6

Chapter Six

RIEMANN INTEGRATION

Important Points from the Chapter

1. **Partition** A finite set $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of points is called a partition or subdivision of the closed interval $[a, b]$, iff
$$a = x_0 < x_1 < x_2 < x_3 < \dots < x_n = b.$$
2. **Refinement of a Partition** A refinement P_2 of a partition P_1 of $[a, b]$ is a partition of the same closed interval $[a, b]$ such that $P_2 \subseteq P_1$.
3. **Darboux Sums** Let f be a bounded function defined on $[a, b]$ and $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a, b]$.
   Let $m_k = \text{glb}\,\{f(x) : x \in [x_{k-1}, x_k]\}$
   and $M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\}$, $\forall\ k = 1, 2, 3, \dots, n$.
   Then, the two sums are $L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$
   and $U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$, where $L(f, P)$ and $U(f, P)$ are called lower and upper Darboux sums respectively of f for the partition P.
4. **Lower Riemann-Integral** The supremum of the set of all lower sums is called the lower Riemann-integral of f on $[a, b]$ and there exist numbers m and M such that $m \le f(x) \le M$, $\forall\ x \in [a, b]$.
   It is defined as $\underline{\int_a^b} f(x)\, dx = \text{lub}\{L(f, P)\}$.
5. **Riemann-Integral** A function f bounded on $[a, b]$ is said to be Riemann- integrable on $[a, b]$, if its upper and lower integrals are equal, i.e. R- integrable $\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx$, it is denoted by $\int_a^b f$.
   (2006, 05)
   The function f is integrand where a, b are the limits of integration. The set of all Riemann-integrable function denoted by $R\,[a, b]$.
6. **Norm** For a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$, there are n-subintervals $[x_{k-1}, x_k]$ where $k = 1, 2, 3, \dots, n$. The length of the greatest subinterval is called the norm of the partition P and is denoted by $\|P\|$.

7. **Oscillatory Sum** For a given bounded function $f:[a,b]\to R$ and a partition P of $[a,b]$, $U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)(x_k-x_{k-1})$ is called the oscillatory sum and (M_k-m_k) is called the oscillation of the function in the subinterval $[x_{k-1},x_k]$.
8. **Darboux's Theorem** Given a bounded function $f:[a,b]\to R$, and a real number $\varepsilon>0$, there exists a real $\delta>0$ such that the relations

   (i) $L(f,P)>\underline{\int_a^b} f(x)\,dx-\varepsilon$ (ii) $U(f,P)<\overline{\int_a^b} f(x)\,dx+\varepsilon$

   hold for every partition P of $[a,b]$ for which $\|P\|<\delta$.

Very Short Answer Questions

Q 1. Prove that the lower Riemann integral cannot exceed the upper Riemann-integral. **(2006)**

Sol. Let us consider the interval $[a,b]$ and P_1,P_2 be two partitions of $[a,b]$.

Then, $$L(f,P_1)\le U(f,P_2) \quad \ldots\text{(i)}$$

fix P_2 and consider the lub $\{L(f,P_1)\}$ for all P_1. Then,

$$\underline{\int_a^b} f(x)\,dx\le U(f,P_2)$$

Taking glb $\{U(f,P_2)\}$ for all P_2, then

$$\underline{\int_a^b} f(x)\,dx\le \overline{\int_a^b} f(x)\,dx$$

Thus, the lower Riemann-integral cannot exceed the upper Riemann-integral. **Hence proved.**

Q 2. Prove that a constant function is R-integrable. **(2015, 1999, 97, 95)**

Sol. Let us consider a constant function $f(x)=k,\ \forall\ x\in[a,b]$ is bounded over $[a,b]$ and $P=\{a=x_0,x_1,x_2,\ldots,x_{k-1},x_k=b\}$ be the partition of $[a,b]$.

Then, $$M_k=\text{lub}\{f(x):x\in[x_{k-1},k_k]\}=k$$

and $$m_k=\text{glb}\{f(x):x\in[x_{k-1},x_k]\}=k$$

Therefore, $$L(f,P)=\sum_{k=1}^{n}m_k\delta_k=k\sum_{k=1}^{n}\delta_k=k(b-a)$$

and $$U(f,P)=\sum_{k=1}^{n}M_k\delta_k=k\sum_{k=1}^{n}\delta_k=k(b-a)$$

for every partition P of $[a,b]$.

Thus, we have

$$\overline{\int_a^b} f=\text{glb of the set of all } U(f,P)=k(b-a)$$

and $\qquad \int_a^b f = \text{lub of the set of all } L(f, P) = k(b-a)$

Hence, a constant function is R-integrable over $[a, b]$

and $\qquad \int_a^b f = k(b-a).$ **Hence proved.**

***Q* 3. Prove that every monotonic function is Riemann-integrable.** **(2006)**

***Or* If f is monotonic on $[a, b]$, then show that f is Riemann-integrable on $[a, b]$.** **(2017)**

Sol. Let f be a monotonic increasing function for a positive integer $\varepsilon > 0$ and partition P of $[a, b]$ such that the length of each subintervals $[x_{k-1}, x_k]$, $\forall\ k = 1, 2, \ldots, n$ is less than

$$\frac{\varepsilon}{f(b) - f(a)} \qquad \ldots\text{(i)}$$

$$M_k = \text{lub of } f(x) \text{ in } (x_{k-1}, x_k) = f(x_k)$$

and $\qquad m_k = \text{glb of } f(x) \text{ in } (x_{k-1}, x_k) = f(x_{k-1})$

Now, $U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\, \delta_k$

$$= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \frac{\varepsilon}{f(b) - f(a)} \qquad \text{[from Eq. (i)]}$$

$$< \sum_{k=1}^{n} \varepsilon < \varepsilon$$

Hence, f is R-integrable on $[a, b]$.
If f be a monotonic decreasing function, then the proof is similar as above.

***Q* 4. If $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$, then show that $f(x)$ is not R-integrable.** **(2005)**

***Or* Let $f(x)$ be defined in [0, 1] as follows**

$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$. Show that f is not Riemann- integrable on [0, 1]. **(2017)**

Sol. Here, $f(x) = 1$, when x is rational and 0 when x is irrational

$$\int_a^{\bar{b}} f(x)\, dx = \sum_{k=1}^{n} M_k \delta_k = 1 \cdot (b-a) = (b-a) \qquad \ldots\text{(i)}$$

and $$\int_{\underline{a}}^{b} f(x)\,dx = \sum_{k=1}^{n} m_k \delta_k = 0 \cdot (b-a) = 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\int_{a}^{\bar{b}} f(x)\,dx \neq \int_{\underline{a}}^{b} f(x)\,dx$$

Hence, f is not R-integrable.

Q 5. Prove that the necessary and sufficient condition for R-integrability of a bounded function $f:[a, b] \to R$ on $[a, b]$ is that $\forall\ \varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. **(2015, 14, 1998)**

Sol. If f is R-integrable on $[a, b]$, then

$$\int_{\underline{a}}^{b} f = \int_{a}^{\bar{b}} f$$

By Darboux's theorem, for $\varepsilon > 0$, there is a partition P of $[a, b]$ such that P and all its refinements

$$L(P, f) > \int_{\underline{a}}^{b} f - \frac{\varepsilon}{2} \text{ and } U(P, f) < \int_{a}^{\bar{b}} f + \frac{\varepsilon}{2}$$

Then, $U(P, f) - L(P, f) < \varepsilon$.

Conversely, if for every $\varepsilon > 0$, there is a partition P such that $U(P, f) - L(P, f) < \varepsilon$, then

$$\int_{a}^{\bar{b}} f \le U(P, f) < L(P, f) + \varepsilon < \int_{\underline{a}}^{b} f + \varepsilon$$

i.e. $$\int_{a}^{\bar{b}} f - \int_{\underline{a}}^{b} f < \varepsilon,\ \forall\ \varepsilon > 0 \Rightarrow \int_{a}^{\bar{b}} f \le \int_{\underline{a}}^{b} f$$

But $$\int_{\underline{a}}^{b} f \le \int_{a}^{\bar{b}} f \text{ and thus } \int_{a}^{\bar{b}} f = \int_{\underline{a}}^{b} f$$

Hence, f is R-integrable. **Hence proved.**

Q 6. Evaluate $\int_0^a x^2\,dx$ and show that $f \in R\,[0, a]$. **(2014)**

Sol. Let $P = \left\{\frac{ra}{n} : r = 0, 1, 2, \ldots, n\right\}$ be a partition of $[0, a]$. Then,

$$m_k = \frac{(r-1)^2 a^2}{n^2},\ M_k = \frac{r^2 a^2}{n^2} \text{ and } \delta_k = \frac{a}{n}.$$

Therefore, $$L(f, P) = \sum_{k=1}^{n} m_k \delta_k = \sum_{k=1}^{n} \frac{(r-1)^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{k=1}^{n} (r-1)^2 = \frac{a^3}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \frac{a^3}{6}\left[\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

and $$U(f, P) = \sum_{k=1}^{n} M_k \delta_k = \sum_{k=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{k=1}^{n} r^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$\therefore$ $$\int_{\underline{0}}^{a} f = \lim_{\|P\|} L(f, P) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) = \frac{a^3}{3}$$

and $$\int_{0}^{\bar{a}} f = \lim_{\|P\| \to 0} U(f, P) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{a^3}{3}$$

Thus, $$\int_{\underline{0}}^{a} f = \int_{0}^{\bar{a}} f$$

Therefore, $f \in R\,[0, a]$ and $\int_0^a f = \dfrac{a^3}{3}$. **Hence proved.**

***Q* 7. A function f is defined on [0, 1] by**

$$f(x) = \begin{cases} 2rx, & \text{where } \dfrac{1}{r+1} \le x \le \dfrac{1}{r}, \ \forall\, r = 1, 2, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

Prove that $f \in R\,[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$. **(2013)**

Sol. Since, $$f\left(\frac{1}{r} - 0\right) = \lim_{h \to 0} 2r\left(\frac{1}{r} - h\right) = 2$$

and $$f\left(\frac{1}{r} + 0\right) = \lim_{h \to 0} 2\,(r-1)\left(\frac{1}{r} + h\right) = 2 - \frac{2}{r}$$

$\therefore$ f is discontinuous at $x = \dfrac{1}{r}$, where $r = 2, 3, \ldots$

However, $f(1) = 2$ and $f(1 - 0) = \lim\limits_{h \to 0} 2(1 - h) = 2$, which shows that f is continuous at $x = 1$.

Since, the set $\left\{\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}$ of points of discontinuity of f has only one limiting point *viz.* 0.

Hence, $f \in R\,[0, 1]$.

Now, $$\int_{1/n+1}^{1} f(x)\,dx = \int_{1/2}^{1} f + \int_{1/3}^{1/2} f + \int_{1/4}^{1/3} f + \ldots + \int_{1/(n+1)}^{1/n} f$$

$$= \sum_{r=1}^{n} \int_{1/(r+1)}^{1/r} f$$

and $\int_{1/(r+1)}^{1/r} f(x)\,dx = \int_{1/(r+1)}^{1/r} 2rx\,dx = r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$

yield $\int_{1/(n+1)}^{1} f(x)\,dx = 1\cdot\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + 2\left(\frac{1}{2^2} - \frac{1}{3^2}\right) + 3\left(\frac{1}{3^2} - \frac{1}{4^2}\right)$

$$+ \ldots + n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} - \frac{n}{(n+1)^2}$$

$$= \left(\sum_{r=1}^{n} \frac{1}{r^2}\right) - \frac{n}{(n+1)^2} = \left(\sum_{r=1}^{n} \frac{1}{r^2}\right) - \frac{1/n}{(1+1/n)^2}$$

Therefore, as $n \to \infty$,

$$\int_0^1 f(x)\,dx = \left(\sum_{r=1}^{\infty} \frac{1}{r^2}\right) - 0 = \frac{\pi^2}{6}$$

***Q* 8. Show that $f(x) = x^3$ is *R*-integrable in [0, a]. Also, find the value of integral.** **(2012, 1998)**

Sol. Let $f(x) = x^3$, $\forall\ x \in [0, a]$ is continuous on $[0, a]$.

Further, if $F(x) = \dfrac{x^4}{4}$ for $x \in [0, a]$

Then, $\qquad F'(x) = x^3 = f(x)$ for $x \in [0, a]$

Hence, by the fundamental theorem of integral calculus,

$$\int_0^a x^3\,dx = F(a) - F(0) = \frac{a^4}{4} - 0 = \frac{a^4}{4}$$

If $P = \{0 = x_0, x_1, x_2, \ldots, x_n = a\}$ be the partition of $[0, a]$ into n congruent subintervals, then $\delta_k = \dfrac{a-0}{n} = \dfrac{a}{n}$

Since, f is increasing on $[0, a]$.

Then, $\qquad m_k = \text{glb}\,\{f(x) : x \in [x_{k-1}, x_k]\} = x_{k-1}^3$

and $\qquad M_k = \text{lub}\,\{f(x) : x \in [x_{k-1}, x_k]\} = x_k^3$

$\therefore \qquad L(f, P) = \sum_{k=1}^{n} m_k \delta_k = \sum_{k=1}^{n} (x_{k-1}^3 \cdot \delta_k)$

$$= \left(0^3 + \frac{a^3}{n^3} + \frac{2^3 a^3}{n^3} + \frac{3^3 a^3}{n^3} + \ldots + \frac{(n-1)^3 a^3}{n^3}\right)\frac{a}{n}$$

$$= \frac{a^4}{n^4}\,(1^3 + 2^3 + 3^3 + \ldots + (n-1)^3)$$

$$= \frac{a^4}{n^4}\left(\frac{(n-1)n}{2}\right)^2 = \frac{a^4}{4}\cdot\left(1-\frac{1}{n}\right)^2$$

and $\quad U(f, P) = \sum_{k=1}^{n} M_k \delta_k = \sum_{k=1}^{n} (x_k^3 \delta_k)$

$$= \left(\frac{a^3}{n^3} + \frac{2^3 a^3}{n^3} + \frac{3^3 a^3}{n^3} + \ldots + \frac{n^3 a^3}{n^3}\right)\frac{a}{n}$$

$$= \frac{a^4}{n^4}(1^3 + 2^3 + 3^3 + \ldots + n^3)$$

$$= \frac{a^4}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2$$

Now, $\quad \int_{\underline{0}}^{a} f = \text{lub of the set of all } L(f, P)$

$$= \text{lub}\left[\frac{a^4}{4}\left(1-\frac{1}{n}\right)^2\right] = \frac{a^4}{4}, \text{ as } \|P\| \to 0 \text{ or } n \to \infty$$

and $\quad \int_{0}^{\overline{a}} f = \text{glb of the set of all } U(f, P)$

$$= \text{glb}\left[\frac{a^4}{4}\left(1+\frac{1}{n}\right)^2\right]$$

$$= \frac{a^4}{4} \text{ as } \|P\| \to 0 \text{ or } n \to \infty$$

Consequently, $\int_{\underline{0}}^{a} f = \int_{0}^{\overline{a}} f$

Hence, f is R-integrable and $\int_0^a f = \frac{a^4}{4}$. **Hence proved.**

***Q* 9. If $f(x) = x, \ \forall \ x \in [0, 1]$, then show that f is R-integrable on $[0, 1]$, and $\int_0^1 f(x)\, dx = \frac{1}{2}$.**

Sol. Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$ be the partition of [0, 1].

Then, $m_r = \frac{r-1}{n}$, $M_r = \frac{r}{n}$ and $\delta_r = \frac{1}{n}$ for $r = 1, 2, \ldots, n$

Therefore, $L(f, P) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{r-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n}(r-1)$

$$= \frac{1}{n^2}[1 + 2 + 3 + \ldots + (n-1)]$$

$$= \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{n-1}{2n}$$

$$= \left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

and $$U(f, P) = \sum_{r=1}^{n} M_r\, \delta_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} r = \frac{1}{n^2}(1 + 2 + 3 + \ldots + n)$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{(n+1)}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

Therefore, $\underline{\int_0^1} x\, dx = \lim_{\|P\| \to 0} L(f, P) = \lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$

and $\overline{\int_0^1} x\, dx = \lim_{\|P\| \to 0} U(f, P) = \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}$

$\because$ $\underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x)\, dx$

Hence, $f(x)$ is R-integrable and the value of $\int_0^1 f(x)\, dx = \frac{1}{2}$, $\forall\, x \in [0, 1]$.

Q 10. Calculate the value of upper and lower integrals for the function f defined on [0, 2] as follows.

$$f(x) = \begin{cases} x + x^2, & \text{when } x \text{ is rational} \\ x^2 + x^3, & \text{when } x \text{ is irrational} \end{cases}$$ **(1996)**

Sol. We have, $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$

Thus, $x + x^2 > x^2 + x^3$ in (0, 1) and $x + x^2 < x^2 + x^3$ in (1, 2)

Now, for every k with usual notations,

$$m_k = x^2 + x^3, \forall\, x \in (0, 1) = x + x^2, \forall\, x \in (1, 2)$$

and $$M_k = x + x^2, \forall\, x \in (0, 1) = x^2 + x^3, \forall\, x \in (1, 2)$$

Hence, $\underline{\int_0^2} f(x)\, dx = \int_0^1 (x^2 + x^3)\, dx + \int_1^2 (x + x^2) dx$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \left[\left(\frac{1}{3} + \frac{1}{4}\right) - (0 + 0)\right] + \left[\left(\frac{4}{2} + \frac{8}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right)\right] = \frac{53}{12} = 4\frac{5}{12}$$

Also, $\overline{\int_0^2} (f)x\, dx = \int_0^1 (x + x^2)\, dx + \int_1^2 (x^2 + x^3) dx$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2 = \left[\left(\frac{1}{2} + \frac{1}{3}\right) - (0 + 0)\right] + \left[\left(\frac{8}{3} + \frac{16}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right)\right]$$

$$\frac{83}{12} = 6\frac{11}{12}$$

Q 11. Prove that a continuous function is Riemann-integrable on $[a, b]$. **(2013, 10, 09, 07)**

Sol. Let us consider a function $f(x)$ is continuous on $[a, b]$ is bounded in $[a, b]$, and $P = \{a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b\}$ is the partition of $[a, b]$. $[x_{k-1}, x_k]$ is a subinterval, where $k = 1, 2, \ldots$

Suppose that the oscillation of $f(x)$ in each of such intervals is less than $\frac{\varepsilon}{b-a}$, for $\varepsilon > 0$ and small.

Now, $U(f, P) = \sum_{k=1}^{n} M_k \delta_k L(f, P) = \sum_{k=1}^{n} m_k \delta_k$

and $U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\, \delta_k$

$$\leq \frac{\varepsilon}{(b-a)} \sum_{k=1}^{n} \delta_k$$

$$\leq \frac{\varepsilon}{(b-a)} (b-a)$$

$$\leq \varepsilon$$

Therefore, the function $f(x)$ is R-integrable. **Hence proved.**

Q 12. State and prove Fundamental theorem of integral calculus. **(2005)**

Sol. Statement Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x), \forall\ a \leq x \leq b$. Then, $\int_a^b f(t)\, dt = \phi(b) - \phi(a)$.

Proof Since, f is continuous on $[a, b]$, then the integral function F of f defined by

$$F(x) = \int_a^x f(t)\, dt,\ x \in [a, b]$$

is differentiable and

$$F'(x) = f(x)\ x \in [a, b] \qquad \ldots(\text{i})$$

But we have $\phi'(x) = f(x),\ x \in [a, b]$ ...(ii)

From Eqs. (i) and (ii), we get

$$F'(x) = \phi'(x), \forall\ x \in [a, b]$$

$$\Rightarrow \quad F'(x) - \phi'(x) = 0,\ \forall\ x \in [a, b]$$

$$\Rightarrow \quad \frac{d}{dx}[F(x) - \phi(x)] = 0,\ \forall\ x \in [a, b]$$

$$\Rightarrow \quad F(x) - \phi(x) = C, \text{ where } C \text{ is constant.}$$

$$\Rightarrow \quad F(x) = \phi(x) + C$$

$$\therefore \quad F(a) = \phi(a) + C \text{ and } F(b) = \phi(b) + C$$

$\Rightarrow \qquad F(b) - F(a) = \phi(b) - \phi(a)$ ...(iii)

But $\qquad F(b) = \int_a^b f(t)\, dt$

and $\qquad F(a) = \int_a^b f(t)\, dt = 0$

$\therefore \qquad F(b) - F(a) = \int_a^b f(t)\, dt - 0 = \int_a^b f(t)\, dt$ ...(iv)

From Eqs. (iii) and (iv), we get

$$\int_a^b f(t)\, dt = \phi(b) - \phi(a)$$

Hence proved.

***Q* 13. If f is R-integrable on $[a, b]$ and $c \in (a, b)$, then prove that f is R-integrable on both $[a, c]$ and $[c, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.**

(2015, 1998)

Sol. f is bounded on $[a, c]$ and $[c, b]$ iff f is bounded on $[a, b]$. Suppose $f \in R[a, b]$. Then, for a real number $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Let $P^* = P \cup \{c\}$ be a refinement of P, then

$$U(f, P^*) - L(f, P^*) \leq U(f, P) - L(f, P) < \varepsilon$$

Let us divide partition P^* of $[a, b]$ into P_1 of $[a, c]$ and P_2 of $[c, b]$.

Then, $U(f, P^*) - L(f, P^*) = \{U(f, P_1) + U(f, P_2)\} - \{L(f, P_1) + L(f, P_2)\}$

$$= \{U(f, P_1) - L(f, P_1)\} + \{U(f, P_2) - L(f, P_2)\}$$

$$< \varepsilon$$

Since, $U(f, P_1) - L(f, P_1) \geq 0$ and $U(f, P_2) - L(f, P_2) \geq 0$.

Also, each of these ε.

Therefore, $f \in R\ [a, c]$ and $f \in R\ [c, b]$

Since, $\qquad U(f, P_1) + U(f, P_2) = U(f, P^*)$

$\therefore \qquad \text{glb}\ [U(f, P_1)] + \text{glb}\ [U(f, P_2)] = \text{glb}\ [U(f, P^*)]$

$\therefore \qquad \int_a^b f = \int_a^c f + \int_c^b f$

Conversely, if $f \in R\ [a, c]$ and $f \in R[c, b]$, the partition P_1 on $[a, c]$ and partition P_2 on $[c, b]$ are such that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2, U(f, P_2) - L(f, P_2) < \varepsilon/2$$

Let $P = P_1 \cup P_2$, then P is a partition on $[a, b]$ and

$$U(f, P) - L(f, P) = \{U(f, P_1) + U(f, P_2)\} - \{L(f, P_1) + L(f, P_2)\}$$

$$= \{U(f, P_1) - L(f, P_1)\} - \{U(f, P_2) - L(f, P_2)\} < \varepsilon$$

Hence, f is R-integrable on $[a, b]$ **Hence proved.**

Short Answer Questions

Q 1. If $f:[a,b] \to R$ is continuous on $[a,b]$ and $F(x)=\int_a^x f(t)\,dt$, then prove that $F(x)$ always possess a derivative $F'(x)$ and moreover $F'=f$ on $[a,b]$. **(2015, 05)**

Sol. Let $x, (x+h) \in [a,b]$. Then,

$$F(x+h)-F(x)=\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt=\int_a^{x+h} f(t)\,dt+\int_x^a f(t)\,dt$$

$$\Rightarrow \quad F(x+h)-F(x)=\int_x^{x+h} f(t)\,dt \qquad \ldots\text{(i)}$$

But f is a continuous function, so $\int_x^{x+h} f(t)\,dt = hf(c)$ ...(ii)

where, $c \in [x, x+h] \subseteq [a,b]$.

From Eqs. (i) and (ii), we get

$$F(x+h)-F(x)=hf(c)$$

$$\therefore \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0} f(c)=\lim_{c\to x} f(c) \qquad [\because \text{if } h\to 0, \text{ then } c \to x]$$

$$= f(x) \qquad [\because f \text{ is continuous}]$$

Hence, $F'(x)=f(x)$ **Hence proved.**

■ **Note** Since, derivability $\Rightarrow$ continuity, the integral function, F is continuous on $[a, b]$.

Q 2. Let f be defined on $[a,b]$ such that

$$f(x)=\begin{cases} 1, & \text{where } x \text{ is a rational number} \\ 3/2, & \text{where } x \text{ is a irrational number} \end{cases}.$$

Show that f is not Riemann-integrable on $[a,b]$. **(2008)**

Sol. Let the subinterval (x_{k-1}, x_k), $\forall\ k=1,2,3,\ldots,n$ for any partition P of $[a,b]$ contains both rational and irrational numbers.

Hence, in each subintervals (x_{k-1},x_k), the upper bound $M_k=\frac{3}{2}$ and the lower bound $m_k=1$.

Then, $\overline{\int_a^b} f(x)\,dx = \text{glb}\,\{U(f,P)=\text{glb}\}\left\{\sum_{k=1}^{n} M_k\delta_k=\text{glb}\left\{\sum_{k=1}^{n}\frac{3}{2}\delta_k\right\}=\frac{3}{2}(b-a)\right.$

and $\underline{\int_a^b} f(x)\,dx=\text{lub}\,\{L(f,P)\}=\text{lub}\left\{\sum_{k=1}^{n} m_k\delta_k\right\}=\text{lub}\left\{\sum_{k=1}^{n}1\cdot\delta_k\right\}=(b-a)$

$$\because \quad \overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$$

Hence, $f(x)$ is not R-integrable on $[a,b]$. **Hence proved.**

Q 3. Prove that a necessary and sufficient condition for Riemann-integrability of a bounded function $f:[a, b] \rightarrow R$ on $[a, b]$ is that $\forall\ \varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. Also, prove that every continuous function is R-integrable on $[a, b]$. **(2015)**

Sol. **Part I** See the solution of Q. 5. of Very Short Answer Questions.
Part II See the Q. 11 of Very Short Answer Questions.

Long Answer Questions

Q 1. Show that the mean value of a continuous function in an interval belong to the range of the function, also evaluates $\int_0^a x^2\, dx$ by Riemann-integration. **(2016)**

Sol. **Part I** Since, $f(x)$ is continuous on $[a, b]$, therefore $f \in R[a, b]$.
Let M and m be the bounds of f on $[a, b]$ and
let $m = f(y), \forall\ y \in [a, b]$ and $M = f(z), \forall z \in [a, b]$.
Then, $m \le f(x) \le M, \forall\ x \in [a, b]$
Also, $(b-a)\ f(y) = \int_a^b f(x)\ dx \le (b-a)\ f(z)$
Now, we consider the following cases
Case **I** If $(b-a)\ f(y) = \int_a^b f(x)\ dx$
Then, there exists $c \in [a, b]$ such that $\int_a^b f(x)\ dx = (b-a)\ f(c)$
Case **II** If $\int_a^b f(x)\ dx = (b-a) f(z)$
Then, there exists $c \in [a, b]$ such that $\int_a^b f(x)\ dx = (b-a)\ f(c)$
Case **III** If $(b-a)\ f(y) < \int_a^b f(x)\ dx < (b-a)\ f(z)$
i.e. $f(y) < \dfrac{1}{b-a} \int_a^b f(x)\ dx < f(z)$
Let $\lambda = \dfrac{1}{(b-a)} \int_a^b f(x)\ dx$
Then, $f(y) < \lambda < f(z)$ and hence by intermediate value theorem, there exists a point $c \in [y, z] \subsetneq [a, b]$ such that $f(x) = \lambda$.
Part II See the solution of Q. 6 of Verty Short Answer Questions.

***Q* 2. Show that the function**

$$f(x) = \begin{cases} 1/2^n, & \text{where } 1/2^{n+1} < x < 1/2^n \\ 0, & \text{where } x = 0 \end{cases} \text{ is}$$

R-integrable and also prove that $\int_0^1 f(x)\,dx = \dfrac{2}{3}$.

(2006, 1996)

Sol. Given function is

$$f(x) = \begin{cases} \dfrac{1}{2^n}, & \text{where } \dfrac{1}{2^{n+1}} < x < \dfrac{1}{2^n}, \forall\, n = 0, 1, 2, 3, \ldots \\ 0, & \text{where } x = 0 \end{cases}$$

$$f(x) = 1, \text{ if } \frac{1}{2} < x \le 1 \text{ at } n = 0$$

$$f(x) = \frac{1}{2}, \text{ if } \frac{1}{2^2} < x < \frac{1}{2} \text{ at } n = 1$$

$$f(x) = \frac{1}{2^2}, \text{ if } \frac{1}{2^3} < x \le \frac{1}{2^2} \text{ at } n = 2$$

....

....

....

$$f(x) = \frac{1}{2^{n-1}}, \text{ if } \frac{1}{2^n} < x < \frac{1}{2^{n-1}} \text{ at } n = n - 1$$

and $f(0) = 0$ at $x = 0$

Hence, $|f(x)| \le 1, \forall\, x \in [0, 1]$

Thus, $f(x)$ is bounded on $[0, 1]$ with sup $(f(x)) = 1$ and inf $(f(x)) = 0$.

Here, $f\left(\dfrac{1}{2^n}\right) = \dfrac{1}{2^n}$ and $f(x) = \dfrac{1}{2^n}, \forall x \in \left[\dfrac{1}{2^{n+1}}, \dfrac{1}{2^n}\right]$

$$\therefore \qquad \int \left(\frac{1}{2^n} - 0\right) dx = \frac{1}{2^n} = f\left(\frac{1}{2^n}\right)$$

So, $f(x)$ is continuous at $x = \dfrac{1}{2^n}$ on the left hand side.

Moreover, $f(x) = \dfrac{1}{2^{n-1}}, \forall\, x \in \left[\dfrac{1}{2^n}, \dfrac{1}{2^{n-1}}\right]$

$$\therefore \qquad f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}} \ne f\left(\frac{1}{2^n}\right) = \frac{1}{2^n}$$

So, $f(x)$ is discontinuous at $x = \dfrac{1}{2^n}$ on the right hand side.

Thus, $f(x)$ is discontinuous at $x = \dfrac{1}{2^n}$.

$\therefore$ The set of discontinuous points of $f(x)$ is $\left\{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^n}, \ldots\right\}$, which is a infinite set with 0 limit point.

Hence, $f(x)$ is R-integrable.

Now, $\sum_{n=0}^{\infty} \int_{1/2^{n-1}}^{1/2^n} f(x)dx = \int_{1/2}^{1} f(x)dx + \int_{1/4}^{1/2} f(x)dx + \ldots + \int_{1/2^{r+1}}^{1/2^r} f(x)dx + \ldots$

$$= \lim_{r \to \infty} \int_{1/2^{r+1}}^{1} f(x)dx = \int_0^1 f(x)dx$$

Thus, $\int_0^1 f(x)dx = \sum_{n=0}^{\infty} \int_{1/2^{n+1}}^{1/2^n} f(x)dx = \sum_{n=0}^{\infty} \int_{1/2^{n+1}}^{1/2^n} \frac{1}{2^n} dx$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}\left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) dx = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$$