Chapter Seven

CALCULUS OF VARIATIONS

Important Points from the Chapter

1. **Normed Linear Space** A linear space V is said to be normed, if each element x of V is assigned a non-negative real number $||x||$, called the norm of x such that

   (i) $||x|| = 0 \Leftrightarrow x = 0$

   (ii) $||\alpha x|| = |\alpha|\,||x||$, where α is any scalar.

   (iii) $||x + y|| \le ||x|| + ||y||$, $\forall\, x, y \in V$

   e.g. Consider the linear space $C(a, b)$ consisting of all continuous functions $y(x)$ defined on $[a, b]$. Norm $C(a, b)$ is defined as the maximum of the absolute value, i.e. $||y||_0 = \max\limits_{a \le x \le b} |y(x)|$.

2. **Functional** A functional is a correspondence which assigns a definite number to each function belonging to some class. Thus, we say that a functional is a kind of function, where the independent variable is itself a functions.

   e.g. Consider the set of all rectifiable plane curves. A definite real number is associated with each such curve, namely, its length. Thus, the length of the curve is a functional defined on the set of rectifiable curves.

3. **Continuity of Functional** The function $J[y]$ is said to be continuous at point $\hat{y}$ of linear space V if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|J[y] - [\hat{y}]| < \varepsilon$ provided that $|(y - \hat{y})| < \delta$.

4. **Linear Functional** Given a normed linear space V, let each element $h \in V$ be assigned a number $\phi[h]$, i.e. let $\phi[h]$ be a functional defined on V. Then, $\phi[h]$ is said to be a linear functional, if

   (i) $\phi[\alpha h] = \alpha\,\phi[h]$, $\forall\, h \in V$ and α is any scalar.

   (ii) $\phi[h_1 + h_2] = \phi[h_1] + \phi[h_2]$, $\forall\, h_1, h_2 \in V$.

   e.g. The integral $\phi[h] = \int_a^b \alpha(x)\, h(x)\, dx$, where $\alpha(x)$ is a fixed function in $C(a, b)$, defines a linear functional on $C(a, b)$. **(2018, 13, 10, 08)**

5. **Variation of a Functional** Let $J\,[y]$ be a functional defined on some normed linear space, and let $\Delta J\,[h] = J\,[y+h] - J\,[y]$

   be its increment, corresponding to the increment $h = h(x)$ in the function $y = y(x)$. If y is fixed, $\Delta J\,[h]$ is a functional of h, in general, a non-linear functional.

   Suppose that $\Delta J\,[h] = \phi\,[h] + \varepsilon \|h\|$, where $\phi\,[h]$ is a linear functional and $\varepsilon \to 0$ as $\|h\| \to 0$. Then, the functional $J\,[y]$ is said to be differentiable, and the principal linear part of the increment $\Delta J[h]$, i.e. the linear functional $\phi\,[h]$, which differs from $\Delta J[h]$ by an infinitesimal of order higher than one relative to $\|h\|$, is called the variation (or differential) of $J[h]$ and is denoted by $\delta J[h]$.
6. **Fixed End Point Problem** Let $J\,[y]$ be a functional of the form $\int_a^b F(x, y, y')\,dx$, defined on the set of functions $y(x)$ which have continuous first derivatives in $[a, b]$ and satisfy the boundary conditions $y(a) = A$, $y(b) = B$. Then, the necessary condition for $J\,[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfies Euler's equation $F_y - \dfrac{d}{dx} F_{y'} = 0$.

   (2016, 13, 07)
7. **Variable End Point Problem** Among all curves whose end points lie on two given vertical lines $x = a$ and $x = b$, find the curve for which the functional $J\,[y] = \int_a^b F(x, y, y')\,dx$ has an extremum.

   (2008)
8. **Brachistochrone** (Shortest Time) **Problem** Starting from the point $P(a, A)$, a heavy particle slides down a curve in the vertical plane, to find the curve such that the particle reaches the vertical line $x = b\ (\neq a)$ in the shortest time.

   (2011, 07)

⏻ Very Short Answer Questions

Q 1. Find the first integral of Euler's equation when the integrand does not depend on x explicitly. **(2008, 06)**

Sol. If the integrand does not depend on x, i.e. if

$$J[y] = \int_a^b F(y, y')\, dx, \text{ then } F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y}\, y' - F_{y'y'}\, y''$$

On multiplying this by y', we get

$$\left(F_y - \frac{d}{dx} F_{y'}\right) y' = F_y\, y' - F_{y'y}\, y'^2 - F_{y'y'} y'\, y'' = \frac{d}{dx}(F - y' F_{y'})$$

Thus, if F does not involve x explicitly, a first integral of Euler's equation is $F - y' F_{y'} = C$, where C is a constant.

Q 2. If $\alpha(x)$ is continuous in $[a, b]$ and $\int_a^b \alpha(x)\, h(x)\, dx = 0$ for every function $h(x) \in C(a, b)$ such that $h(a) = h(b) = 0$, then prove that $\alpha(x) = 0$, $\forall\, x \in [a, b]$. **(2007)**

Sol. Suppose the function $\alpha(x)$ is non-zero, say positive, at some point x_0 in $[a, b]$. Then, $\alpha(x)$ is also positive in some interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ contained in $[a, b]$. Let this neighbourhood of x_0 be denoted by I.

If the set $h(x) = 0$ outside I and $h(x) = [(x - x_0)^2 - \varepsilon^2]^2$ inside I, then $h(x)$ satisfies the conditions of the given problem. However, in this case, the integral $\int_a^b \alpha(x)\, h(x)\, dx$ reduces to an integral over I and is obviously positive, as the integrand D itself is positive.

This contradicts the fact that $\int_a^b \alpha(x)\, h(x)\, dx = 0$ for every function $h(x) \in C(a, b)$ such that $h(a) = h(b) = 0$. This contradiction proves the result.

Q 3. If $\phi[h]$ is a linear functional and if $\dfrac{\phi[h]}{\|h\|} \to 0$ as $\|h\| \to 0$, then prove that $\phi[h] \equiv 0$.

Sol. Suppose, $\phi[h_0] \neq 0$ for some $h_0 \neq 0$.

On putting $h_n = \dfrac{h_0}{n}$, we get $\lambda = \dfrac{\phi[h_0]}{\|h_0\|}$

We see that $\|h_n\| \to 0$ as $n \to \infty$ but

$$\lim_{n\to\infty} \frac{\phi[h_n]}{\|h_n\|} = \lim_{n\to\infty} \frac{\phi\left[\dfrac{h_0}{n}\right]}{\left\|\dfrac{h_0}{n}\right\|} = \lim_{n\to\infty} \frac{\dfrac{1}{n}\phi[h_0]}{\dfrac{1}{n}\|h_0\|}$$

$$= \lim_{n \to \infty} \frac{n\phi\,[h_0]}{n\,||\,h_0\,||} = \lambda \neq 0$$

which contradicts the hypothesis that $\frac{\phi\,[h]}{||\,h\,||} \to 0$, as $||\,h\,|| \to 0$. This contradiction proves the result.

Q 4. Find the stationary value (extremal) of the functional $\int_{x_0}^{x_1} \frac{y'^2}{x^3}\, dx$.

(2019, 16)

Sol. Let $f = \frac{y'^2}{x^3}$, which is independent of y, i.e. $\frac{\partial f}{\partial y} = 0$.

Also, $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 2\left(\frac{x^3 y'' - y' 3x^2}{x^6}\right) = \frac{2}{x^4}\,(xy'' - 3y')$

Therefore, Euler's equation reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow \quad \frac{2}{x^4}\,(xy'' - 3y') = 0 \quad \Rightarrow \frac{y''}{y'} = \frac{3}{x}$

On integrating, we get

$$\int \frac{y''}{y'}\, dy = 3\int \frac{1}{x}\, dx + \log C$$

$\Rightarrow$
$$\log y' = 3 \log x + \log C,\; y' = Cx^3$$

On integrating, we get $y = \frac{Cx^4}{4} + C_1$

Q 5. Find out Euler's equation, corresponding to the integral $J\,[y] = \int_a^b F(y, y')\, dx$.

(2018, 14)

Sol. If the integrand does not depend on x, i.e. if

$$J\,[y] = \int_a^b F(y, y')\, dx, \text{ then}$$

$$F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y}\, y' - F_{y'y'}\, y''$$

On multiplying this by y', we get

$$\left[F_y - \frac{d}{dx} F_{y'}\right] y' = F_y y' = F_{y'y}\, y'^2 - F_{y'y'}\, y'\, y''$$

$$= \frac{d}{dx}\,(F - y'\, F_{y'})$$

Thus, if F does not involve x explicitly, then first integral of Euler's equation is $F - y' F_{y'} = C$, where C is constant.

Q 6. Obtain Euler's equation to determine the extremals of the integral $\int_0^1 (y'^2 + k^2 \cos y)\, dx$. **(2012)**

Sol. Let $\quad f = y'^2 + k^2 \cos y \quad$ ...(i)

Then, $\quad \dfrac{\partial f}{\partial y} = -k^2 \sin y, \dfrac{\partial f}{\partial y'} = 2y'$

Now, Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \text{ reduced to } -k^2 \sin y - \frac{d}{dx}(2y') = 0$$

i.e. $\quad -k^2 \sin y - 2y'' = 0$

$$2y'' + k^2 \sin y = 0$$

$$\Rightarrow \quad 2y'' = -k^2 \sin y \Rightarrow \frac{\partial^2 y}{\partial x^2} = -\frac{k^2}{2} \sin y$$

On integrating, we get $\dfrac{dy}{dx} = -\dfrac{k^2 \sin y}{2} \cdot x + C_1$

Again, on integrating, we get

$$y = -\frac{k^2 x^2}{2} \sin y + C_1 x + C_2$$

Q 7. If $\alpha(x)$ is continuous on (a, b) and if $\int_a^b \alpha(x)\, h'(x)\, dx = 0$, for every function $h(x) \in D_1(a, b)$ such that $h(a) = h(b) = 0$, then prove that $\alpha(x) = c, \forall\ x \in [a, b]$, where c is constant.

(2015, 13, 11)

Sol. We have, $0 = \int_a^b \alpha(x)\, h'(x)\, dx = [\alpha(x)\, h(x)]_a^b - \int_a^b \alpha'(x)\, h(x)\, dx$

$$= -\int_a^b \alpha'(x)\, h(x)\, dx \qquad [\because h(a) = h(b) = 0]$$

Thus, $\int_a^b \alpha'(x)\, h(x)\, dx = 0$ for every function $h(x) \in D_1(a, b)$ such that $h(a) = h(b) = 0$.

Hence, from Q. 2, $\alpha'(x) = 0, \forall\ x \in [a, b]$, which implies that $\alpha(x) = c, \forall\ x \in [a, b]$, where c is a constant.

Short Answer Questions

Q 1. Determine the function $y(x)$, which minimizes the integral $J = \int_0^1 (1 - y'^2)\, dx$ such that $y(0) = 0$, $y(1) = 1$. **(2014)**

Sol. Since, the integrand $F = 1 - y'^2$ does not contain y explicitly, then we have

$$F_{y'} = c\,(\text{constant}) \Rightarrow -2y' = C \text{ or } y' = -\frac{C}{2}$$

On integrating, we get $y = -\frac{C}{2}x + C_1$ ...(i)

where, C and C_1 are constants to be determined by the given condition $y(0) = 0$, $y(1) = 1$.

On putting $x = 0$, $y = 0$ in Eq. (i), we get

$$C_1 = 0$$

On putting $x = 1$, $y = 1$ in Eq. (i), we get

$$1 = -\frac{C}{2} + C_1$$

which on putting $C_1 = 0$, gives $C = -2$.

Hence, $y = x$ is the required value of $y(x)$.

Q 2. Define a linear functional and illustrate it with an example. Also, prove that the differential of a differentiable function is unique. **(2018, 13, 10, 08, 06)**

Or **Show that the differential of a differentiable functional is unique.** **(2009)**

Sol. **Part I Linear Functional** Given a normed linear space V, let each element $h \in V$ be assigned a number $\phi[h]$, i.e. let $\phi[h]$ be a functional defined on V. Then, $\phi[h]$ is said to be a linear functional, if

(i) $\phi[\alpha h] = \alpha\phi[h]$, $\forall\, h \in V$ and α is any scalar.

(ii) $\phi[h_1 + h_2] = \phi[h_1] + \phi[h_2]$, $\forall\, h_1, h_2 \in V$.

e.g. The integral $\phi[h] = \int_a^b \alpha(x)\, h(x)\, dx$, where $\alpha(x)$ is a fixed function in $C(a, b)$, defines a linear functional on $C(a, b)$.

Part II Suppose, the differential of the functional $J[y]$ is not uniquely defined, so that

$$\Delta J[h] = \phi_1[h] + \varepsilon_1 \,||\, h \,||,$$
$$\Delta J[h] = \phi_2[h] + \varepsilon_2 \,||\, h \,||,$$

where $\phi_1[h]$ and $\phi_2[h]$ are linear functionals, and $\varepsilon_1, \varepsilon_2 \to 0$ as $||\, h \,|| \to 0$. This implies that

$$\phi_1[h] - \phi_2[h] = (\varepsilon_2 - \varepsilon_1)\,||h||$$

$$\Rightarrow \qquad \frac{\phi_1[h] - \phi_2[h]}{||h||} = \varepsilon_2 - \varepsilon_1$$

Now, $\phi_1[h] - \phi_2[h]$ is a linear functional and $\dfrac{\phi_1[h] - \phi_2[h]}{||h||} \to 0$ as $||h|| \to 0$, therefore $\phi_1[h] - \phi_2[h] \equiv 0$, i.e. $\phi_1[h] \equiv \phi_2[h]$.
Hence, the differential of a differentiable functional is unique.

Q 3. Use the calculus of variables to find the shortest distance between the line $y = x$ and parabola $y^2 = x - 1$. **(2014)**

Sol. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points and S the length of the arc joining these points, then

$$\therefore \qquad S = \int_{x_1}^{x_2} dS \Rightarrow S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 - y'^2}\, dx \qquad \text{...(i)}$$

where, $y(x_1) = y_1 = x$ and $y(x_2) = y_2 = \sqrt{x-1}$.
If S satisfies the Euler's equation, then it will be minimum.

$$\therefore \qquad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{...(ii)}$$

Here, from Eq. (i), we get $f = \sqrt{1 + y'^2}$

$$\therefore \qquad \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial y'} = \frac{1}{2}(1 + y'^2)^{\frac{1}{2}-1} \cdot 2y = \frac{y'}{\sqrt{1 + y'^2}}$$

From Eq. (ii), we have

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

On integrating, we get

$$\frac{y'}{\sqrt{1 + y'^2}} = C_1 \Rightarrow y'^2 = C_1^2(1 + y'^2)$$

$$\Rightarrow \qquad y'^2(1 - C_1^2) = C_1^2, \; y'^2 = \frac{C_1^2}{1 - C_1^2} = C_2^2 \Rightarrow y' = C_2$$

Again, integrating, we get

$$y = C_2 x + C_3 \qquad \text{...(iii)}$$

$$\because \qquad y(x_1) = x_1 \text{ and } y(x_2) = \sqrt{x_2 - 1}$$

$$\therefore \qquad x_1 = C_2 x_1 + C_3 \qquad \text{...(iv)}$$

$$\text{and} \qquad \sqrt{x_2 - 1} = C_2 x_2 + C_3 \qquad \text{...(v)}$$

Using transversality condition, we have

$$[(1 + y'^2)^{1/2} + (2x - y') \cdot y'(1 + y'^2)^{-1/2}]_{x = x_1} = 0$$

and $\quad [(1+y'^2)^{1/2}+(1-y')\,y'(1+y'^2)^{-1/2}]_{x=x_2}=0$

$\therefore \quad [(1+C_2^2)^{1/2}+(2x_1-C_2)\,(C_2(1+C_2^2)^{-1/2}]=0$

$\Rightarrow \quad 1+C_2^2+2x_1C_2-C_2^2=0$

$\Rightarrow \quad 2x_1C_2=-1 \quad \Rightarrow \quad x_1=-\dfrac{1}{2C_2}$

and $\quad [(1+C_2^2)^{1/2}+(1-C_2)\,C_2(1+C_2^2)^{-1/2}]=0$

$\Rightarrow \quad 1+C_2^2+C_2-C_2^2=0 \Rightarrow C_2=-1$

$\therefore \quad x_1=-x_1+C_3 \;\Rightarrow\; C_3=2x_1 \Rightarrow x_1=\dfrac{C_3}{2}$

$$\sqrt{x_2-1}=-x_2+C_3$$

$\therefore \quad C_2=-1$, then $x_1=\dfrac{1}{2}$

$\therefore \quad C_3=1$

Now, $\quad x_2-1=C_2^2x_2^2+C_3^2+2C_2C_3x_2$

$\Rightarrow \quad x_2-1=x_2^2+1-2x_2 \Rightarrow x_2^2-3x_2+2=0$

$\Rightarrow \quad x_2(x_2-2)-1(x_2-2)=0$

$\because \quad x_2=1,2 \Rightarrow y'=C_2=-1$

So, $I=\int_{1/2}^{1}\sqrt{2}\,dx$ and $\int_{1}^{2}\sqrt{2}\,dx=\sqrt{2}\,[x]_{1/2}^{1}$

and $\quad \sqrt{2}\,[x]_1^2=\sqrt{2}\left[1-\dfrac{1}{2}\right]$ and $\sqrt{2}\,[2-1]$

$\therefore \quad I=\dfrac{\sqrt{2}}{2}$ and $\sqrt{2}$, $\;I=\dfrac{1}{\sqrt{2}}$ and $\sqrt{2}$

Hence, the shortest distance between the line $y=x$ and parabola $y^2=x-1$ is $\dfrac{1}{\sqrt{2}}$.

Q 4. Use Euler's equation, find the extremal of the functional $J[y]=\int_{1/2}^{1}x^2y'^2\,dx$ subject to the conditions $y\left(\dfrac{1}{2}\right)=1$, $y(1)=2$, $y=y(x)$. **(2012, 10, 07)**

Sol. Let $F(x_1, y_1, y')=x^2y'^2$ ...(i)

The Euler's equation $F_y-\dfrac{d}{dx}F_{y'}=0$ may written as

$$F_{y'y'}\frac{d^2y}{dx^2}+F_{y'y'}\frac{dy}{dx}+(F_{y'x}-F_y)=0 \qquad \text{...(ii)}$$

From Eq. (i), we get

$$F_{y'}=2x^2y',\; F_{y'}=0$$
$$F_{y'y'}=2x^2, F_{y'x}=4x'y', F_{y'y}=0$$

Therefore, Eq. (ii), becomes

$$2x^2\frac{d^2y}{dx^2}+4xy'=0 \Rightarrow x\frac{d^2y}{dx^2}+2y'=0 \Rightarrow \frac{y''}{y'}+\frac{2}{x}=0$$

On integrating, we get

$$\log y'+2\log x=\log C \Rightarrow y'=\frac{C}{x^2}, \text{ where } C \text{ is constant.}$$

Again integrating, we get

$$y=-\frac{C}{x}+C' \qquad \text{...(iii)}$$

where, C' is a constant.

On putting $x=\frac{1}{2}$, $y=1$, we get

$$1=-2C+C' \qquad \text{...(iv)}$$

On putting $x=1$, $y=2$, we get $2=-C+C'$ ...(v)

On solving Eqs. (iv) and (v), we get $C=1$, $C'=3$

Thus, from Eq. (iii), it follows that the external of the functional is

$$y=-\frac{1}{x}+3.$$

***Q 5.* With the help of Euler's equation, show that the solution of the functional $J(y)=\int_1^2 \frac{\sqrt{(1+y'^2)}}{x}\,dx$ subject to the conditions $y(1)=0$, $y(2)=1$ is equal to $(y-2)^2+x^2=5$.** **(2016, 15, 11, 08, 06)**

***Or* Find the stationary value (extremal) of the functional $\int_1^2 \frac{\sqrt{(1+y'^2)}}{x}\,dx$, where $y(1)=0$ and $y(2)=1$.** **(2016)**

***Or* Using Euler's equation, find the extremal of the functional $J(y)=\int_1^2 \frac{\sqrt{1+y'^2}}{x}\,dx$ subject to the conditions $y(1)=0$, $y(2)=1$.** **(2017)**

Sol. Since, the integrand $F(x,y,y')=\frac{\sqrt{(1+y'^2)}}{x}$ does not contain y explicitly, therefore the first integral of Euler's equation is $F_{y'}=C$.

Thus, $\frac{y'}{x\sqrt{(1+y'^2)}}=C$. So that $y'^2(1-C^2x^2)=C^2x^2 \Rightarrow y'=\frac{Cx}{\sqrt{(1-C^2x^2)}}$

On taking the positive square root, we get

$$\frac{dy}{dx} = \frac{Cx}{\sqrt{(1 - C^2x^2)}}$$

On integrating, we get $y = \int \frac{Cx}{\sqrt{(1 - C^2x^2)}} dx + C_1$

$$\Rightarrow \qquad y = -\frac{1}{C}\sqrt{(1 - C^2x^2)} + C_1$$

$$\Rightarrow \qquad (y - C_1)^2 + x^2 = \frac{1}{C^2}$$

Thus, the solution is a circle with its centre on the Y-axis. From the given conditions $y(1) = 0$, $y(2) = 1$, we find that $C = \frac{1}{\sqrt{5}}$, $C_1 = 2$.

Hence, the final solution is $x^2 + (y - 2)^2 = 5$.

***Q* 6. Among all curves of length l in the upper half plane passing through the point $(-a, 0)$ and $(a, 0)$, find the one which together with the interval $[-a, a]$ encloses the largest area.**
(2018, 14, 12, 09)

***Or* In the family of all curves of length l in the upper half plane passing through the points $(-a, 0)$ and $(a, 0)$. Find the one which together with the interval $[-a, a]$ encloses the largest area.**
(2011)

Sol. We are looking for the function $y = y(x)$ for which the integral

$$J[y] = \int_{-a}^{a} y\, dx$$

takes the largest value subject to the conditions

$$y(-a) = y(a) = 0,\ K[y] = \int_{-a}^{a} \sqrt{1 + y'^2}\, dx = l$$

Thus, we are dealing with an isoperimetric problem.
We form the functional

$$J[y] + \lambda K[y] = \int_{-a}^{a} (y + \lambda \sqrt{1 + y'^2})\, dx$$

and write the corresponding Euler equation

$$1 - \lambda \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0$$

which on integration gives $x - \lambda \frac{y'}{\sqrt{(1 + y'^2)}} = C_1$

$$\Rightarrow \qquad (x - C_1) = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

$$\Rightarrow \qquad (x - C_1)\sqrt{1 + y'^2} = \lambda y'$$

$$\Rightarrow \qquad (x - C_1)^2 + y'^2 (x - C_1)^2 = \lambda^2 y'^2$$

$$\Rightarrow \qquad y'^2 \{\lambda^2 - (x - C_1)^2\} = (x - C_1)^2$$

$$\Rightarrow \qquad y' = \frac{x - C_1}{\sqrt{\{\lambda^2 - (x - C_1)^2\}}}$$

On integrating, we get $y - C_2 = -\sqrt{\{\lambda^2 - (x - C_1)^2\}}$

$$\Rightarrow \qquad (x - C_1)^2 + (y - C_2)^2 = \lambda^2 \qquad \text{...(i)}$$

which is the equation of a family of circles.

The values of C_1, C_2 and λ are then determined from the conditions

$$y(-a) = y(a) = 0, K[y] = l$$

Now,
$$y(-a) = 0$$

$$\Rightarrow \qquad (a + C_1)^2 + C_2^2 = \lambda^2 \qquad \text{...(ii)}$$

and
$$y(a) = 0$$

$$\Rightarrow \qquad (a - C_1)^2 + C_2^2 = \lambda^2 \qquad \text{...(iii)}$$

Here, Eqs. (ii) and (iii) gives

$$C_1 = 0, a^2 + C_2^2 = \lambda \qquad \text{...(iv)}$$

$$\therefore \qquad l = \int_{-a}^{a} \sqrt{\left\{1 + \frac{x^2}{(y - C_2)^2}\right\}}\, dx = \int_{-a}^{a} \frac{\lambda}{\sqrt{\lambda^2 - x^2}}\, dx$$

$$= 2\lambda \left[\sin^{-1} \frac{x}{\lambda}\right]_0^a = 2\lambda \sin^{-1} \frac{a}{\lambda}$$

$$\therefore \qquad \lambda = a \operatorname{cosec} \frac{l}{2\lambda} \qquad \text{...(v)}$$

From Eq. (iv) and (v), we get $C_2 = a \cot \dfrac{l}{2\lambda}$

Hence, from Eq. (i), the required curve is

$$x^2 + \left(y - a \cot \frac{l}{2\lambda}\right)^2 = a^2 \operatorname{cosec}^2 \frac{l}{2\lambda}$$

$$\Rightarrow \qquad x^2 + y^2 - 2ay \cot \frac{l}{2\lambda} = a^2, \text{ where } \lambda \text{ is given by Eq. (v).}$$

***Q 7.* Prove mathematically that the shortest distance between two points in a plane is a straight line.** **(2015, 13, 09)**

Sol. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in XY-plane and let S be the length of the arc joining these points.

Then,
$$S = \int_{x_1}^{x_2} dS \qquad \text{...(i)}$$

$$\left[\because \frac{dS}{dx} = \sqrt{1 + y'^2} \Rightarrow dS = \sqrt{1 + y'^2}\, dx\right]$$

$$y(x_1) = y_1 \text{ and } y(x_2) = y_2$$

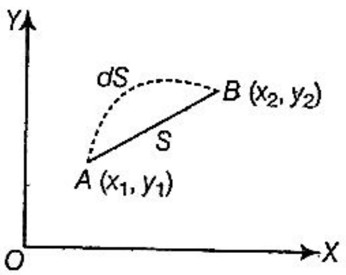
To find the shortest distance between these points, we use Euler's equation, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{...(ii)}$$

Let $$F \equiv (1 + y'^2)^{1/2}$$

Then, Eq. (ii), we get $0 - \dfrac{d}{dy}\left(\dfrac{y'}{\sqrt{1 + y'^2}}\right) = 0$

On integrating, we get $\dfrac{y'}{\sqrt{1 + y'^2}} = C$



$$\Rightarrow \quad y'^2 = C^2(1 + y'^2)$$

$$\Rightarrow \quad y'^2 = \frac{C^2}{1 - C^2} = m^2 \text{ (say)}$$

$$\Rightarrow \quad y'^2 = m^2 \Rightarrow y' = m$$

$$\Rightarrow \quad \frac{dy}{dx} = m \Rightarrow \quad dy = m\, dx$$

On integrating, we get

$$y = mx + C, \text{ where } C \text{ is constant.}$$

which show that the shortest distance between two points in a plane is a straight line.

***Q* 8. If $J[y]$ is the functional of the form $J[y] = \int_a^b F(x, y, y')\, dx$ where the curves $y = y(x)$ are continuous and possess continuous derivatives of the first order in $[a, b]$ and $y(a) = A$ and $y(b) = B$. Then, show that a necessary condition for $J[y]$ to be extremum is that $y(x)$ satisfies the Euler's equation $F_y - \dfrac{d}{dx}[F_{y'}] = 0$.**

***Or* Establish the necessary condition for $\int_a^b f(x, y, y')\, dx$ to be maximum or minimum (extremal).** **(2018, 16)**

Sol. Suppose, we give $y(x)$ an increment $h(x)$. Then, in order for the function $y(x) + h(x)$ to continue to satisfy the boundary conditions, we must have

$$y(a) + h(a) = A, \; y(b) + h(b) = B$$

$$\Rightarrow \quad h(a) = h(b) = 0 \qquad [\because y(a) = A, \; y(b) = B]$$

The corresponding increment of the functional is given by

$$\Delta J = J[y+h] - J[y]$$
$$= \int_a^b F(x, y+h, y'+h')\,dx - \int_a^b F(x, y, y')\,dx$$
$$= \int_a^b [F(x, y+h, y'+h') - F(x, y, y')]\,dx$$

Using Taylor's theorem to expand $F(x, y+h, y'+h')$, we find that

$$\Delta J = \int_a^b [F_y(x, y, y')\cdot h + F_{y'}(x, y, y')\cdot h']\,dx + \dots \qquad \text{...(i)}$$

where, the subscripts denote partial derivatives with respect to the corresponding arguments, and the dots denote terms of order higher than one relative to h and h'.

The integral on the right hand side of Eq. (i) represents the principal linear part of the increment ΔJ, and hence the variation of $J[y]$ is given by

$$\delta J = \int_a^b [F_y(x, y, y')\cdot h + F_{y'}(x, y, y')\cdot h']\,dx$$

A necessary condition for $J[y]$ to have an extremum for $y = y(x)$ is that

$$\delta J = \int_a^b (F_y \cdot h + F_{y'}\cdot h')\,dx = 0 \qquad \text{...(ii)}$$

for all admissible h.

The Eq. (ii) implies that

$$\frac{d}{dx}F_{y'} = F_y \Rightarrow F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{...(iii)}$$

Here, Eq. (iii) is known as Euler's equation.

Thus, $J[y]$ is extremum for the curve which satisfies Euler's equation.

Note *Or* Same do as above and a replace by x_1 and b replace by x_2 respectively.

Q 9. **Prove that the sphere is the solid revolution, which for a given surface area, has maximum volume.** **(2016, 13, 12, 10, 08)**

Sol. Let the surface of revolution is obtained by the revolution of the curve $y = y(x)$ about X-axis between the points $(-a, 0)$ and $(a, 0)$. Then, the volume of revolution is $J\,[y] = \int_{-a}^{a} \pi y^2 dx$

Now, we have to find the maximum of $J\,[y]$ subject to the conditions

$$y(-a) = y(a) = 0,$$
$$K\,[\bar{y}] = \int_{-a}^{a} 2\pi y\sqrt{1 + y'^2} = S \text{ (surface area)}$$

Thus, we are cleaning with an isoperimetric problem. We have the functional

$$J\,[y] = +\,\pi K[y] = \pi\int_{-a}^{a} 2\pi y\sqrt{1 + y'^2}\,dx$$

Here, integrand is $F = y^2 + 2\pi y\sqrt{1 + y'^2}$

which does not involve x explicitly.

Euler's equation is $F - y' F_{y'} = C$, where C is a constant.

So, $y^2 + 2\pi y\sqrt{1+y'^2} - y'2\pi y\dfrac{y'}{\sqrt{1+y'^2}} = C \Rightarrow y^2 + \dfrac{2\pi y}{\sqrt{1+y'^2}} = C$ ...(i)

But $y(-a) = y(a) = 0$ and $\sqrt{1+y'^2} \neq 0$, therefore $c = 0$ and the Eq. (i) gives $y + \dfrac{2\pi}{\sqrt{1+y'^2}} = 0 \Rightarrow \dfrac{4\pi^2}{1+y'^2} = y^2$

$$\Rightarrow \quad 1 + y'^2 = \frac{4\pi^2}{y^2} \quad \Rightarrow \quad y'^2 = \frac{4\pi^2 - y^2}{y^2}$$

$$\Rightarrow \quad y' = \frac{\sqrt{4\pi^2 - y^2}}{y} \Rightarrow \frac{y}{\sqrt{4\pi^2 - y^2}}\,dy = dx$$

On integrating, we get $\displaystyle\int \frac{y}{\sqrt{4\pi^2 - y^2}}\,dy = \int dx + C'$

Put $\quad 4\pi^2 - y^2 = t \Rightarrow -2ydy = dt \Rightarrow ydy = \dfrac{-dt}{2}$

$$\Rightarrow \quad \int \frac{-dt/2}{\sqrt{t}} = x + C' \Rightarrow -\sqrt{t} = x + C'$$

$$\therefore \quad -\sqrt{4\pi^2 - y^2} = x + C'$$

$$\Rightarrow \quad 4\pi^2 - y^2 = (x + C')^2 \qquad \text{...(ii)}$$

From the condition $y(-a) = y(a) = 0$, we get

$$4\pi^2 = (a + C')^2 \text{ and } 4\pi^2 = (-a + C')^2$$

which gives $C' = 0, 4\pi^2 = a^2$.

$\therefore$ Eq. (ii) becomes $x^2 + y^2 = a^2$, which is equation of a circle.

Hence, the volume of revolution is sphere, which has maximum volume of revolution for given surface area.

Q 10. Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the X-axis. **(2018, 17, 15, 11, 09, 07)**

Sol. The area of the surface of revolution generated by rotating the curve $y = y(x)$ about the X-axis is equal to

$$2\pi \int_{x_0}^{x_1} y\sqrt{(1 + y'^2)}\,dx$$

So, we want to find that curve for which the functional

$\displaystyle\int_{x_0}^{x_1} y\sqrt{(1+y'^2)}\,dx$ is extremum.

Since, the integrand does not depend explicitly on x.

The Euler's equation is $F - y' \cdot F_{y'} = C$

$$\Rightarrow \quad y\sqrt{(1+y'^2)} - y\frac{y'^2}{\sqrt{(1+y'^2)}} = C$$

$$\Rightarrow \qquad y = C\sqrt{(1+y'^2)}$$

$$\Rightarrow \qquad y' = \frac{\sqrt{y^2 - C^2}}{C} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{y^2-C^2}}{C} \Rightarrow dx = \frac{C}{\sqrt{(y^2-C^2)}}\, dy$$

On integrating, we get $x + C_1 = C\cosh^{-1}\left(\frac{y}{C}\right) \Rightarrow y = C\cosh\left(\frac{x+C_1}{C}\right)$

Hence, the required curve is a catenary passing through the two given points.

Q 11. Prove that invariance of Euler's equation under transformation of coordinates. **(2013, 09)**

Or **Explain the invariance of Euler's equation under transformation of coordinates.** **(2016, 06)**

Sol. Suppose, that instead of the rectangular plane coordinates x and y, we introduce curvilinear coordinates u and v, where $x = x(u, v)$, $y = y(u, v)$

and $$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \qquad \text{...(i)}$$

Then, the curve given by the equation $y = y(x)$ in the XY-plane corresponds to the curve given by some equation $v = v(u)$ in the UV-plane. When we make the change of variables Eq. (i), the functional $J[y] = \int_a^b F(x, y, y')\, dx$ goes into the functional

$$J_1[v] = \int_{a_1}^{b_1} F\left[x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right](x_u + x_v v')\, du = \int_{a_1}^{b_1} F_1(u, v, v')\, du$$

where, $F_1(u, v, v') = F\left[x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right](x_u + x_v v')$

We now prove that if $y = y(x)$ satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \text{...(ii)}$$

corresponding to the original functional $J[y]$, then $v = v(u)$ satisfies the Euler's equation $\frac{\partial F_1}{\partial v} - \frac{d}{du}\frac{\partial F_1}{\partial v'} = 0$ ...(iii)

corresponding to the new functional $J_1[v]$.

To prove this, we use the concept of the variational derivative. Let $\Delta\sigma$ denotes the area bounded by the curves $y = y(x)$ and $y = y(x) + h(x)$, and let $\Delta\sigma_1$ denote the area bounded by the corresponding curves $v = v(u)$ and $v = v(u) + \eta(u)$ in the UV-plane. By the standard formula for the transformation of areas, the limit as $\Delta\sigma, \Delta\sigma_1 \to 0$ of the ratio $\frac{\Delta\sigma}{\Delta\sigma_1}$ tends to the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ which by hypothesis is non-zero.

Therefore, if $\lim\limits_{\Delta\sigma\to 0}\dfrac{J[y+h]-J[y]}{\Delta\sigma}=\lim\limits_{\Delta\sigma\to 0}\dfrac{\Delta J}{\Delta\sigma}=0$, then

$$\Rightarrow \qquad \lim_{\Delta\sigma_1\to 0}\frac{J_1[v+\eta]-J_1[v]}{\Delta\sigma_1}=\lim_{\Delta\sigma_1\to 0}\frac{\Delta J_1}{\Delta\sigma_1}=\lim_{\Delta\sigma_1\to 0}\frac{\Delta J_1}{\Delta J}\frac{\Delta J}{\Delta\sigma}\frac{\Delta\sigma}{\Delta\sigma_1}=0$$

Thus, $v(u)$ satisfies Eq. (iii), if $y(x)$ satisfies Eq. (ii).
In other words, whether or not a curve is an extremal is a property which is independent of the choice of the coordinate system.

Q 12. Among all curves whose end points lie on two given vertical lines $x=a$ and $x=b$, find the curve for which the functional $J[y]=\int_a^b F(x,y,y')\,dx$ has an extremum. Also, indicate the natural boundary conditions. **(2008, 06)**

Sol. We have, $J[y]=\int_a^b F(x,y,y')\,dx$ ...(i)

The increment in the functional corresponding to an increment $h(x)$ in $y(x)$ is given by $\Delta J=J[y+h]-J[y]$

$$=\int_a^b F(x,y+h,y'+h')\,dx-\int_a^b F(x,y,y')\,dx$$

$$=\int_a^b \{F\{x,y+h,y'+h')-F(x,y,y'\}\,dx$$

By using Taylor's theorem, we get

$$\Delta J=\int_a^b \{F_y(x,y,y')\cdot h+F_{y'}(x,y,y')\cdot h'\}\,dx+\ldots$$

where, the dots denote terms of order higher than one relative to h and h'.
The variation δJ of the functional Eq. (i), being the principal linear part of the increment ΔJ is given by $\delta J=\int_a^b (F_y h+F_{y'}h')\,dx$

Here, unlike the fixed end point problem, $h(x)$ need no longer vanish at the points a and b, so that integration by parts, we get

$$\delta J=\int_a^b\left(F_y-\frac{d}{dx}F_{y'}\right)h(x)\,dx+[F_{y'}h(x)]_{x=a}^{x=b}$$

$$=\int_a^b\left(F_y-\frac{d}{dx}F_{y'}\right)h(x)\,dx+[F_{y'}]_{x=b}\,h(b)-[F_{y'}]_{x=a}\,h(a) \qquad \text{...(ii)}$$

First of all, we consider functions $h(x)$ such that $h(a)=h(b)=0$. Then, as in the fixed end point problem, the condition $\delta J=0$ implies that

$$F_y-\frac{d}{dx}F_{y'}=0 \qquad \text{...(iii)}$$

Therefore, in order for the curve $y=y(x)$ to be a solution of the variable end point problem, y must be an extremal, that is a solution of Euler's equation.

But if y is an extremal, the integral in Eq. (ii) for δJ vanishes, and then the condition $\delta J=0$ takes the form

$$[F_{y'}]_{x=b}\, h(b) - [F_{y'}]_{x=a}\, h(a) = 0$$

Since, $h(x)$ is arbitrary, therefore $[F_{y'}]_{x=a} = 0$, $[F_{y'}]_{x=b} = 0$ ...(iv)

Thus, to solve the variable end point problem, we must first find a general integral of Euler's Eq. (iii), and then use the conditions (iv) to determine the values of the arbitrary constants. Also, the condition (iv) are often called the natural boundary conditions.

***Q* 13. Find the extremals of the functional**

$$J(y(x)) = \int_{x_0}^{x_1} \frac{1+y^2}{\left(\frac{dy}{dx}\right)^2}\, dx. \qquad \textbf{(2017)}$$

Sol. We have, $J\{y(x)\} = \int_{x_0}^{x_1} \frac{1+y^2}{y'^2}\, dx$ ...(i)

Here, F is dependent on y and y' alone, so that $F = F\{y, y'\}$, i.e. F is independent of x, so that $\frac{\partial F}{\partial x} = 0$. Euler's equation is given by

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0 \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0 \qquad \left[\because \frac{\partial F}{\partial x} = 0\right]$$

On integrating, we get $F - y'\frac{\partial F}{\partial y'} = C_1$ ...(iii)

Now, $F = \frac{1+y^2}{y'^2}$, then $\frac{\partial F}{\partial y'} = -\frac{2(1+y^2)}{y'^3}$

On putting these values in Eq. (iii), we get

$$\frac{1+y^2}{y'^2} + 2y'\frac{(1+y^2)}{y'^3} = C_1 \Rightarrow \frac{(1+y^2)+2(1+y^2)}{y'^2} = C_1$$

$$\Rightarrow \quad 3(1+y^2) = C_1 y'^2 \Rightarrow \sqrt{3(1+y^2)} = \sqrt{C_1}\, y' = C_2 y' \quad [\because \sqrt{C_1} = C_2, \text{say}]$$

$$\Rightarrow \quad \sqrt{3(1+y^2)} = C_2 \frac{dy}{dx}$$

On seperating the variables, we have

$$\frac{dx}{C_2} = \frac{dy}{\sqrt{3(1+y^2)}} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{1+y^2}}\, dy$$

On integrating, we get

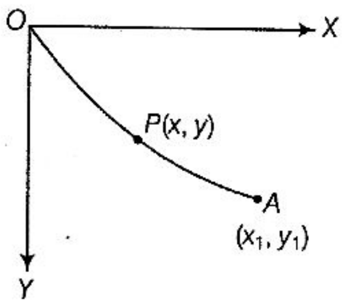
$$\frac{\sqrt{3}}{C_2} x + C_3 = \sinh^{-1} y \Rightarrow C_4 x + C_3 = \sinh^{-1} y \qquad \left[\because \frac{\sqrt{3}}{C_2} = C_4, \text{say}\right]$$

$$\therefore \quad y = \sinh(C_4 x + C_3)$$

Long Answer Questions

Q 1. Find the curve connecting two points (not on a vertical line), such that a particle sliding down this curve under gravity (in absence of resistance) from one point to another reaches in the shortest time. **(2017)**

Sol. Let the particle slide on the curve OA from O with zero velocity.
Again, let $OP = s$ and the time taken from O to $P = t$.
By the law of conservation of energy, we have
K.E. at P – K.E. at O = Potential energy at P



$$\Rightarrow \qquad \frac{1}{2}mv^2 - 0 = mgh$$

$$\Rightarrow \qquad \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mgy \Rightarrow \frac{ds}{dt} = \sqrt{2gy}$$

Time taken by the particle to move from O to A,

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}}\int_0^{x_1}\frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{2g}}\int_0^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx$$

Here, $f = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$, which is independent of x, i.e. $\dfrac{\partial f}{\partial x} = 0$

and $\dfrac{\partial f}{\partial y'} = \dfrac{1}{2\sqrt{y}}\dfrac{2y'}{\sqrt{1+y'^2}} = \dfrac{y'}{\sqrt{y}\sqrt{1+y'^2}}$.

The solution of Euler's equation is

$$f - y'\frac{\partial f}{\partial y'} = \text{constant } c \Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{y'}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \qquad \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} = c\sqrt{y} \Rightarrow 1 + y'^2 - y'^2 = c\sqrt{1+y'^2}\,\sqrt{y}$$

$$\Rightarrow \qquad 1 = c\sqrt{y(1+y'^2)}$$

$$\Rightarrow \quad 1+\left(\frac{dy}{dx}\right)^2=\frac{1}{yc^2} \Rightarrow \frac{dy}{dx}=\sqrt{\frac{1-yc^2}{yc^2}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\sqrt{\frac{\frac{1}{c^2}-y}{y}}=\frac{\sqrt{a-y}}{\sqrt{y}} \qquad \left[\frac{1}{c^2}=a \text{ (say)}\right]$$

$$\Rightarrow \quad dx=\sqrt{\frac{y}{a-y}}\,dy \Rightarrow \int_0^x dx=\int_0^y \sqrt{\frac{y}{a-y}}\,dy$$

Put $y=a\sin^2\theta \Rightarrow dy=2a\sin\theta\cos\theta\,d\theta$

$$\therefore x=\int_0^\theta \sqrt{\left(\frac{a\sin^2\theta}{a-a\sin^2\theta}\right)}\,2a\sin\theta\cos\theta\,d\theta=\int_0^\theta \frac{\sin\theta}{\cos\theta}\times 2a\sin\theta\cos\theta\,d\theta$$

$$=\int_0^\theta 2a\sin^2\theta\,d\theta=a\int_0^\theta (1-\cos 2\theta)\,d\theta=a\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^\theta$$

$$\Rightarrow x=\frac{a}{2}(2\theta-\sin 2\theta) \text{ and } y=a\sin^2\theta=\frac{a}{2}(1-\cos 2\theta)$$

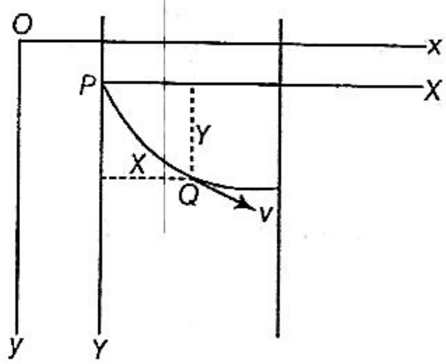
Again, put $\frac{a}{2}=A$ and $2\theta=\phi$

$\therefore \quad x=A(\phi-\sin\phi)$ and $y=A(1-\cos\phi)$, which is a cycloid.

Q 2. Find the curve connecting two points (not on a vertical line) such that a particle sliding down this curve under gravity (in absence of resistance) from one point to another reaches in the shortest time. **(2011)**

Or **State and prove Brachistochrone problem.** **(2017)**

Sol. **Statement** Starting from the point $P(a, A)$, where $y=y(x)$ and $y(a)=A$, a heavy particle slides down a curve in the vertical plane to find the curve such that the particle reaches the vertical line $x=b\ (\neq a)$ in the shortest time.



Proof With respect to the fixed coordinate axes Ox and Oy, the starting point P has coordinates (a, A). Let us choose the point P as new origin and PX, PY as new coordinate axes, which are parallel to old ones, then we have

$$\left.\begin{aligned} x&=X+a \\ y&=Y+A \end{aligned}\right\} \qquad \text{...(i)}$$

where, (x, y) and (X, Y) are the coordinates of the same variable point Q with respect to the coordinate axes Oxy and PXY, respectively.

The velocity of the particle at Q, after sliding an arc s in time t, is given by

$$v = \frac{ds}{dt} = \frac{ds}{dX}\frac{dX}{dt} = \sqrt{(1+Y'^2)}\,\frac{dX}{dt} \Rightarrow dt = \frac{\sqrt{(1+Y'^2)}}{v}\,dX \qquad \text{...(ii)}$$

But the velocity v is the particle after falling a distance Y is given by

$$v^2 = 2gY \qquad \text{...(iii)}$$

Using Eq. (iii) in Eq. (ii), we get

$$dt = \frac{\sqrt{(1+Y'^2)}}{\sqrt{2gY}}\,dX$$

Thus, the total time T is given by

$$T = \int_0^{b-a} \frac{\sqrt{(1+Y'^2)}}{\sqrt{2gY}}\,dX \qquad \text{...(iv)}$$

We have to find the curve for which T is minimum.

Here, $F(x, y, y') = \dfrac{\sqrt{(1+Y'^2)}}{\sqrt{2gY}}$, which does not contain X explicitly.

So, the first integral of Euler's equation is $F - Y' F_{Y'} = C$

$$\Rightarrow \qquad \frac{\sqrt{(1+Y'^2)}}{\sqrt{2gY}} - Y'\frac{Y'}{\sqrt{2gY}\,\sqrt{(1+Y'^2)}} = C$$

$$\Rightarrow \qquad \frac{1}{\sqrt{2gY}\,\sqrt{(1+Y'^2)}} = C \Rightarrow \sqrt{1+Y'^2} = \frac{1}{C\sqrt{2gy}}$$

On squaring both the sides, we get

$$1 + Y'^2 = \frac{1}{2C^2gY} \Rightarrow Y'^2 = \frac{1}{2C^2gY} - 1$$

$$\Rightarrow \qquad Y'^2 = \frac{1-2C^2gY}{2C^2gY} \Rightarrow Y'^2 = \frac{K-Y}{Y}, \text{ where } K = \frac{1}{2C^2g}$$

Now, $$\left(\frac{dY}{dX}\right)^2 = \frac{K-Y}{Y}$$

$$\therefore \qquad \frac{dY}{dX} = \frac{\sqrt{(K-Y)}}{\sqrt{Y}} \Rightarrow dX = \frac{\sqrt{Y}}{\sqrt{(K-Y)}}\,dY$$

On integrating, we get

$$X + C_1 = \int \frac{\sqrt{Y}}{\sqrt{K-Y}}\,dY = \int \frac{\sqrt{K}\sin\theta\, 2K\sin\theta\cos\theta}{\sqrt{K}\cos\theta}\,d\theta$$

[put $Y = K\sin^2\theta$]

$$= K\int (2\sin^2\theta)\,d\theta = K\int (1-\cos 2\theta)\,d\theta = K\left[\theta - \frac{\sin 2\theta}{2}\right]$$

$$= \frac{K}{2}(2\theta - \sin 2\theta) = \frac{K}{2}(\phi - \sin\phi) \qquad [\text{put } 2\theta = \phi]$$

Thus, $$X = \frac{K}{2}(\phi - \sin\phi) - C_1 \quad \text{...(v)}$$

and $$Y = K\sin^2\theta = \frac{K}{2}(1 - \cos 2\theta) = \frac{K}{2}(1 - \cos\phi) \quad \text{...(vi)}$$

Initially, when $X = 0$, then we have $Y = 0$.
But $Y = 0$, then $\theta = 0 \Rightarrow \phi = 0$.

On putting $X = 0$, $\phi = 0$ in the Eq. (v), we get $C_1 = 0$
Thus, Eqs. (v) and (vi), becomes

and $$\left.\begin{aligned} X &= \frac{K}{2}(\phi - \sin\phi) \\ Y &= \frac{K}{2}(1 - \cos\phi) \end{aligned}\right\} \quad \text{...(vii)}$$

The boundary condition is $[F_{Y'}]_{X = b - a} = 0$, which gives

$$\frac{Y'}{\sqrt{2gY}\sqrt{(1 + Y'^2)}} = 0 \Rightarrow Y' = 0 \Rightarrow \frac{dY}{dX} = 0$$

which shows that the tangent to the curve at $X = b - a$ is parallel to the X-axis and hence $\phi = \pi$ at $X = b - a$.

This is obvious from the fact that $\dfrac{dY}{dX} = \dfrac{dY/d\phi}{dX/d\phi} = \dfrac{\sin\phi}{1 - \cos\phi}$

Thus, Eq. (vii) gives

$$b - a = \frac{K}{2}(\pi - 0) \Rightarrow \frac{K}{2} = \frac{b - a}{\pi}$$

On putting value of $\dfrac{K}{2}$ in Eq. (vii), we get

and $$\left.\begin{aligned} X &= \frac{b - a}{\pi}(\phi - \sin\phi) \\ Y &= \frac{b - a}{\pi}(1 - \cos\phi) \end{aligned}\right\} \quad \text{...(viii)}$$

Eq. (viii) represents an inverted cycloid with its base along X-axis and its cusp at P. Referred to the original axes, the Eq. (viii) becomes

and $$\left.\begin{aligned} x - a &= \frac{b - a}{\pi}(\phi - \sin\phi) \\ y - A &= \frac{b - a}{\pi}(1 - \cos\phi) \end{aligned}\right\}$$

which is the required equations of the curve.

Q 3. Prove that inveriance of Euler's equation under the transformation of Coordinates. **(2019)**

Sol. For solution refer to Q.11 (S.A.).

Q 4. Prove that the sphere is the solid revolution, which for a given surface area, has maximum volume. **(2019)**

Sol. For solution refer to Q.9 (S.A.).

Q 5. Establish necessary condition for $\int_{x_1}^{x_2} f(x, y, y')dx$ to be maximum or minimum (Extremal) **(2019)**

Sol. For solution refer to Q. 8 (S.A.).

Q 6. Find the stationary value (Extremal) of the functional $\int_1^2 \frac{(1+y'^2)^{1/2}}{x} dx$ where $y(1) = 0,\ y(2) = 1$ **(2019)**

Sol. For solution refer to Q.5 (S.A.).