Chapter Five

THEOREMS BASED ON RESIDUES AND ARGUMENT PRINCIPLE

Important Points from the Chapter

1. If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles, and has no zero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$$

where, N is the number of zeroes and P the number of poles of $f(z)$ inside C.
A pole or zero of order n must be counted n times. **(2009, 1993, 91)**

2. If $g(z)$ is analytic function, regular inside and on a simple closed contour C and if $f(z)$ is also analytic inside and on C, (except for a finite number of poles) having zeroes at $z_1, z_2, \dots, z_m$ and poles at $p_1, p_2, \dots, p_n$, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)\,g(z)}{f(z)}\,dz = \sum_{r=1}^{m} g(z_r) - \sum_{s=1}^{n} g(p_s).$$

(2011, 04, 02, 1999, 95, 94, 92, 90)

3. **Rouche's Theorem** Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside C.
(2014, 09, 03, 1999, 97, 92)

4. **Fundamental Theorems of Algebra** Every polynomial of degree n has exactly n zeroes. **(2013, 11, 08, 03, 01, 2000, 1998, 96, 93)**

5. **Argument Principle** If $f(z)$ is meromorphic inside a closed contour C and has no zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$, where N is the number of zeroes and P the number of poles inside C, (a pole or zero of order m must be counted m times)

- **Note** $N - P = \frac{1}{2\pi}\Delta_C \arg f(z)$, where Δ_C denotes the variation in $\arg f(z)$ as z moves once round C.

Very Short Answer Questions

Q 1. Using Rouche's theorem, determine the number of zeroes of the polynomial $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ in $|z| < 1$.

Sol. Here, $P(z) = z^{10} - 6z^7 + 3z^3 + 1$

Let $f(z) = -6z^7$, $g(z) = z^{10} + 3z^3 + 1$

Then, $\quad P(z) = f(z) + g(z)$

Consider the circle C defined by $|z| = 1$.

Then, $f(z)$ and $g(z)$ both are analytic within and upon C and

$$\left|\frac{g}{f}\right| = \left|\frac{z^{10} + 3z^3 + 1}{-6z^7}\right|$$

$$\leq \frac{|z|^{10} + 3|z|^3 + 1}{6|z|^7}$$

$$= \frac{1^{10} + 3(1)^3 + 1}{6(1)^7} = \frac{5}{6} < 1$$

$\Rightarrow$
$$\left|\frac{g}{f}\right| < 1 \text{ or } |g| < |f|$$

On applying Rouche's theorem, we get $f(z) + g(z) = P(z)$ has the same number of zeroes inside C as $f(z) = -6z^7$.

But $f(z)$ has seven zeroes inside C.

Hence, $P(z)$ has seven zeroes inside C.

Q 2. Apply Rouche's theorem to determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$, that lie inside the circle $|z| = 1$.

Sol. Consider the circle C defined by $|z| = 1$.

Take $f(z) = -4z^5$, $g(z) = z^8 + z^2 - 1$

Then,
$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^8 + z^2 - 1}{-4z^5}\right| \leq \frac{|z|^8 + |z|^2 + 1}{4|z|^5} = \frac{1^8 + 1^2 + 1}{4 \cdot 1^5} = \frac{3}{4} < 1$$

or
$$\left|\frac{g}{f}\right| < 1 \text{ or } |g| < |f|$$

Now, f and g are analytic functions within and upon the contour C such that $|g| < |f|$.

On applying Rouche's theorem, we find $f + g = z^8 - 4z^5 + z^2 - 1$ has the same number of zeroes inside C as $f(z)$, but $f(z) = -4z^5$ has five zeroes all located at the origin. It follows that $f + g$ has 5 zeroes inside C.

Hence, the equation has 5 roots inside $|z| = 1$.

Q 3. For which value of the real number of 'a', the function $z^n e^a - e^z$ will have n zeroes inside $|z| = 1$?

(2008, 05, 04, 01, 2000, 1997, 95, 94, 92, 90)

Sol. We have to find zeroes of

$$z^n e^a - e^z = 0 \quad \text{...(i)}$$

Let us take $f(z) = z^n e^a$ and $g(z) = -e^z$

On the unit circle, $|z| = 1$, we have

$$|f(z)| = |z^n e^a| \leq e^a \text{ and } |g(z)| = |-e^z| = e$$

Now, $f(z)$ and $f(z) + g(z)$ will have of same number of zeroes inside $|z| = 1$, provided $|g(z)| < |f(z)|$ and thus $e < e^a$, i.e. $a > 1$.

Thus, if $a > 1$, then $f(z)$ and $f(z) + g(z)$ have same number of zeroes inside $|z| = 1$. But $f(z) = z^n e^a$ has n zeroes inside $|z| = 1$.

Hence, $f(z) + g(z)$, i.e. $z^n e^a - e^z$ has n zeroes inside the circle $|z| = 1$ provided $a > 1$.

Short Answer Questions

Q 1. If $g(z)$ is analytic function, regular inside and on a simple closed contour C and if $f(z)$ is also analytic inside and on C, (except for a finite number of poles) having zeroes at $z_1, z_2, \ldots, z_m$ and poles at $p_1, p_2, \ldots, p_n$, then prove that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)\, g(z)}{f(z)} dz = \sum_{r=1}^{m} g(z_r) - \sum_{s=1}^{n} g(p_s).$$

(2011, 04, 02, 1999, 95, 94, 92, 90)

Sol. Let $z = z_r$ be a simple zero of $f(z)$, so that the function $f(z)$ can be written as $f(z) = (z - z_r)\, \phi(z)$, where $\phi(z)$ being analytic inside and on C, in the neighbourhood of $z = z_r$, where $\phi(z_r) \neq 0$.

Taking logarithmic differentiation, we get

$$\frac{f'(z)}{f(z)} = \left\{\frac{1}{(z - z_r)}\right\} + \left\{\frac{\phi'(z)}{\phi(z)}\right\} \quad [\because \log f(z) = \log (z - z_r) + \log \phi(z)]$$

where, $\phi'(z)$ is analytic at $z = z_r$ thus

$$\left\{\frac{f'(z)}{f(z)}\right\} g(z) = \left\{\frac{g(z)}{(z - z_r)}\right\} + \left\{\frac{g(z)\, \phi'(z)}{\phi(z)}\right\}$$

Since, $g(z)\, \phi'(z) / \phi(z)$ is analytic and regular at $z = z_r$ and $\{f'(z) / f(z)\}\, g(z)$ has a simple pole at $z = z_r$ with residue $g(z_r)$, taking into account all the zeroes of $f(z)$ inside C, we have

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} g(z)\, dz = \sum_{r=1}^{m} g(z_r)$$

Similarly, if $z = p_s$ is a simple pole of $f(z)$, then we can write
$f(z) = \phi(z)/(z - p_s)$, $\phi(z)$ has no poles at $z = p_s$, which gives

$$\frac{f'(z)}{f(z)} = -\left\{\frac{1}{z - p_s}\right\} + \left\{\frac{\phi'(z)}{\phi(z)}\right\} \qquad [\because \log f(z) = \log(z - p_s) + \log \phi(z)]$$

$$\therefore \qquad \left\{\frac{f'(z)}{f(z)}\right\} g(z) = -\left\{\frac{g(z)}{z - p_s}\right\} + \left\{\frac{\phi'(z)\, g(z)}{\phi(z)}\right\},$$

where, $\dfrac{\phi'(z)\, g(z)}{\phi(z)}$ is analytic and regular at $z = p_s$.

Thus, $\left\{\dfrac{f'(z)}{f(z)}\right\} g(z)$ has a simple pole at $z = p_s$ with residue $-g(p_s)$.

Taking into account all the poles of $f(z)$, we have

$$\frac{1}{2\pi i}\int_C \left\{\frac{f'(z)}{f(z)}\right\} g(z)\, dz = -\sum_{s=1}^{n} g(p_s)$$

Now, combining both the results, we get

$$\frac{1}{2\pi i}\int_C \left(\frac{f'(z)}{f(z)}\right) g(z)\, dz = \sum_{r=1}^{m} g(z_r) - \sum_{s=1}^{n} g(p_s) \qquad \textbf{Hence proved.}$$

Q 2. **State and prove Rouche's theorem on zeroes of an analytic function.** **(1999)**

Or **State and prove Rouche's theorem.** **(2014, 09)**

Or **If $f(z)$ and $g(z)$ are analytic within and on a simple closed contour C, and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ have same number of zeroes inside C.** **(2010, 07, 03, 1997, 92)**

Sol. **Statement Rouche's Theorem** Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside C.

Proof Since, $|g(z)| < |f(z)|$, therefore

$$\left|\frac{g(z)}{f(z)}\right| < 1 \text{ on } C, \text{ where } |f(z)| \neq 0.$$

Otherwise, $\left|\dfrac{g(z)}{f(z)}\right|$ will be infinity and not less than 1.

Also, $|f(z) + g(z)| = |f(z) - \{-g(z)\}| > |f(z)| - |g(z)| \neq 0 \qquad [\because |g(z)| < |f(z)|]$

$\therefore$ $f(z) + g(z) \neq 0$ and hence neither $f(z)$ nor $\{f(z) + g(z)\}$ has zero on C.

Let $F(z) = \dfrac{g(z)}{f(z)} \Rightarrow |F(z)| = \left|\dfrac{g(z)}{f(z)}\right| < 1$ on C

$\Rightarrow \qquad |F(z)| < 1$ on C

If immediately follows that $g(z)$ and $f(z)$ are not zero on C, then we have

$$F(z) = \frac{g(z)}{f(z)} \text{ on } C, \text{ i.e. } g = fF$$

$$\therefore \qquad g' = f'F + fF'$$

Let N_1 and N_2 be the number of zeroes of $f(z)$ and $f(z) + g(z)$ respectively inside C, as these functions have no poles inside C.

By using the formula,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = N - P$$

We have, $N_1 = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz$ and $N_2 = \frac{1}{2\pi i}\int_C \frac{f' + g'}{f + g}\, dz$

$$\therefore \quad N_2 - N_1 = \frac{1}{2\pi i}\int_C \frac{f' + g'}{f + g}\, dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f' + f'F + fF'}{f + fF}\, dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f'(1 + F) + fF'}{f(1 + F)}\, dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz + \frac{1}{2\pi i}\int_C \frac{F'}{1 + F}\, dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{F'}{1 + F}\, dz$$

Again, $1 + F \neq 0$ on C, because $|F(z)| < 1$ and $F'(z)$ is analytic on C, since F is analytic on C. [$\because$ derivative on an analytic function is analytic].

Thus $\frac{F'}{1 + F}$ is analytic on C, hence by Cauchy's theorem, we have

$$\int_C \frac{F'}{1 + F}\, dz = 0$$

$$\therefore \qquad N_2 - N_1 = 0 \Rightarrow N_1 = N_2$$

which proves the theorem.

***Q* 3.** **State and prove Fundamental theorem of algebra.** **(2008)**

Or **Prove that every polynomial of degree *n* has exactly *n* zeroes.**
(2017, 13, 11, 06, 03, 01, 2000, 1998, 96, 93, 91)

Sol. **Statement** Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$, where $a_n \neq 0$ be a polynomial equation of degree n.

Proof We will prove this by contradiction method. We suppose that the result is not true, i.e. $P(z)$ has no zero or $P(z) = 0$ has no root.

Let $$f(z) = \frac{1}{P(z)}$$

Since, $P(z)$ has no zero, therefore $f(z)$ is analytic everywhere in the domain.

Now, we have $f(z) = \frac{1}{P(z)} = \frac{1}{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}$

$$= \frac{1}{z^n}\left\{\frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \ldots + a_n}\right\} \to 0, \text{ as } z \to \infty$$

Thus, for every $\varepsilon > 0$, there exists $a\ \delta > 0$, such that

$$|f(z)| = \left|\frac{1}{P(z)}\right| < \varepsilon \text{ for } |z| > \delta$$

Also, $f(z)$ is continuous in the bounded domain $|z| \le \delta$, therefore it is bound in this domain.

Thus, there exists a number k such that $|f(z)| \le k$ for $|z| \le \delta$,

i.e. $|f(z)| \le \max(k, \varepsilon)$ for every z.

Now, let $\max(k, \varepsilon) = m$, then

$$|f(z)| = \left|\frac{1}{P(z)}\right| \le m, \ \forall\ z$$

Hence, by Liouville's theorem, $f(z)$ is constant, i.e. $P(z)$ must be constant, which is a contradiction as $P(z)$ cannot be constant, when $n \ge 1$ and $a \ne 0$. Therefore, $P(z)$ must have a zero.

Hence, the polynomial $P(z)$ has at least one zero or the polynomial equation $P(z) = 0$ has at least one root.

Q 4. If $a > e$, then prove by the help of Rouche's theorem that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$.

(1997, 95, 94)

Sol. Let C denote the circle $|z| = 1$ with centre at the origin and radius unity.

The given equation is $az^n - e^z = 0$

Take $f(z) = az^n$, $g(z) = -e^z$

It is evident that both $f(z)$ and $g(z)$ are analytic inside and on C.

Now, $$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-e^z}{az^n}\right| = \left|\frac{e^z}{az^n}\right| = \frac{|e^z|}{|a|\cdot|z|^n} = \frac{e^z}{a|z|^n} \quad \text{[since, } a \text{ is positive]}$$

$$= \frac{\left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right|}{a|z|^n} \le \frac{1 + |z| + \frac{1}{2!}|z|^2 + \frac{1}{3!}\cdot|z|^3 + \ldots}{a|z|^n}$$

$$= \frac{1}{a}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) \qquad [\because |z| = 1]$$

$$= \frac{e}{a} < 1 \qquad [\because a > e]$$

$\therefore |g(z)| < |f(z)|$ on C. Thus, all the conditions of Rouche's theorem are satisfied, so $f(z) + g(z)$ has the same number of zeroes inside C as $f(z)$. But $f(z) = az^n$ has n zeroes, all located at the origin, consequently $az^n - e^z$ has n zeroes inside C.

Hence, the equation $e^z = az^n$ has n roots inside $|z| = 1$.

***Q* 5. State and prove Fundamental theorem of Algebra. Find real number a for which the function $(z^n e^a - e^z)$ will have n zeroes inside $|z| = 1$.** (2008)

Sol. **Part I** See the solution of Q. 3.

Part II See the solution of Q. 3 of Very Short Answer Questions.

***Q* 6. State and prove Hurwitz's theorem.** (2017)

Sol. **Statement** Let $f_n(z)$ be a sequence of analytic functions defined on a domain D such that $f_n(z) \neq 0, \forall z \in D, n = 1, 2, 3, \ldots$. Assume that $f_n(z)$ converges uniformly to $f(z)$ on every bounded and closed subset of D. Then, the limit function is either identically zero or nowhere zero in D.

Proof Suppose $f(z)$ is not identically zero in D. Then, we have to show that $f(z)$ is never zero in D. Assume the contrary to hold, i.e. $f(z_0) = 0$ for some z_0 in D. Since, zeroes of an analytic function are isolated, therefore there exist a deleted neighbourhood $N_\delta(z_0)$ of z_0 in which the function is non-zero. Therefore,

$$f(z) \neq 0, z \in 0 < |z - z_0| < \delta, \delta > 0$$

In particular, $f(z)$ is non-zero on the circle.

$$C : 0 < |z - z_0| < \delta_1, \delta_1 < \delta.$$

Let $$\varepsilon = \text{Min}\{|f(z)| : z \in C\}$$

Since, C is bounded and closed, it follows by the given hypothesis that $f_n(z)$ converges uniformly to $f(z)$ on C. Hence, for above $\varepsilon > 0$, there exist n_0 such that

$$|f_n(z) - f(z)| < \varepsilon, \forall n > n_0 \quad \ldots(i)$$

Due to the definition of ε, note that

$$\varepsilon \leq |f(z)| \text{ for } z \in C$$

Hence, on using Eq. (i), we have

$$|f_n(z) - f(z)| < |f(z)|, \forall n > n_0 \text{ for all points on } C.$$

Now, Rouche's theorem asserts that the functions $f(z)$

and $\{f_n(z) - f(z)\} + f(z) = f_n(z)$

have the same number of zeroes in C. But $f(z)$ has a zero at z_0.

So, $f_n(z)$ must also have a zero in C.

This contradicts the hypothesis, hence we conclude that $f(z)$ can never be zero in D (in case it is not identically zero).

Long Answer Questions

***Q* 1. State Rouche's theorem. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.** **(2014)**

***Or* Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between circles $|z| = 1$ and $|z| = 2$.** **(2018)**

***Sol.* Part I Rouche's Theorem** Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside C.

Part II Let C_1 represents the circle $|z| = 1$ and C_2 represents the circle $|z| = 2$.

Suppose that $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

We observe that both $f(z)$ and $g(z)$ are analytic within and on C_1.

Now, we have $$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7 - 5z^3}{12}\right| \leq \frac{|z|^7 + |-5z^3|}{12}$$
$$= \frac{|z|^7 + 5|z|^3}{12} = \frac{1+5}{12} = \frac{1}{2}$$

Since, $|z| = 1$ on C, therefore
$$\left|\frac{g(z)}{f(z)}\right| < 1 \Rightarrow |g(z)| < |f(z)| \text{ on } C_1$$

$\therefore$ By Rouche's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeroes inside C_1 as $f(z) = 12$.

Since, $f(z) = 12$ has no zeroes inside C_1, therefore $f(z) + g(z) = z^7 - 5z^3 + 12$ has no zero inside C_1.

Let $F(z) = z^7$, $\phi(z) = 12 - 5z^3$, we observe that both $F(z)$ and $\phi(z)$ are analytic within and on C_2, we have
$$\left|\frac{\phi(z)}{F(z)}\right| = \frac{|12 - 5z^3|}{|z|^7} \leq \frac{12 + |-5z^3|}{|z|^7}$$
$$\leq \frac{|12| + 5|z|^3}{|z|^7} = \frac{12 + 5 \cdot 2^3}{2^7} = \frac{52}{128} < 1 \qquad [\because |z| = 2]$$

Thus, on C_2, $|\phi(z)| < |F(z)|$

Now, by Rouche's theorem, $F(z)$ and $\phi(z) = z^7 - 5z^3 + 12$ has the same number of zeroes as $F(z) = z^7$ inside C_2.

Since, $F(z) = z^7$ has all the seven zeroes inside the circle $|z| = 2$, as they are all located at the origin, therefore all the zeroes of $z^7 - 5z^3 + 12$ lie inside the circle C_2.

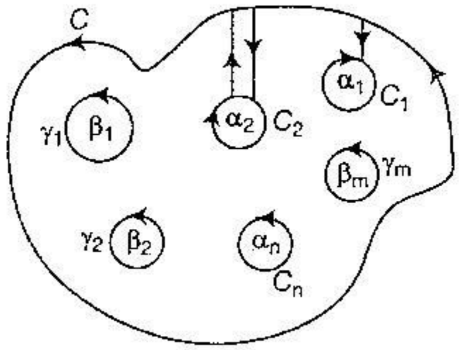
Hence, all the roots of the equation $z^7 - 5z^3 - 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Q 2. If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$, where N is the number of zeroes and P the number of poles of $f(z)$ inside C. A pole or zero of order n must be counted n times.

(2012, 09, 1993, 91)

Or **State and prove argument principle.**

Sol. Let $f(z)$ be an analytic function within and on a simple closed contour C except for a joint number of poles inside C. Suppose that $f(z) \neq 0$ on C.



Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the poles of order $p_1, p_2, \ldots, p_n$ respectively and $\beta_1, \beta_2, \ldots, \beta_m$ be the zeroes of order $q_1, q_2, \ldots, q_m$ respectively of $f(z)$ lying inside C.

Enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_n$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ each of radii ρ. This can always be done, since the poles and zeroes are isolated.

$\therefore$ By extension of Cauchy's theorem to multi-connected region, we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{r=1}^{n} \frac{1}{2\pi i}\int_{C_r} \frac{f'(z)}{f(z)}\,dz + \sum_{i=1}^{m} \frac{1}{2\pi i}\int_{\gamma_i} \frac{f'(z)}{f(z)}\,dz \qquad \text{...(i)}$$

Since, α_r is a pole of order p_r of $f(z)$, therefore we may write

$$f(z) = \frac{\phi_r(z)}{(z-\alpha_r)^{p_r}}$$

where, $\phi_r(z)$ is analytic and non-zero at α_r.

$$\therefore \qquad \log f(z) = \log \phi_r(z) - p_r \log(z - \alpha_r)$$

On differentiating both the sides, we have

$$\frac{f'(z)}{f(z)} = \frac{\phi'_r(z)}{\phi_r(z)} - \frac{p_r}{z-\alpha_1}$$

Since, $\phi_r(z)$ is analytic at α_r therefore $\phi'_r(z)$ and $\frac{\phi'_r(z)}{\phi_r(z)}$ are analytic.

$$\therefore \qquad \int_{C_r} \frac{\phi'_r(z)}{\phi_r(z)}\,dz = 0$$

Now, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{\phi'(z)}{\phi(z)}\,dz - \frac{p_r}{2\pi i}\int_C \frac{dz}{(z-\alpha_r)}$

$$= 0 - \frac{pr}{2\pi i}\int_0^{2\pi} \frac{pie^{i\theta}d\theta}{pe^{i\theta}}$$

$\because$ $z - \alpha_r = \rho e^{i\theta}$ on C_r

$0 \le \theta \le 2\pi$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = -p_r \qquad \text{...(ii)}$$

Also, since β_s is a zero of order q_s of $f(z)$, therefore we may write

$$f(z) = (z-\beta_s)^{q_s}\,\psi_s(z)$$

where, $\psi_s(z)$ is analytic and non-zero at β_s.

$$\therefore \quad \log f(z) = q_s \log(z-\beta_s) + \log \psi_s(z)$$

On differentiating both the sides, we get

$$\frac{f'(z)}{f(z)} = \frac{q_s}{z-\beta_s} + \frac{\psi'_s(z)}{\psi_s(z)}$$

Since, $\psi_s(z)$ is analytic at β_s.

Therefore, ψ'_s are analytic at β_s and $\frac{\psi_s(z)}{\psi_s(z)}$ are analytic at β_s.

$$\therefore \quad \int_{\gamma_s} \frac{\psi'_s(z)}{\psi_s(z)}\,dz = 0$$

Now, $\frac{1}{2\pi i}\int_{\gamma_s} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma_s} \frac{q_s}{(z-\beta_s)}\,dz + \frac{1}{2\pi i}\int \frac{\psi'_s(z)}{\psi_s(z)}\,dz$

$$= \frac{q_s}{2\pi i}\int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}\,d\theta + 0$$

$\because$ $z - \beta_s = \rho e^{i\theta}$ on γ_s

$0 \le \theta \le 2\pi$

$$\therefore \quad \frac{1}{2\pi i}\int_{\gamma_s} \frac{f'(z)}{f)z)}\,dz = q_s \qquad \text{...(iii)}$$

Using Eqs. (ii) and (iii) in Eq. (i), we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = -\sum_{r=1}^{n} p_r + \sum_{s=1}^{m} q_s = N - P$$

where, $N = \sum_{s=1}^{m} q_s$ = Number of zeroes and $P = \sum_{r=1}^{n} p_r$ = Number of poles.