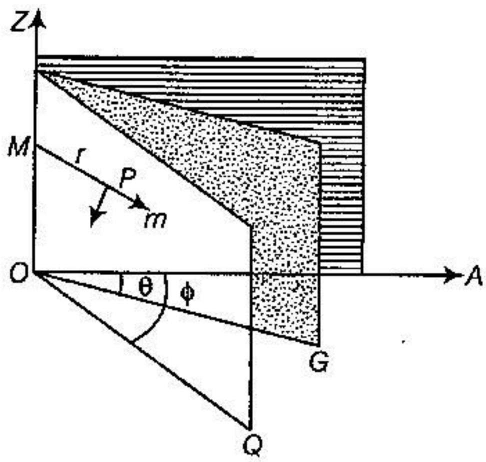
Chapter Six

MOTION ABOUT A FIXED AXIS

Important Points from the Chapter



1. **Moment of the Effective Forces about the Axis of Rotation** Let a rigid body rotates about a fixed axis OZ, perpendicular to the plane of paper. Consider an element of mass m at point P of the body and θ is an angle between the planes ZOG and ZOA, where G is centre of gravity of the body.

   Then, the moment of effective forces of the whole body about OZ is

   $$\Sigma mr^2 \frac{d^2\theta}{dt^2} = \frac{d^2\theta}{dt^2} \Sigma mr^2 = Mk^2 \cdot \frac{d^2\theta}{dt^2}$$

   where, r is the perpendicular distance of particle P from OZ and k is radius of gyration of the body about OZ.

2. **Equation of Motion of the Body about Axis of Rotation** When a body rotates about a fixed axis OZ, then

   Moment of the effective forces about OZ

   = Moment of all the external forces about OZ

   $$\therefore \qquad Mk^2 \frac{d^2\theta}{dt^2} = L$$

   where, L is the moment of all the external forces about OZ.

3. **Moment of Momentum about Axis of Rotation** The moment of momentum of the whole body about OZ is

   $$\Sigma mr^2 \frac{d\theta}{dt} = \frac{d\theta}{dt} \cdot \Sigma mr^2 = Mk^2 \frac{d\theta}{dt}$$

   where, k is the radius of gyration of the body about OZ.

4. **Kinetic Energy of the Body** K.E. of the whole body

$$= \Sigma \frac{1}{2} mr^2 \left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2} \left(\frac{d\theta}{dt}\right)^2 \Sigma mr^2 = \frac{1}{2} Mk^2 \left(\frac{d\theta}{dt}\right)^2$$

where, k is the radius of gyration of the body about OZ.

5. **Compound Pendulum** A compound pendulum is a rigid body of any form and constitution, which is free to turn about a fixed horizontal axis. **(2012, 07)**

∴ Time of a complete oscillation of the compound pendulum is

$$T = 2\pi \sqrt{\frac{k^2}{gh}}$$

where, h is the distance of the C.G. of the body from the axis of rotation and k is the radius of gyration of the body about fixed axis.

6. **Simple Equivalent Pendulum** A simple pendulum having the same periodic time as the compound pendulum is called a simple equivalent pendulum.

A simple pendulum of length l is the simple equivalent pendulum, if its period is

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{k^2}{gh}} \Rightarrow l = \frac{k^2}{h}$$

Thus, $\frac{k^2}{h}$ is the length of simple, equivalent pendulum as that of a compound pendulum.

7. **Centre of Suspension** The point O, where the perpendicular from the centre of gravity 'G' of the body, cuts the axis of rotation or the vertical plane through the centre of gravity of the body meets the axis of rotation, is called the centre of suspension. **(2010)**

8. **Centre of Oscillation** If O is the centre of suspension and G the centre of gravity of the body, then the point O' on OG produced such that $OO' = \frac{k^2}{h}$ (i.e. the length of simple equivalent pendulum) is called the centre of oscillation of the body.

Centre of suspension and centre of oscillation are convertable.

9. **Centre of Percussion** If a body, rotating about a fixed axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of action of the stricking force is called the centre of percussion.

In the case of a rod, the centre of percussion coincides with its centre of oscillation. **(2010)**

Very Short Answer Questions

Q 1. Write down the equation of motion about an axis of rotation.

(2011)

Sol. When a body rotates about a fixed axis say OZ, the impressed forces include, besides the external forces, the reaction on the axis of rotation OZ. The moment of the reaction on the axis OZ, about OZ will be zero.

Since, by D' Alembert's principle, the reversed effective forces and the impressed forces are in equilibrium, therefore the algebraic sum of their moments about OZ will be zero, i.e. moment of the effective forces about OZ = moment of all the external forces about OZ,

i.e. $$Mk^2\ddot{\theta} = L$$

where L is the moment of all the external forces about the axis of rotation OZ.

This equation is called the equation of motion of the body about the axis of rotation.

Q 2. Find the position of centre of percussion in the following cases.

(i) Uniform circular plate, axis a horizontal tangent.

(ii) A uniform rod with one end fixed.

Sol.

(i) Here, $\bar{x} = a$, $k^2 = \frac{1}{4}a^2 + a^2 = \frac{5}{4}a^2$

$\therefore$ Distance of centre of percussion below the highest point is

$$\frac{k^2}{\bar{x}} = \frac{5}{4}a^2/a = \frac{5}{4}a.$$

(ii) In this case, if rod be of length $2a$, then $\bar{x} = a$, $k^2 = \frac{1}{3}a^2 + a^2 = \frac{4}{3}a^2$

$\therefore$ Distance of the centre of percussion below fixed end is $\frac{k^2}{\bar{x}} = \frac{4}{3}a.$

Short Answer Questions

Q 1. Find the moment of momentum of the body about the fixed axis.

(2017, 13, 11, 07)

Sol. Refer to the figure of Q. 1 (i) of Long Answer Questions the velocity of the particle m at P is $r\frac{d\phi}{dt}$ perpendicular to MP.

$\therefore$ Moment of momentum of the particle m at P about OZ

$$= MP \cdot \left\{ mr \left(\frac{d\phi}{dt} \right) \right\} = r \cdot mr \frac{d\phi}{dt} = mr^2 \cdot \frac{d\theta}{dt} \qquad \left[\because \frac{d\phi}{dt} = \frac{d\theta}{dt} \right]$$

$\therefore$ Moment of momentum of the whole body about OZ

$$= \Sigma mr^2 \frac{d\theta}{dt} = \frac{d\theta}{dt} \cdot \Sigma mr^2 = Mk^2 \frac{d\theta}{dt}$$

where, k is the radius of gyration of the body about OZ.

Q 2. Find the minimum time of oscillation of a compound pendulum. **(2008)**

Sol. Let K be the radius of gyration of the body about a line through the centre of gravity G, parallel to the axis of rotation. Then, the moment of inertia of the body about the axis of rotation

$= Mk^2 = MK^2 + Mh^2$, so that $k^2 = K^2 + h^2$

[$\because$ k is the radius of gyration of the body about the axis of rotation]

$\therefore$ Length of the simple equivalent pendulum is

$$l = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = \frac{K^2}{h} + h \qquad \ldots\text{(i)}$$

Now, the time of oscillation of a compound pendulum will be least when the length l of the simple equivalent pendulum is minimum, i.e. when

$$\frac{dl}{dh} = \frac{d}{dh}\left(\frac{K^2}{h} + h \right) = -\frac{K^2}{h^2} + 1 = 0 \text{ or } h = K$$

Also, $\dfrac{d^2 l}{dh^2} = \dfrac{2K^2}{h^3}$, which is positive when $h = k$.

$\therefore$ l is minimum when $h = K$.

Hence, the time of oscillation of a compound pendulum is minimum when $h = K$.

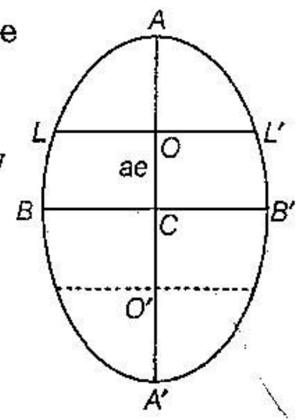
On putting $h = K$ in Eq. (i), the minimum value of $l = 2K$.

Thus, the minimum time of oscillation of a compound pendulum is given by

$$T = 2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{2K}{g}}.$$

Q 3. Find the length of the simple equivalent pendulum for an elliptic lamina oscillating about horizontal axis through a latus rectum. **(2009)**

Sol. Let C be the centre, O, O' the two foci and LL', the latus rectum of an elliptic lamina.



Here, the latus rectum LL' is taken as the axis of rotation. If h is the depth of the C.G. of the lamina below LL' and k is the radius of gyration about it, then

$$h = OC = ae$$

and $\qquad Mk^2 =$ Moment of inertia about LL'

$$= M\left(\frac{1}{4}a^2 + OC^2\right) = M\left[\frac{1}{4}a^2 + a^2e^2\right]$$

$$\Rightarrow \qquad k^2 = a^2\left(\frac{1}{4} + e^2\right)$$

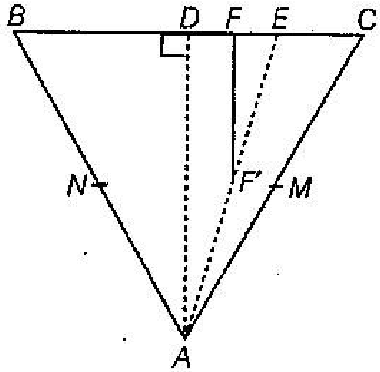
$\therefore$ Length of the simple equivalent pendulum $= \dfrac{k^2}{h}$

$$= \frac{a^2\left(\frac{1}{4} + e^2\right)}{ae} = \left(\frac{1}{4e} + e\right)a$$

Q 4. If a ΔABC is free to move about its side BC, find the centre of percussion. **(2009)**

Sol. In a ΔABC, first we draw AD perpendicular to BC, and let E, M, N be the mid-points of the sides BC, CA and AB, respectively.

Let F be the mid-point of DE, which is the point at which BC is a principle axis.



In order to find the moment of inertia of the ΔABC of mass M, we put three particles each of mass $\dfrac{M}{3}$ at the points E, M and N, whose moment of inertia is same as that of the ΔABC about BC.

The M.I. of ΔABC about BC

$$= \frac{M}{3}\left[0 + \left(\frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^2\right] = \frac{1}{6}Mp^2$$

Therefore, $k^2 = \dfrac{1}{6}p^2$, where $p = AD$

In the ΔABC, draw FF' perpendicular to BC to meet AE in F', so that

$$FF' = \frac{1}{2}AD = \frac{p}{2}.$$

Hence, the point F' is the centre of percussion.

Q 5. A uniform rod of mass 'm' and length '$2a$' can turn freely about a fixed end. Show that least angular velocity with which it must be started from the lowest position, so that it may just make complete revolution is $\sqrt{\dfrac{3g}{a}}$. **(2008)**

Sol. Do same as Q. 1 (ii) of Long Answer Questions.

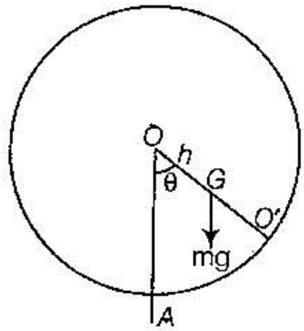
Q 6. Discuss the motion of heavy body moving about a smooth fixed horizontal axis. **(2008)**

Sol. Let us take the vertical plane through the horizontal axis as the plane of reference and the plane passing through the axis and C.G. 'G' of the body as the plane fixed in the body. Let θ be the angle between these two planes at any time t and h, the distance of G from the axis of rotation.

Now, the equation of motion is

$$Mk^2\ddot{\theta} = -Mgh\sin\theta \quad \text{...(i)}$$

where, Mk^2 is the moment of inertia of the body about the axis of rotation.



From Eq. (i), $\ddot{\theta} = -\dfrac{gh}{k^2}\sin\theta$

For small value of θ, $\ddot{\theta} = -\dfrac{gh}{k^2}\cdot\theta$

which is the equation of S.H.M. and the period is $2\pi\sqrt{\dfrac{k^2}{gh}}$, this is the period of the simple pendulum of length $l = \dfrac{k^2}{h}$.

Through G, draw a perpendicular to the axis of rotation. Cutting it at O. The point O is called the centre of suspension. Produce OG to O' such that $OO' = \dfrac{k^2}{h}$, then O' is called centre of oscillation. Thus, if the whole mass of the body is gathered at O' and suspended form O by a string, then its angular motion and time of oscillation will be the same as those of the body under the same initial conditions.

If Mk_1^2 is the moment of inertia about parallel axis through G, then

$$Mk^2 = Mk_1^2 + Mh^2 \text{ or } k^2 = k_1^2 + h^2 \quad \text{...(ii)}$$

$$\therefore \quad l = \frac{k_1^2 + h^2}{h} \text{ or } lh = k_1^2 + h^2 \quad \text{...(iii)}$$

Also, $\quad OG \cdot GO' = h(l-h) = hl - h^2 = k_1^2$ [from Eq. (iii)]

$OG \cdot OO' = hl = k_1^2 + h^2$ [from Eq. (ii)]

= Square of the radius of gyration about O

and $\quad O'G \cdot O'O = (l-h)l = l^2 - hl = (l-h)^2 + lh - h^2$

$= (l-h)^2 + k_1^2$ [from Eq.(ii)]

= Square of the radius of gyration about O'

The symmetry of these relations shows that if the the body were suspended from a parallel axis through O', then point O' will be the new centre of oscillation.

$\Rightarrow$ Centre of suspension and centre of oscillation are convertible and the times of oscillation in either cases are the same.

The time of oscillation is known when l is known and hence the simple equivalent pendulum is of minimum length and therefore the time of oscillation is minimum, when $\frac{dl}{dh} = 0$ or $1 - \frac{k_1^2}{h^2} = 0$ or $h = k_1$, i.e. the length of the simple equivalent pendulum is then $2k_1$.

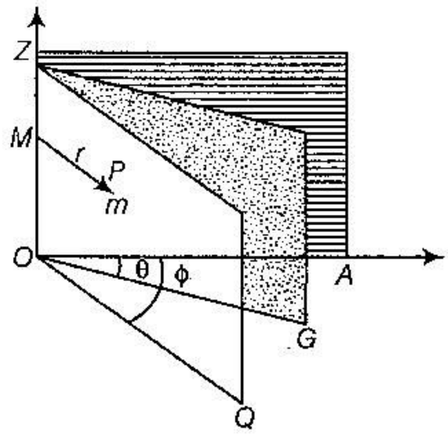
So, these conditions are realised if a heavy body is moving about a smooth fixed horizontal axis.

Long Answer Questions

Q 1. (i) Find the kinetic energy of a body which is moving with respect to a fixed axis. **(2016, 14, 12)**

(ii) A uniform rod of mass m and length $2a$ can turn freely about one end which is fixed; it is started with angular velocity ω from the position in which it hangs vertically. Find the motion. **(2015, 12, 06)**

Sol.



(i) The velocity of the particle m at P is $r\frac{d\phi}{dt}$ perpendicular to M.

$\therefore$ Kinetic energy of the particle m at

$$P = \frac{1}{2} m \left(r \frac{d\phi}{dt} \right)^2$$

$$= \frac{1}{2} mr^2 \left(\frac{d\theta}{dt} \right)^2 \qquad \left[\because \frac{d\phi}{dt} = \frac{d\theta}{dt} \right]$$

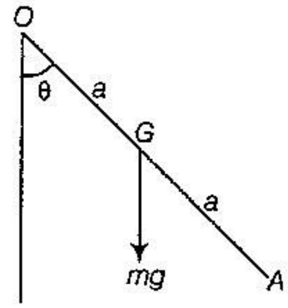
$\therefore$ Kinetic energy of the whole body

$$= \Sigma \frac{1}{2} mr^2 \left(\frac{d\theta}{dt} \right)^2 = \frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 \Sigma mr^2$$

$$= \frac{1}{2} Mk^2 \left(\frac{d\theta}{dt} \right)^2$$

where k is the radius of gyration of the body about OZ.

(ii) Let OA be the rod of length $2a$ and mass m, which can turn freely about O. The only external force is the weight mg whose moment L about the fixed axis is $Mg \cdot a \sin\theta$, when the rod has revolved through an angle θ, and this moment tends to decrease θ.



Now, the equation of motion is

$$Mk^2 \frac{d^2\theta}{dt^2} = -Mga \sin\theta$$

Since, $\quad k^2 = \frac{4a^2}{3}, \frac{d^2\theta}{dt^2} = \frac{-3g}{4a} \sin\theta$

On integrating, we have

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{4a}\cos\theta + C \qquad \left[\text{where, } \frac{1}{2}\omega^2 = \frac{3g}{4a} + C\right]$$

$$\Rightarrow \quad \left(\frac{d\theta}{dt}\right)^2 = \omega^2 - \frac{3g}{2a}(1 - \cos\theta) \qquad \text{...(i)}$$

gives the angular velocity at any instant.

In general, the Eq. (i) cannot be integrated further, so that t cannot be found in terms of θ.

The angular velocity $\frac{d\theta}{dt}$ gets less and less as θ gets bigger, and just vanishes, when $\theta = \pi$, i.e. when the rod is in its highest position, if $\omega = \sqrt{\frac{3g}{a}}$, which is the least angular velocity with which the rod must begin to move to perform a complete revolution in the vertical plane.

With this particular angular velocity, the Eq. (i) gives

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{2a}(1 + \cos\theta) = \frac{3g}{a}\cos^2\frac{\theta}{2}$$

$$\therefore \quad t\sqrt{\frac{3g}{a}} = \int_0^\theta \frac{d\theta}{\cos\frac{\theta}{2}} \Rightarrow t\sqrt{\frac{3g}{a}} = 2\left[\log\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right)\right]$$

$$\Rightarrow \quad t\sqrt{\frac{3g}{a}} = 2\log\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right) \Rightarrow t = 2\sqrt{\frac{a}{3g}}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right)$$

where t the time of describing an angle θ.

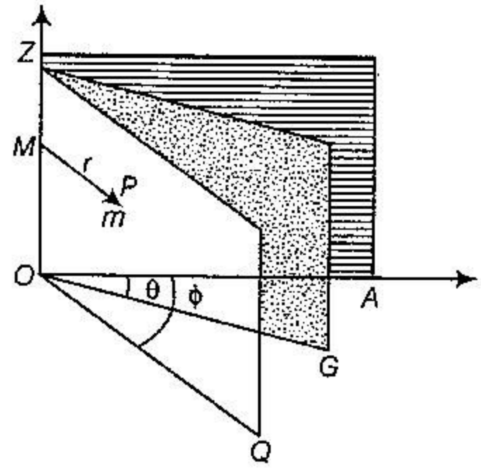
Q 2. (i) Find the moment of the effective forces of a body about a fixed axis. **(2010, 08, 07)**

(ii) Show that the time of oscillation of a compound pendulum is $2\pi\sqrt{\frac{k^2}{gh}}$. **(2010)**

Sol.

(i) Let us choose OZ as the axis of rotation and a plane AOZ be fixed in the space and it may be taken as the plane of reference. Let any other plane ZOG fixed in the body makes an angle θ with the plane ZOA.

$\therefore \qquad \angle AOG = \theta$



Consider a particle of mass m of the body at P and take a plane QOZ through P and axis OZ makes an angle ϕ with the plane AOZ

i.e. $\qquad \angle AOQ = \phi$

Again, if α be the angle between the plane ZOQ and the plane ZOG fixed in the body then this angle α will remain constant as the body rotates about OZ.

From the figure, $\phi = \theta + \alpha$

$$\therefore \qquad \frac{d\phi}{dt} = \frac{d\theta}{dt} \text{ and } \frac{d^2\phi}{dt^2} = \frac{d^2\theta}{dt^2}$$

i.e. the rate of change of ϕ is same as the rate of change of θ.

If $PM = r$ be the distance of P from the axis of rotation OZ, the point P describes a circle of radius r about M as centre.

Hence, the components of acceleration of mass m at P are $r\left(\frac{d\phi}{dt}\right)^2$ and $r\frac{d^2\phi}{dt^2}$ along and perpendicular to PM, respectively.

$\therefore$ The effective forces on m at P along and perpendicular to PM are $mr\left(\frac{d\phi}{dt}\right)^2$ and $mr\frac{d^2\phi}{dt^2}$, respectively.

But $\qquad \dfrac{d\phi}{dt} = \dfrac{d\theta}{dt}$ and $\dfrac{d^2\phi}{dt^2} = \dfrac{d^2\theta}{dt^2}$

Therefore, the effective forces on the particle m are $mr\left(\frac{d\theta}{dt}\right)^2$ and $mr\frac{d^2\theta}{dt^2}$ along and perpendicular to PM, respectively.

The effective force $mr\left(\frac{d\theta}{dt}\right)^2$ along PM cuts OZ at M and hence its moment about OZ is zero, whereas the moment of other effective force $mr\frac{d^2\theta}{dt^2}$ about OZ is $rmr\frac{d^2\theta}{dt^2} = mr^2\frac{d^2\theta}{dt^2}$

Therefore, moment of all the effective force of the whole body

$$= \Sigma mr^2 \frac{d^2\theta}{dt^2} = \frac{d^2\theta}{dt^2} \Sigma mr^2$$

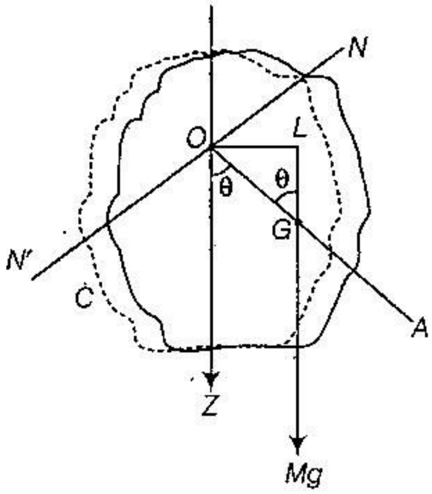
$$\left[\because \frac{d^2\theta}{dt^2} \text{ is same for all particles of the body}\right]$$

$$= \frac{d^2\theta}{dt^2} \text{(Moment for inertia about } OZ) = \frac{d^2\theta}{dt^2} \times Mk^2$$

[where, k is the radius of gyration of the body about OZ]

$$= Mk^2 \frac{d^2\theta}{dt^2}$$

(ii) Let a body be free to turn about a fixed horizontal axis NON'. Let the vertical plane through the centre of gravity 'G' of the body meets the axis of rotation NON' at the point O. This plane is the plane of rotation of the line OG and let this plane be taken as the plane of the paper.



Let $OG = h$, so that h is the distance of the centre of gravity G of the body from the axis of rotation. Let OZ be the vertical line through O. At any time t, let $\angle GOZ = \theta$. Then, θ is the inclination at time t of the plane through G and the fixed axis (i.e. the plane fixed in the body) to the vertical plane through the fixed axis (i.e. the plane fixed in space).

The impressed forces on the body are
 (i) its weight Mg acting vertically downwards at G and
 (ii) the reaction of the fixed axis.

Let k be the radius of gyration of the body about the fixed axis.

To avoid the reaction of the fixed axis, taking moments about the fixed axis NON', we have $Mk^2 \frac{d^2\theta}{dt^2} = -Mg \cdot OL = -Mgh \sin\theta$

[$\because$ '– ve' sign is taken as the moment of Mg is in the direction of θ decreasing]

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\left(\frac{gh}{k^2}\right)\theta \qquad \text{[taking } \sin\theta = \theta, \text{ as } \theta \text{ is very small]}$$

which shows that the motion of the body about the fixed axis is simple harmonic motion.

Hence, the time of a complete oscillation of the compound pendulum is given by

$$T = \frac{2\pi}{\sqrt{gh/k^2}} \Rightarrow T = 2\pi\sqrt{\frac{k^2}{gh}}.$$