quad y = e^{1+z} = e \cdot e^z = e \cdot \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)^2 + \ldots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{1}{8}x^2 + \text{Terms containing power of } x \text{ higher than } 3\right]$$

$$= e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \ldots\right)$$

Now, $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0} \frac{e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \ldots\right) - e}{x}$

$$= \lim_{x\to 0} \frac{e\left(-\frac{1}{2}x + \frac{11}{24}x^2 - \ldots\right)}{x} = \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{11}{24}x - \ldots\right) = -\frac{1}{2}e$$

Q 2. Evaluate $\lim_{x\to 0} \frac{\log\log(1-x^2)}{\log\log\cos x}$.

Sol. We have, $\lim_{x\to 0} \frac{\log\log(1-x^2)}{\log\log\cos x}$ $\quad\left[\text{form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\frac{1}{\log(1-x^2)} \cdot \frac{1}{1-x^2} \cdot (-2x)}{\frac{1}{\log\cos x} \cdot \frac{1}{\cos x} \cdot (-\sin x)}$$

$$= 2 \lim_{x\to 0} \frac{x\cos x \log\cos x}{\sin x(1-x^2)\log(1-x^2)}$$

$$= 2 \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{\cos x}{1-x^2} \cdot \lim_{x\to 0} \frac{\log\cos x}{\log(1-x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x\to 0} \frac{\log\cos x}{\log(1-x^2)} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= 2 \lim_{x\to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\frac{1}{1-x^2} \cdot (-2x)} = 2 \times \frac{1}{2} \cdot \lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \frac{1-x^2}{\cos x}\right)$$

$$= 1$$