Chapter Seven

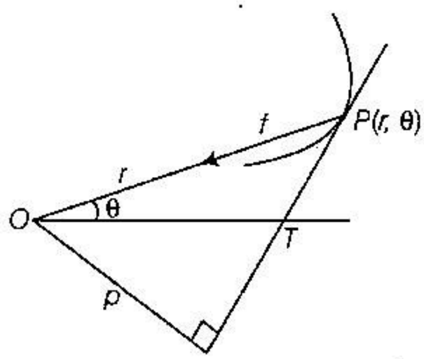
CENTRAL ORBITS

Important Points from the Chapter

1. **Central Force** A force whose line of action always passes through a fixed point is called a central force. The fixed point is called as the centre of force. **(2018, 15, 13)**
2. **Central Orbit** A central orbit is the path described by a particle moving under the action of central force. The motion of a planet about the Sun is an important example of a central orbit.
3. **Differential Equation of a Central Orbit** A particle moves in a plane with an acceleration which is always directed towards a fixed point O in the plane, then the differential equation of central orbit.

   (i) **In polar form** $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2}$

   where, $u = \frac{1}{r}$, F is central acceleration of moving particle, $h = \frac{1}{u^2}\frac{d\theta}{dt}$ and $P(r, \theta)$ be the position of a moving particle at any time t.

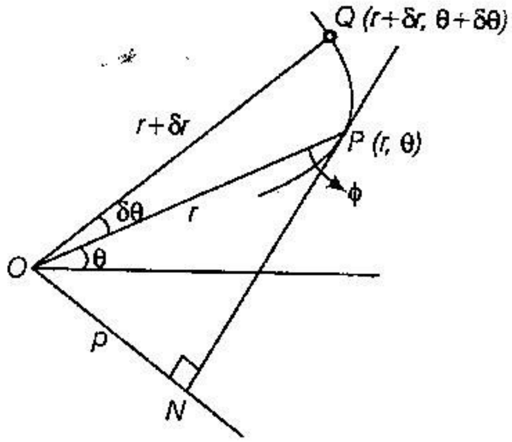


   (2015, 12, 09, 06, 05, 04, 02, 1996, 93, 91)

   (ii) **In pedal form** $F = \frac{h^2}{p^3}\frac{dp}{dr}$, where p is the length of perpendicular form the origin (pole) to the tangent at $P(r, \theta)$. **(2009, 08, 03, 01)**
4. **Sectorial Area** The area traced out by the radius vector to the centre of force is called sectorial area.
5. **Aerial Velocity** When a particle moves in a plane, the rate of description of sectorial area is called the aerial velocity of the particle about the fixed point.
6. **Rate of Description of Sectorial Area** The aerial velocity of the particle about the fixed point O is called the rate of description of sectorial area. Its value is contant and is equal to $h/2$, i.e.

   (i) sectorial area described by the particle increases uniformly.

(ii) $v \propto \frac{1}{p}$, i.e. the linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.



7. **Linear Velocity at any Point of the Path of a Central Orbit** The linear velocity at any point of the path of a central orbit is

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

8. **Apse** An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a maximum or minium value. **(2017, 14)**

9. **Apsidal Distance** The length of the radius vector at an apse is called an apsidal distance. **(2017)**

10. **Apsidal Angle** The angle between two consecutive apsidal distances is called an apsidal angle.

    At an apse the radius vector is perpendicular to the tangent, $p = r$.

11. **Velocity in a Circle** If v be the velocity of a particle in a circle at a distance $r = a$ from the centre at right angle to the radius vector r, then

$$v^2 = af(a) \text{ where } F = f(a).$$

12. **Velocity from Infinity** If a particle is falling from infinity to $r = a$ under an attractive force F towards the centre of force then acquired velocity in falling from infinity to $r = a$ is the velocity from infinity, thus $v^2 = -2\int_0^a f(r)dr$.

13. **Velocity of Fall to the Point of Projection** If the particle falls from the centre of repulsion under a force F to a point at a distance r from the centre, then $v^2 = 2\int_0^r f(r)\, dr$.

14. **Time in a path** The time t of passing from P to another point Q of a central orbit (path) is given by $t = \frac{1}{h}\int_a^b r^2\, d\theta$.

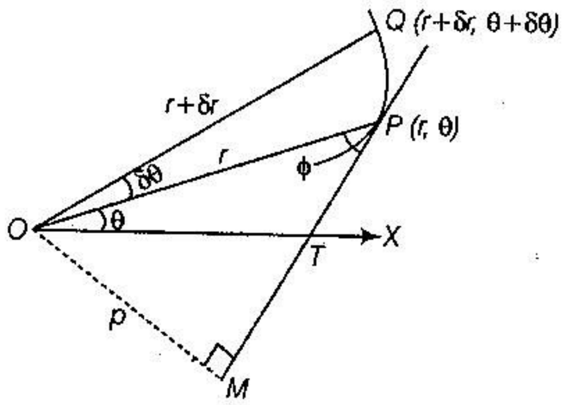
Very Short Answer Questions

Q 1. Write down the equation of central orbit in pedal form. (2016)

Sol. The differential equation of central orbit in pedal form is $F = \frac{h^2}{p^3}\frac{dp}{dr}$, where F is an acceleration towards a fixed point O, p is the length of perpendicular from the origin (pole) to the tangent at $P(r, \theta)$ and h is any constant equal to $r^2 \frac{d\theta}{dt}$.

Q 2. Prove that the rate of description of sectional area in a central orbit is constant. (2013)

Sol. Take the centre of force O as the pole and OX as the initial line. Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be the positions of a particle moving in a central orbit at times t and $(t + \delta t)$, respectively.



Sectional area OPQ described by the particle in time δt

$= \text{Area of the } \Delta OPQ$

[∵ point Q is very closed to P and ultimately we have to take limit as $Q \to P$]

$$= \frac{1}{2} OP \cdot OQ \sin \angle POQ = \frac{1}{2} r(r + sr) \sin \delta\theta$$

∴ Rate of description of the sectorial area

$$= \lim_{\delta t \to 0} \frac{\text{Sectorial area } OPQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r(r + \delta r)\sin \delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{2} r(r + \delta r) \cdot \frac{\sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}$$

Hence, the rate of description of the sectorial area is constant and equal to $h/2$.

Hence proved.

***Q* 3. Write down the equation of the central orbit in polar form.**

Sol. If a particle $P\ (r, \theta)$ is moving in a plane with an acceleration F which is directed towards a fixed point O in the plane. Then, the differential equation of its path is $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$, where $u = \frac{1}{r}$

***Q* 4. Find the time t of a particle P in a path, passing from one point P to another point.**

Sol. The time of passing from one point P to another point Q of a central orbit (path) is obtained from the relation

$$r^2 \frac{d\theta}{dt} = h \Rightarrow r^2\, d\theta = h\, dt$$

On integrating between the proper limits, we get

$$ht = \int_a^b r^2\, d\theta \Rightarrow t = \frac{1}{h}\int_a^b r^2\, d\theta$$

Short Answer Questions

***Q* 1. Show that the rate of description of sectorial area in a central orbit is constant and velocity varies inversely as the perpendicular from centre to the tangent.** **(2018, 16)**

Sol. **Part I** See the solution of Q. 2 of Very Short Answer Questions.

Part II For a central orbit, $r^2 \frac{d\theta}{dt} = h$

$$\therefore \qquad r^2 \frac{d\theta}{ds}\frac{ds}{dt} = h \Rightarrow r^2 \frac{d\theta}{ds} \cdot v = h \qquad \ldots\text{(i)}$$

But from the differential calculus, we know that $r \frac{d\theta}{ds} = \sin\phi$, where ϕ is the angle between the radius vector and tangent.

$$\therefore \qquad r^2 \frac{d\theta}{ds} = r \sin\phi = p$$

where, p is length of the perpendicular drawn from the pole O on the tangent at P.

On putting $r^2 \left(\frac{d\theta}{ds}\right) = p$ in Eq. (i), we get

$$vp = h \Rightarrow v = \frac{h}{p}$$

$$\therefore \qquad v \propto \frac{1}{p}$$

i.e. the linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.

Q 2. A particle moves with a central acceleration which varies inversely as cube of the distance. If it be projected from apse at a distance a from the origin with a velocity $\sqrt{2}$ times the the velocity for a circle of radius a, show that its path is $r\cos(\theta/\sqrt{2}) = a$. **(2016, 14, 12, 06, 02, 1992)**

Sol. Here, the central acceleration varies inversely as the cube of the distance, i.e. $F = \mu/r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [F]_{r=a} = \mu/a^3$$

$$\Rightarrow \quad V = \sqrt{(\mu/a^2)}$$

$\therefore$ Velocity of the projection, $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{F}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

On multiplying both the sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \qquad \text{[where, } A \text{ is a constant.]} \quad ...(i)$$

But initially, when $r = a$, i.e. $u = \frac{1}{a}, \frac{du}{d\theta} = 0$ (at an apse)

and $\quad v = v_1 = \sqrt{(2\mu/a^2)}$

From Eq. (i), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$$\therefore \quad h^2 = 2\mu \text{ and } A = \frac{\mu}{a^2}$$

On putting the values of h^2 and A in Eq. (i), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$\Rightarrow \quad 2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

$$\Rightarrow \quad \sqrt{2}a\frac{du}{d\theta} = \sqrt{1 - a^2u^2} \Rightarrow \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}$$

On integrating, we get

$$\frac{\theta}{\sqrt{2}} + B = \sin^{-1}(au), \text{ where } B \text{ is a constant.}$$

But initially, when $u = 1/a, \theta = 0$

$$\therefore \qquad B = \sin^{-1} 1 = \frac{1}{2}\pi$$

Hence, $(\theta/\sqrt{2}) + \frac{1}{2}\pi = \sin^{-1}(au) \Rightarrow au = \frac{a}{r} = \sin\left\{\frac{\pi}{2} + \left(\frac{\theta}{\sqrt{2}}\right)\right\}$

$\Rightarrow \quad a = r\cos\left(\frac{\theta}{\sqrt{2}}\right)$, which is the required equation of path.

Q 3. Find the law of force towards the pole, under which the curve $r = \dfrac{2a}{1+\cos\theta}$ can be described. **(2014, 02)**

Sol. The equation of the curve is $r = \dfrac{2a}{1+\cos\theta}$.

On replacing r by $\frac{1}{u}$, we get

$$\frac{1}{u} = \frac{2a}{1+\cos\theta} \Rightarrow u = \frac{1+\cos\theta}{2a}$$

On differentiating w.r.t. θ, we get

$$\frac{du}{d\theta} = \frac{1}{2a}(-\sin\theta) \Rightarrow \quad \frac{du}{d\theta} = -\frac{\sin\theta}{2a}$$

On differentiating again w.r.t. θ, we get

$$\frac{d^2u}{d\theta^2} = -\frac{\cos\theta}{2a}$$

$$\Rightarrow \qquad u + \frac{d^2u}{d\theta^2} = \frac{1+\cos\theta}{2a} - \frac{\cos\theta}{2a} = \frac{1}{2a}$$

But differential equation of the path is

$$F = h^2 u^2\left[u + \frac{d^2u}{d\theta^2}\right] = h^2 u^2 \cdot \frac{1}{2a} = \frac{h^2}{2a} \cdot \frac{1}{r^2}$$

Thus, $\qquad F \propto \dfrac{1}{r^2}$

i.e. force varies inversely as second power of the distance from the pole.

Q 4. Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described. **(2013, 06, 05, 03)**

Sol. Given equation of the curve is $r^n = a^n \cos n\theta$.

On replacing r by $\frac{1}{u}$, we get

$$\frac{1}{u^n} = a^n \cos n\theta \Rightarrow a^n u^n = \sec n\theta$$

On taking log both the sides, we get

$$n\log a + n\log u = \log\sec n\theta$$

On differentiating w.r.t. θ, we get

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} \times n \sec n\theta \cdot \tan n\theta .$$

$$\Rightarrow \qquad \frac{du}{d\theta} = u \tan n\theta$$

On differentiating again w.r.t. θ, we get

$$\frac{d^2u}{d\theta^2} = u \sec^2 n\theta \cdot n + \tan n\theta \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + \tan n\theta \cdot u \tan n\theta$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta$$

The differential equation of the central orbit is

$$F = h^2 u^2 \left[u + \frac{d^2u}{d\theta^2}\right]$$

$$\therefore \qquad F = h^2 u^2 \,[u + nu \sec^2 n\theta + u \tan^2 n\theta]$$

$$= h^2 u^3 \,[\sec^2 n\theta + n \cdot \sec^2 n\theta]$$

$$= h^2 u^3 \,(1 + n) \sec^2 n\theta = h^2 (1+n) u^3 \,(a^n u^n)^2$$

$$= h^2 a^{2n} \cdot (1+n)\, u^{2n+3} = \frac{h^2 a^{2n}(1+n)}{r^{2n+3}}$$

$$\therefore \qquad F \propto \frac{1}{r^{2n+3}}$$

i.e. the force varies inversely as the $(2n + 3)$th power of the distance from the pole.

***Q* 5.** **A particle describes the equiangular spiral $r = ae^{\theta \cot \alpha}$ under a force to the pole. Find the law of force.** **(2007)**

Sol. Given equation of the curve is $r = ae^{\theta \cot \alpha}$

$$\Rightarrow \qquad u = \frac{1}{a} e^{-\theta \cot \alpha} \qquad \left[\text{put } r = \frac{1}{u}\right] \ldots\text{(i)}$$

On differentiating w.r.t. θ, we get

$$\frac{du}{d\theta} = -\frac{1}{a} \cot\alpha \; e^{-\theta \cot \theta} = -u \cot\alpha$$

and

$$\frac{d^2u}{d\theta^2} = -\cot\alpha \cdot \frac{du}{d\theta} = u \cot^2\alpha$$

Now,

$$u + \frac{d^2u}{d\theta^2} = u + u \cot^2\alpha = u\,(1 + \cot^2\alpha) = u \operatorname{cosec}^2\alpha$$

But

$$F = h^2 u^2 \left[u + \frac{d^2u}{d\theta^2}\right] = h^2 u^2 \,(u \operatorname{cosec}^2\alpha) = \frac{h^2 u^3}{\sin^2\alpha}$$

$$= \frac{h^2}{\sin^2\alpha} u^3 = \frac{h^2}{\sin^2\alpha} \cdot \frac{1}{r^3} \Rightarrow F \propto \frac{1}{r^3}$$

Q 6. Find the law of force towards the pole under which the curve $r = \frac{a}{2}(1 + \cos\theta)$ can be described. **(2018, 15, 1996)**

Sol. Do same as Q. 3.

Ans. $F \propto \frac{1}{r^4}$, i.e. force varies inversely as the fourth power of the distance from pole.

Q 7. A particle moves in an ellipse under a force which is directed towards its focus, find the law of force and velocity at any point on the path. **(2015, 12, 02, 1993, 90)**

Or **A particle moves, under a force which is always directed towards focus, in an ellipse. Find the law of force.** **(2017)**

Sol. $\because$ Equation to the ellipse with focus as pole is $\frac{l}{r} = 1 + e\cos\theta$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{l} + \frac{e\cos\theta}{l} \Rightarrow u = \frac{1}{l} + \frac{e\cos\theta}{l}$$

$$\therefore \quad \frac{du}{d\theta} = -\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

Hence,
$$F = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right] = h^2u^2\left[\frac{1}{l} + \frac{e\cos\theta}{l} - \frac{e\cos\theta}{l}\right]$$

$$= \frac{h^2u^2}{l} = \frac{\mu}{r^2} \qquad \left[\text{where, } \mu = \frac{h^2}{l}\right]$$

i.e. $\quad h = \sqrt{\mu l}$

$$\therefore \quad F \propto \frac{1}{r^2}$$

Thus, the central force varies inversely as the square of the distance from the focus.

Also,
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(-\frac{e}{l}\sin\theta\right)^2\right]$$

$$= \mu l\left[\frac{1}{l^2} + \frac{e^2}{l^2}(\cos^2\theta + \sin^2\theta) + \frac{2e}{l^2}\cos\theta\right]$$

$$= \mu\left[\frac{2}{l} - \frac{1}{l} + \frac{e^2}{l} + \frac{2e}{l}\cos\theta\right]$$

$$= \mu\left[\frac{2}{l}(1 + e\cos\theta) - \frac{1}{l}(1 - e^2)\right] = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

as $\quad l = \frac{b^2}{a} = a(1 - e^2) \quad$ [where, $2a$ is major axis of the ellipse]

Q 8. Prove that if the system is conservative, then

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{h^2}. \qquad \textbf{(2017)}$$

Sol. By the principle of conservation of energy

$$\text{K.E.} + \text{P.E.} = \text{E (constant)}$$

or $$\frac{1}{2}mv^2 + V = E$$

or $$\frac{1}{2}mv^2 = E - V$$

$\Rightarrow$ $$mv^2 = 2(E-V) \Rightarrow m(\dot{r}^2 + r^2\dot{\theta}^2) = 2(E-V) [\because \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}]$$

$\Rightarrow$ $$m\left[\left(h\frac{du}{d\theta}\right)^2 + \frac{r^4}{r^2}\dot{\theta}^2\right] = 2(E-V) \qquad \left[u = \frac{1}{r}\right]$$

$\Rightarrow$ $$m\left[h^2\left(\frac{du}{d\theta}\right)^2 + \frac{h^2}{r^2}\right] = 2(E-V)$$

$\Rightarrow$ $$m\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2(E-V)}{h^2} \qquad \ldots\text{(i)}$$

For unit mass Eq. (i) becomes $\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2(E-V)}{h^2}$ **Hence proved.**

Long Answer Questions

Q 1. Obtain the differential equation of central orbit. (2018, 16)

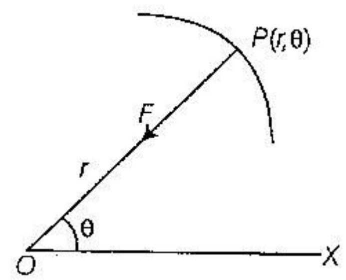
Or **Find the equation of the central orbit in polar coordinates (r, θ).** (2014, 07)

Or **Prove that the differential equation of a central orbit is**

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}. \qquad \textbf{(2006, 02)}$$

Sol. Let a particle moves in a plane with an acceleration F which is always directed to a fixed point O in the plane. Taking the centre of force O as the pole.

Again, let OX be the initial line and (r, θ) the polar coordinates of the position P of the moving particle at any instant t. Since, the acceleration of the particle is always directed towards the pole O, therefore the particle has only the radial acceleration and the transverse component of the acceleration of the particle is always zero.



So, the equations of motion of the particle at point P are the radial acceleration

$$\Rightarrow \quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F \quad \text{...(i)}$$

[$\because$ – ve sign has been taken, because the radial acceleration F is in the direction of r decreasing]

and the transverse acceleration, i.e. $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ ...(ii)

From Eq. (ii), we have $\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$

On integrating, we get $r^2\frac{d\theta}{dt} = \text{constant} = h$ (say) ...(iii)

Let $\quad r = \frac{1}{u}.$

Now, from Eq. (iii), we have

$$\frac{d\theta}{dt} = \frac{h}{r^2} = h\,u^2$$

Also, $\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{1}{u^2}\cdot\frac{du}{d\theta}\cdot u^2h = -h\frac{du}{d\theta}$

and $\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\cdot(u^2h) = -h^2u^2\frac{d^2u}{d\theta^2}$

On putting these values in Eq. (i), we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(u^2h)^2 = -F$$

$$\Rightarrow \quad h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = F$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

which is differential equation of a central orbit in polar form referred to the centre of force as the pole.

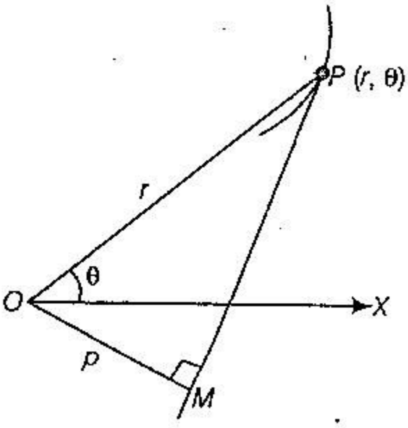
Q 2. Obtain the differential equation of central orbit in pedal form. **(2016)**

Or **Derive equation of central orbit in terms of *p* and *r*.** **(2017)**

Sol. Let p be the length of the perpendicular drawn from the origin upon the tangent at the point P, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

But $u = \frac{1}{r}$, therefore



$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\Rightarrow \quad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\therefore \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \qquad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. θ, we get

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}\cdot\frac{F}{h^2u^2} \qquad \left[\because \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}\right]$$

$$\Rightarrow \quad -\frac{1}{p^3}\frac{dp}{dr}\cdot\frac{dr}{d\theta} = \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{F}{h^2u^2}\right)$$

$$\Rightarrow \quad \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^2}\cdot\frac{F}{h^2u^2} = u^2\cdot\frac{F}{h^2u^2} = \frac{F}{h^2}$$

$$\therefore \quad F = \frac{h^2}{p^3}\frac{dp}{dr}$$

which is required differential equation of a central orbit in pedal form or in terms of p and r.

***Q* 3. A particle moves in a plane with an acceleration which is always directed towards a fixed point. Find the equation of central orbit.** **(2012, 1996, 93, 91)**

Sol. See the solution of Q. 1 and 2.

***Q* 4. A particle subject to an acceleration $\mu\cdot\frac{(r+2a)}{r^5}$ towards origin, is projected from the point $(a, 0)$ with a velocity equal to the velocity from infinity at an angle $\cot^{-1} 2$ with initial line. Show that path is $r = a(1 + 2\sin\theta)$** **(2015)**

Sol. Here, the central acceleration is

$$F = \frac{\mu(r+2a)}{r^5} = \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right) = \mu(u^4 + 2au^5)$$

Let V be the velocity of the particle acquired in falling from rest from infinity under the same acceleration to the point of projection which is at a distance 'a' from the centre. Then,

$$V^2 = -2\int_\infty^a F\,dr = -2\int_\infty^a \mu\left[\frac{1}{r^4} + \frac{2a}{r^5}\right]dr$$

$$= -2\mu\left[-\frac{1}{3r^3} - \frac{2a}{4r^4}\right]_{\infty}^{a} = 2\mu\left[\frac{1}{3a^3} + \frac{1}{2a^3}\right] = \frac{5\mu}{3a^3}$$

$\Rightarrow \qquad V = \sqrt{\frac{5\mu}{3a^3}}$

According to the question, the velocity of projection of the particle is equal to V, i.e. $\sqrt{\frac{5\mu}{3a^3}}$.

Now, the differential equation of the path is

$$h^2\left[u^2 + \frac{d^2u}{d\theta^2}\right] = \frac{F}{u^2} = \frac{\mu}{u^2}(u^4 + 2au^5) = \mu(u^2 + 2au^3)$$

On multiplying both the sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2u^3}{3} + au^4\right) + A \qquad \text{...(i)}$$

where, A is a constant.

But initially, when $r = a$, i.e. $u = \frac{1}{a}$, $v = \sqrt{\frac{5\mu}{3a^3}}$

Also, initially $\phi = \cot^{-1} 2 \Rightarrow \cot\phi = 2 \Rightarrow \sin\phi = \frac{1}{\sqrt{5}}$

But $\quad p = r\sin\phi$

Therefore, initially $p = a\left(\frac{1}{\sqrt{5}}\right) = \frac{a}{\sqrt{5}} \Rightarrow \frac{1}{p^2} = \frac{5}{a^2}$

But $\qquad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Therefore, initially when $r = a$, we have $u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{5}{a^2}$

Applying the above initial conditions in Eq. (i), we have

$$\frac{5\mu}{3a^3} = h^2\frac{5}{a^2} = \mu\left(\frac{2}{3a^3} + \frac{a}{a^4}\right) + A$$

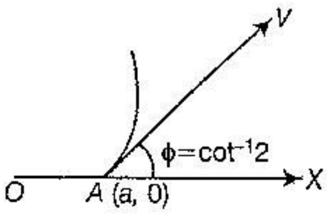
$\therefore \qquad h^2 = \frac{\mu}{3a}, \; A = 0$

On putting the values of h^2 and A in Eq. (i), we have

$$\frac{\mu}{3a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 + au^4\right)$$

$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2$

Put $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$



$$\therefore \quad \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$$

$$\Rightarrow \quad \left(\frac{dr}{d\theta}\right)^2 = 2ar + 3a^2 - r^2$$

$$= 3a^2 - (r^2 - 2ar) = 3a^2 - (r-a)^2 + a^2$$

$$= 4a^2 - (r-a)^2$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \sqrt{(2a)^2 - (r-a)^2}$$

[$\because \frac{dr}{d\theta}$ has been taken with positive sign as the particle starts moving from A, r increases as θ increases]

$$\therefore \quad d\theta = \frac{dr}{\sqrt{(2a)^2 - (r-a)^2}}$$

On integrating, we get

$$\theta + B = \sin^{-1}\frac{r-a}{2a}$$

But intially, when $r = a$, then $\theta = 0$

$$\therefore \quad B = \sin^{-1} 0 = 0$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{r-a}{2a}\right) \Rightarrow \sin\theta = \frac{r-a}{2a}$$

$$\Rightarrow \quad r = a + 2a\sin\theta \Rightarrow r = a(1 + 2\sin\theta)$$

which is the required equation of the path.

***Q* 5. A particle is moving with central acceleration $\mu\,(r^5 - c^4 r)$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}\,c^3$, show that its path is the curve $x^4 + y^4 = c^4$.** **(2001)**

Sol. Here, the central acceleration is

$$F = \mu\,(r^5 - c^4 r) = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)$$

$\therefore$ The differential equation of the path is

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{F}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u^5} - \frac{c^4}{u}\right) = \mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)$$

On multiplying both the sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right) + A \quad \text{...(i)}$$

where, A is a constant.

But initially, when $r = c$, i.e. $u = \frac{1}{c}$, $\frac{du}{d\theta} = 0$ (at an apse) and $v = c^3\sqrt{\left(\frac{2\mu}{3}\right)}$

$\therefore$ From Eq. (i), we have

$$\frac{2\mu c^6}{3} = h^2 \cdot \frac{1}{c^2} = \mu\left(-\frac{c^6}{3} + c^6\right) + A$$

$$\therefore \quad h^2 = \frac{2}{3}\mu c^8, \quad A = 0$$

On putting the values of h^2 and A in Eq. (i), we have

$$\frac{2}{3}\mu c^8\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right)$$

$$\Rightarrow \quad c^8\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6} + \frac{3c^4}{2u^2} - c^8u^2$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} + \frac{3}{2}c^4u^4 - c^8u^8\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left\{c^8 \cdot u^8 - \frac{3}{2}c^4u^4\right\}\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left\{c^4u^4 - \frac{3}{4}\right\}^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left\{c^4u^4 - \frac{3}{4}\right\}^2\right]$$

$$\therefore \quad c^4u^3\frac{du}{d\theta} = \sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}$$

$$\Rightarrow \quad d\theta = \frac{c^4u^3\,du}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}}$$

Put $c^4u^4 - \frac{3}{4} = z \Rightarrow 4c^4u^3\,du = dz$

$$\therefore \quad 4\,d\theta = \frac{dz}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - z^2\right]}}$$

On integrating, we get

$$4\theta + B = \sin^{-1}\left(\frac{z}{1/4}\right) = \sin^{-1}(4z) \qquad \text{[where, } B \text{ is a constant]}$$

$$\Rightarrow \quad 4\theta + B = \sin^{-1}[4c^4u^4 - 3]$$

But initially, when $u = \frac{1}{c}, \theta = 0$

$$\therefore \quad B = \sin^{-1} 1 = \frac{\pi}{2}$$

$$\therefore \quad 4\theta + \frac{\pi}{2} = \sin^{-1}(4c^4u^4 - 3)$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{2} + 4\theta\right) = 4c^4u^4 - 3$$

$$\Rightarrow \quad \cos 4\theta = 4c^4u^4 - 3 \text{ or } 4c^4u^4 = 3 + \cos 4\theta$$

$$\Rightarrow \quad 4c^4/r^4 = 3 + \cos 4\theta$$

$$\Rightarrow \quad 4c^4 = r^4\,[3 + (2\cos^2 2\theta - 1)] = 2r^4\,[1 + \cos^2 2\theta]$$

$$= 2r^4\,[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2]$$

$$= 4r^4[\cos^4\theta + \sin^4\theta]$$

$$\therefore \quad c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$$

$$\Rightarrow \quad c^4 = x^4 + y^4 \qquad [\because x = r\cos\theta,\ y = r\sin\theta]$$

which is the required equation of the path.

Q 6. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^3}$. Find the path in different possible cases.

(2004)

Sol. Given that, $F = \frac{\mu}{r^3}$ or $F = \mu u^3$

$\therefore$ The equation of the central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2u^2} \Rightarrow \frac{d^2u}{d\theta^2} = \left(\frac{\mu}{h^2} - 1\right)u \qquad \text{...(i)}$$

***Case* I** Let $h^2 < \mu$, so that

$$\frac{\mu}{h^2} - 1 = k^2 \text{ (positive)}$$

$$\Rightarrow \quad \frac{d^2u}{d\theta^2} = k^2u \text{ or } (D^2 - K^2)\,u = 0$$

Its general solution is $u = Ae^{k\theta} + Be^{-k\theta}$

where, A and B are arbitrary constants.

This is a spiral curve with an infinite number of convolutions about pole.

In the particular case, when A and B vanishes, it is an equiangular spiral.

***Case* II** Let $h^2 = \mu$, so that Eq. (i) becomes $\frac{d^2u}{d\theta^2} = 0$. Its solution is $u = A\theta + B$, where A and B are arbitrary constants.

This represents a reciprocal spiral in general. In the particular case, when $A = 0$, it is a circle.

***Case* III** Let $h^2 > \mu$, so that $\left(\frac{\mu}{h^2} - 1\right)$ is negative.

$$\therefore \qquad \frac{\mu}{h^2} - 1 = -k^2$$

Therefore, Eq. (i) becomes

$$\frac{d^2u}{d\theta^2} = -k^2u$$

Its solution is $u = A \cos (k\theta + B)$,
where A and B are arbitrary constants, which represents a conic.

Q 7. In a central orbit the force is $\mu u^3(3 + 2a^2u^2)$, if the particle be projected at a distance with a velocity $\sqrt{\left(\frac{5\mu}{a^2}\right)}$ in a direction making an angle $\tan^{-1}\frac{1}{2}$ with the radius, prove that the equation to the path is $r = a\tan\theta$. **(1996)**

Sol. Given, central acceleration, $F = \mu u^3\ (3 + 2a^2u^2)$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{F}{u^2} = \frac{\mu u^3}{u^2}\ (3 + 2a^2u^2) = \mu\ (3u + 2a^2u^3)$$

On multiplying both the sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[\frac{3u^2}{2} + \frac{2a^2u^4}{4}\right] + A$$

where A is a cosntant.

$$\Rightarrow \qquad v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\ (3u^2 + a^2u^4) + A \qquad \text{...(i)}$$

But initially, when $r = a$, i.e. $u = \frac{1}{a}$, then

$$v = \sqrt{\left(\frac{5\mu}{a^2}\right)},\ \phi = \tan^{-1}\frac{1}{2}$$

$$\Rightarrow \qquad \tan\phi = \frac{1}{2} \text{ or } \sin\phi = \frac{1}{\sqrt{5}} \text{ or } p = r\sin\phi = \frac{a}{\sqrt{5}}$$

$$\Rightarrow \qquad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{5}{a^2}$$

$\therefore$ From Eq. (i), we have

$$\frac{5\mu}{a^2} = h^2 \cdot \frac{5}{a^2} = \mu\left(\frac{3}{a^2} + \frac{a^2}{a^4}\right) + A$$

$$\therefore \qquad h^2 = \mu \text{ and } A = \frac{\mu}{a^2}$$

On putting the values of h^2 and A in Eq. (i), we have

$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu(3u^2 + a^2u^4) + \frac{\mu}{a^2}$$

$$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = 2u^2 + a^2u^4 + \frac{1}{a^2} = \frac{1}{a^2}(2a^2u^2 + a^4u^4 + 1)$$

Put $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$\therefore \qquad \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{a^2}\left(\frac{2a^2}{r^2} + \frac{a^4}{r^4} + 1\right)$$

$$\Rightarrow \qquad \left(\frac{dr}{d\theta}\right)^2 = \frac{1}{a^2}(2a^2r^2 + a^4 + r^4) = \frac{1}{a^2}(a^2 + r^2)^2$$

$$\Rightarrow \qquad \frac{dr}{d\theta} = \frac{1}{a}(r^2 + a^2) \Rightarrow d\theta = \frac{a\,dr}{r^2 + a^2}$$

On integrating, we get

$$\theta + B = \tan^{-1}\left(\frac{r}{a}\right) \qquad \text{[where, } B \text{ is a constant.]} \quad \text{...(ii)}$$

But initially, when $r = a$, let $\theta = \frac{\pi}{4}$.

Then, $\frac{\pi}{4} + B = \tan^{-1} 1 = \frac{\pi}{4}$, so that $B = 0$

On putting $B = 0$ in Eq. (ii), we get

$$\theta = \tan^{-1}\left(\frac{r}{a}\right)$$

$$\therefore \qquad r = a\tan\theta$$

which is the required equation of the path. **Hence proved.**