Chapter Seven

RIEMANN-STIELTJES INTEGRALS

Important Points from the Chapter

1. **Riemann-Stieltjes Sum** Let f be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$.
Again, let $m_k = \text{glb}\{f(x) : x \in [x_{k-1}, x_k]\}$, $M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\}$ and g be monotonically non-decreasing bounded function on $[a, b]$. Since, $g(a)$ and $g(b)$ both are finite, we write $\delta_{g_k} = g(x_k) - g(x_{k-1})$. Then, $\delta_{g_k} > 0$.
We write the following two sums
$$L(f, g, P) = \sum_{k=1}^{n} m_k \delta_{g_k} \text{ and } U(f, g, P) = \sum_{k=1}^{n} M_k \delta_{g_k}$$
These sums are respectively called lower and upper Riemann-Stieltjes sums (or simply lower and upper RS-sums). **(2012)**

2. **Lower and Upper *RS*-Integrals** Let f be a bounded function on $[a, b]$ and g is a monotonically non-decreasing function on $[a, b]$.
Now, let P be the class of all partition of $[a, b]$. Then,
$$\underline{\int_a^b} f\, dg = \text{lub}\{L(f, g, P) : P \in \mathbf{P}\} \text{ and } \overline{\int_a^b} f\, dg = \text{glb}\{U(f, g, P) : P \in \mathbf{P}\}.$$
These integrals are respectively called the lower and upper RS-integrals of relative to g over $[a, b]$. **(2012, 07)**

3. **Riemann-Stieltjes Integrable** A bounded function $f : [a, b] \to R$ is said to be Riemann-Stieltjes integrable (or RS-integrable) relative to a monotonically non-decreasing function g on $[a, b]$, if $\underline{\int_a^b} fdg = \overline{\int_a^b} fdg$ and the common value is called the RS-integral of f relative to g over $[a, b]$ and it is denoted by $(RS)\int_a^b f\, dg$ or $(S)\int_a^b f\, dg$.
The function f is called the **integrand** and g is called **integer**.
The class of all RS-integrable functions relative to g over $[a, b]$ is denoted by $RS([a, b], g)$ or $RS(g)$.

- **Note** The statement "$\int_a^b fdg$ exists" means that the function f is bounded and g is monotonically non-decreasing and f is integrable relative to g over $[a, b]$.

4. **Mean Value Theorem** Let $f \in RS(g)$ on $[a, b]$. Then,

$$m[g(b) - g(a)] \leq (S)\int_a^b f\,dg \leq M[g(b) - g(a)]$$

where, m and M are the bounds of f on $[a, b]$. **(2007)**

Very Short Answer Questions

***Q* 1. Show that Riemann-Stieltjes integral is generalisation of Riemann-integral.** **(2014)**

***Or* Define Riemann-Stieltjes integral. Show that Riemann integral is particular case of it.** **(2012)**

Sol. **Part I Riemann-Stieltjes Integral** A bounded function $f : [a, b] \rightarrow R$ is said to be Riemann-Stieltjes integrable (or RS-integrable) relative to a monotonically non-decreasing function g on $[a, b]$, if $\underline{\int_a^b} f dg = \overline{\int_a^b} f dg$ and the common value is called the RS-integral of f relative to g over $[a, b]$ and it is denoted by $(RS)\int_a^b f\,dg$ or $(S)\int_a^b f\,dg$.

The function f is called the intergrand and g is called integer.

The class of all RS-integrable functions relative to g over $[a, b]$ is denoted by $RS([a, b], g)$ or $RS(g)$.

Part II If $f(x)$ be a bounded function and $g(x)$ be a monotonic non-decreasing function on $[a, b]$. Then, $f(x)$ is called RS-integrable on $[a, b]$ relative to $g(x)$, if

$$\underline{\int_a^b} f\,dg(x) = \overline{\int_a^b} f\,dg(x) \quad \text{...(i)}$$

Now, if we replace $g(x)$ by x, then Eq. (i) becomes

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$$

which is R-integral.

Hence, RS-integral is the generalisation of Riemann-integral.

***Q* 2. Prove that the lower *RS*-sum for a partition is always less than or equal to the upper *RS*-sum for any partition.**

Sol. Let P_3 be a partition of $[a, b]$, which is a refinement of both P_1 and P_2.

Then, $L_1(f, g, P_1) \leq L_3(f, g, P_3)$ and $U_3(f, g, P_3) \leq U_2(f, g, P_2)$

But $L_3(f, g, P_3) \leq U_3(f, g, P_3)$

Hence, $L_1(f, g, P_1) \leq L_3(f, g, P_3) \leq U_3(f, g, P_3) \leq U_2(f, g, P_2)$

i.e. $L_1(f, g, P_1) \leq U_2(f, g, P_2)$

Similarly, we can show that

$$L_2(f, g, P_2) \leq U_1(f, g, P_1).$$

Hence proved

Short Answer Questions

Q 1. Prove that for a bounded function, the upper *RS*-integral is never less than the lower *RS*-integral. **(2016)**

Or **Define upper and lower *RS*-integrals. Prove that the lower *RS*-integral cannot exceed the upper *RS*-integral.** **(2010)**

Sol. **Part I Lower and Upper *RS*-integrals** Let f be a bounded function on $[a, b]$ and g is a monotonically non-decreasing function on $[a, b]$.

Now, let P be the class of all partition of $[a, b]$. Then,

$\underline{\int_a^b} f\, dg = \text{lub}\{L(f, g, P) : P \in \mathbf{P}\}$ and $\overline{\int_a^b} f\, dg = \text{glb}\{U(f, g, P) : P \in \mathbf{P}\}$.

These integrals are respectively called the lower and upper RS-integrals of relative to g over $[a, b]$.

Part II Let f be a bounded function on $[a, b]$ and g is monotonic non-decreasing function on$[a, b]$.

Since, $\overline{\int_a^b} fdg =$ glb of the set of upper RS-sums, we choose an upper RS-sum $s(f, g, P)$ for a partition P of $[a, b]$ such that

$$\overline{\int_a^b} f\, dg > S(f, g, P) - \frac{\varepsilon}{2}, \text{ for } \varepsilon > 0$$

Similarly, $\underline{\int_a^b} f\, dg =$ lub of the set of lower RS sums, we choose a lower RS-sum $s(f, g, P)$ such that

$$\underline{\int_a^b} f\, dg < S(f, g, P) + \frac{\varepsilon}{2}, \text{ for } \varepsilon > 0$$

$$\therefore \quad \overline{\int_a^b} f\, dg - \underline{\int_a^b} fdg > U(f, g, P) - L(f, g, P) - \varepsilon$$

But $U(f, g, P) - L(f, g, P) \geq 0$, therefore

$$\overline{\int_a^b} f\, dg - \underline{\int_a^b} f\, dg > -\varepsilon, \text{ i.e. } \underline{\int_a^b} f\, dg < \overline{\int_a^b} f\, dg + \varepsilon$$

Since, $\varepsilon > 0$ is arbitrary, then

$$\underline{\int_a^b} fdg \leq \overline{\int_a^b} fdg$$

Hence proved.

Q 2. Let f be continuous and g be monotonic, non-decreasing on $[a, b]$, then $f \in RS$ on $([a, b], g)$. **(2014)**

Sol. Since, f is continuous on $[a, b]$, so it is bounded on $[a, b]$ and attains its glb and lub on $[a, b]$ and on every closed subintervals of it. Also, f is uniform continuous on $[a, b]$.

Then, for $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that

$$| f(x') - f(x'') | < \frac{\varepsilon}{g(b) - g(a)}, \text{ for } | x' - x'' | < \delta, \forall\ x', x'' \in [a, b]$$

Now, $[a, b]$ divided in n equal parts such that $n > \frac{b-a}{\delta}$.

We denote this partition by P.
Let $m_k = \text{glb}\{f(x) : x \in [x_{k-1}, x_k]\}$
and $M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\}$.
Next, we put $f(x'_k) = m_k$ and $f(x''_k) = M_k$. Then,

$$M_k - m_k < \frac{\varepsilon}{g(b) - g(a)} \qquad [\because |x'_k - x''_k| < \delta]$$

$$\therefore U(f, g, P) - L(f, g, P) = \sum_{k=1}^{n} (M_k - m_k)\ (g(x_k) - g(x_{k-1}))$$

$$< \sum_{k=1}^{n} \frac{\varepsilon}{g(b) - g(a)} (g(x_k) - g(x_{k-1}))$$

$$= \frac{\varepsilon}{g(b) - g(a)} \sum_{k=1}^{n} (g(x_k) - g(x_{k-1}))$$

$$= \frac{\varepsilon}{g(b) - g(a)} (g(b) - g(a)) = \varepsilon$$

Hence, f is RS-integrable on $([a, b], g)$. **Hence proved.**

***Q* 3. If f is bounded function on $[a, b]$ and α is monotonic increasing function, then $f \in R(\alpha)$ iff for every $\varepsilon > 0$, there exists a partition P such that $U(p, f, \alpha) - L(P, f, \alpha) < \varepsilon$.**
(2012, 08)

Sol. **Necessary condition** Let f be RS-integrable relative to α over $[a, b]$.

Then, $$\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha \qquad \text{...(i)}$$

Since, $\underline{\int_a^b} f\, d\alpha = \sup L(P, f, \alpha)$ over all partitions P, there exists a partition P_1 such that

$$\underline{\int_a^b} f\, d\alpha < L(P, f, \alpha) + \frac{\varepsilon}{2}, \varepsilon > 0 \qquad \text{...(ii)}$$

Similarly, $\overline{\int_a^b} f\, d\alpha = \inf U(P, f, \alpha)$

Therefore, $$U(P_2, f, \alpha) < \overline{\int_a^b} f\, d\alpha + \frac{\varepsilon}{2}, \varepsilon > 0 \qquad \text{...(iii)}$$

Let $P = P_1 \cup P_2$. Then, P is the common refinement of P_1 and P_2
Therefore, from Eqs. (ii) and (iii),

$$\underline{\int_a^b} fd\alpha < L\ (P, f, \alpha) + \frac{\varepsilon}{2} \qquad \text{...(iv)}$$

and $$U\ (P, f, \alpha) < \overline{\int_a^b} f\, dg + \frac{\varepsilon}{2} \qquad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get

$$\int_{\underline{a}}^{b} f\,dg + U(P, f, \alpha) < L(P, f, \alpha) + \int_{a}^{\bar{b}} f\,d\alpha + \varepsilon \quad \text{...(iv)}$$

From Eq. (i), we get

$$U(P, f, \alpha) < L(P, f, \alpha) + \varepsilon$$

$$\Rightarrow \quad U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Hence, the condition is necessary.

Sufficient condition Let for a positive integer $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \quad \text{...(vii)}$$

By definition, we get

$$\int_{a}^{\bar{b}} f\,d\alpha = \inf U(P, f, \alpha) \quad \text{...(viii)}$$

and

$$\int_{\underline{a}}^{b} f\,d\alpha = \sup L(P, f, \alpha) \quad \text{...(ix)}$$

Therefore ,

$$\int_{a}^{\bar{b}} f\,d\alpha \le U(P, f, \alpha) \quad \text{...(x)}$$

and

$$\int_{\underline{a}}^{b} f\,d\alpha \ge L(P, f, \alpha) \quad \text{...(xi)}$$

Hence,

$$L(P, f, \alpha) \le \int_{\underline{a}}^{b} f\,d\alpha \le \int_{a}^{\bar{b}} f\,d\alpha \le U(P, f, \alpha)$$

$$\Rightarrow \quad \int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f d\alpha < U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow \quad 0 < \int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha < \varepsilon \quad \text{...(xii)}$$

Since, ε is arbitrary, then we have

$$\int_{a}^{\bar{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha \quad \text{...(xiii)}$$

Hence, f is RS-integrable relative to α over $[a, b]$. **Hence proved.**

***Q* 4. Let f be monotonic and g be continuous and monotonic non-decreasing on $[a, b]$. Then, prove that f is RS-integrable relative to g on $[a, b]$.**

Sol. Let f is monotonic non-decreasing, then $f(b) > f(a)$.

Again, let $\varepsilon > 0$.

Since, g is continuous in $[a, b]$, then takes all the values between $g(a)$ and $g(b)$.

Also, g is monotonic non-decreasing, we can choose a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ such that

$$\delta_{g_k} = g(x_k) - g(x_{k-1}) = \frac{g(b) - g(a)}{n}, \text{ for } k = 1, 2, \dots, n$$

and $$n > \frac{\{g(b) - g(a)\}\{f(a) - f(a)\}}{\varepsilon}$$

Let $M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\}$ and $m_k = \text{glb}\{f(x) : x \in [x_{k-1}, x_k]\}$.

Then, $M_k = f(x_k), m_k = f(x_{k-1})$

Thus, $U(P, f, g) - L(P, f, g) = \sum_{k=1}^{n} (M_k - m_k)\{g(x_k) - g(x_{k-1})\}$

$$= \sum_{k=1}^{n} \{f(x_k) - f(x_{k-1})\} \frac{g(b) - g(a)}{n}$$

$$= \frac{\{f(b) - f(a)\}\{g(b) - g(a)\}}{n}$$

$$< \varepsilon$$

Hence, f is RS-integrable relative to g over $[a, b]$. **Hence proved.**

Q 5. If $f \in RS(g)$ on $[a, b]$ and c is a real number, then prove that $cf \in RS(g)$ on $[a, b]$ and $\int_a^b cf\, dg = c\int_a^b f\, dg$.

Sol. Since, f is RS-integrable on $[a, b]$, $\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$ and f is bounded on $[a, b]$. Then, $|cf| = |c||f| \Rightarrow cf$ is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ and M_k, m_k be the lub and glb of f on $[x_{k-1}, x_k]$. Then, cM_k and cm_k are the bounds of cf on $[x_{k-1}, x_k]$.

Again, let $\delta_{g_k} = g(x_k) - g(x_{k-1})$

Now, let $c > 0$.

Then, $$\underline{\int_a^b} cfdg = \text{lub}\left\{\sum_{k=1}^{n} cm_k\delta_{g_k}\right\} = c\,\text{lub}\left\{\sum_{k=1}^{n} m_k\delta_{g_k}\right\}$$

$$= c_{g_k}\underline{\int_a^b} fdg = c\overline{\int_a^b} fdg \qquad \ldots\text{(i)}$$

[$\because f$ is RS-integrable]

$$= c\,\text{glb}\left\{\sum_{k=1}^{n} M_k\delta g_k\right\} = \text{glb}\left\{\sum_{k=1}^{n} cM_k\delta g_k\right\}$$

$$= \overline{\int_a^b} cf\, dg \qquad \ldots\text{(ii)}$$

Hence, cf is RS-integrable on $[a, b]$ and in view of Eqs. (i) and (ii), we get

$$\int_a^b cf\, dg = c\int_a^b f\, dg$$

For $c = 0$, the result is obvious.

Suppose $c < 0$. In this case cM_k and cm_k, are respectively denote the glb and lub of cf of $[x_{k-1} - 1, x_k]$.

Thus, $\int_{\underline{a}}^{b} c\, f\, dg = \text{lub}\left\{\sum_{k=1}^{n} cM_k\delta_{g_k}\right\} = c\cdot\text{glb}\left\{\sum_{k=1}^{n} M_k\delta_{g_k}\right\}$ $[\because c<0]$

$$= c\int_{\underline{a}}^{b} fdg = c\cdot\text{lub}\left\{\sum_{k=1}^{n} m_k\delta g_k\right\} = \text{glb}\left\{\sum_{k=1}^{n} cM_k\delta g_k\right\} \quad [\because c<0]$$

$$= \int_{a}^{\bar{b}} c\cdot f\, dg \quad \ldots\text{(iii)}$$

Hence, cf is RS-integrable on $[a, b]$ and in view of Eq. (iii).

$$\therefore \quad \int_a^b c\, f\, dg = c\int_a^b f\, dg$$

which completes the proof of this theorem.

Q 6. If $f \in RS(g_1)$ on $[a, b]$ and $f \in RS(g_2)$ on $[a, b]$, then prove that $f \in RS(g_1 + g_2)$ and $\int_a^b fd(g_1 + g_2) = \int_a^b f\, dg_1 + \int_a^b f\, dg_2$. **(2007)**

Sol. Since, $f \in RS(g_1)$ on $[a, b]$, for given $\varepsilon > 0$, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1\ f, g_1) - L(P_1, f, g_1) < \frac{\varepsilon}{2}$$

Similarly, there exists a partition P_2 of $[a, b]$ such that

$$U(P_2, f, g_2) - L(P_2, f, g_2) < \frac{\varepsilon}{2}$$

Let $\quad P = P_1 \cup P_2 = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$

Then, P is a common refinement of P_1 and P_2, therefore

$$U(P, f, g_1) - L(P, f, g_1) < \frac{\varepsilon}{2}$$

and $\quad U(P, f, g_2) - L(P, f, g_2) < \frac{\varepsilon}{2}$

Let M_k and m_k are lub and glb of f in $[x_{k-1}, x_k]$.

Then, $\quad \delta_{g_{1k}} = g_1(x_k) - g_1(x_{k-1})$ and $\delta_{g_{2k}} = g_2(x_k) - g_2(x_{k-1})$

Let $g = g_1 + g_2$. Then, $\delta_{g_k} = \delta_{g_{1k}} + \delta_{g_{2k}}$

and $\sum_{k=1}^{n}(M_k - m_k)\,\delta_{g_k} = \sum_{k=1}^{n}(M_k - m_k)\,\delta_{g_{1k}} + \sum_{k=1}^{n}(M_k - m_k)\,\delta_{g_{2k}}$

$$\Rightarrow \quad U(P, f, g) - L(P, f, g) = [U(P, f, g_1) - L(P, f, g_1)] + [U(P, f, g_2) - L(P, f, g_2)] < \varepsilon$$

$$\therefore \quad f \in RS(g)$$

Since, $\quad U(P, f, g) = \sum_{k=1}^{n} M_k\delta g_{1k} + \sum_{k=1}^{n} M_k\delta g_{2k}$

$$= U(P, f, g_1) - U(P, f, g_2)$$

$\therefore \quad \text{glb}\,[U(P, f, g)] = \text{glb}\,[U(P, f, g_1) + U(P, f, g_2)]$

$$\geq \text{glb}\,[U(P, f, g_1) + U(P, f, g_2)]$$

$\therefore$ $$\int_a^b f\,dg \geq \int_a^b f\,dg_1 + \int_a^b f\,dg_2 \quad \text{...(i)}$$

Similarly, $L(P, f, g) = L(P, f, g_1) + L(P, f, g_2)$ implies that

$$\text{lub}\,[L(P, f, g)] = \text{lub}\,[L(P, f, g_1] + L(P, f, g_2)]$$
$$\leq \text{lub}\,[L(P, f, g_1)] + \text{lub}\,[L(P, f, g_2)]$$

$\therefore$ $$\int_a^b f\,dg \leq \int_a^b f\,dg_1 + \int_a^b f\,dg_2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2$$ **Hence proved.**

Q 7. If $f \in RS(g)$ on $[a, b]$, then prove that $|f| \in RS(g)$ on $[a, b]$ and $\left| \int_a^b f\,dg \right| \leq \int_a^b |f|\,dg$.

Sol. If $f \in RS(g)$ on $[a, b]$, there exists $k > 0$ such that

$$|f(x)| \leq k, \forall x \in [a, b].$$

Therefore, $|f|$ is bounded on $[a, b]$.

Let $\varepsilon > 0$

Then, there exists a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f, g) - L(P, f, g) < \varepsilon, \text{ i.e. } \sum_{k=1}^{n} (M_k - m_k)\,\delta_{g_k} < \varepsilon$$

where, M_k and m_k are lub and glb respectively, of f on $[x_{k-1}, x_k]$ and $\delta_{g_k} = g(x_k) - g(x_{k-1}) > 0$.

Let M'_k and m'_k are lub and glb of $|f|$ on $[x_{k-1}, x_k]$ and $x', x'' \in [x_{k-1}, x_k]$.

Then, $$||f(x')| - |f(x'')|| \leq |f(x') - f(x'')|$$

$\Rightarrow$ $$M'_k - m'_k \leq M_k - m_k$$

So that $U(P, |f|, g) - L(P, |f|, g) \leq U(P, f, g) - L(P, f, g) < \varepsilon$

Therefore, $|f| \in RS(g)$

Now, let $f_1 = \frac{1}{2}(|f| + f)$, $f_2 = \frac{1}{2}(|f| - f)$

Then, for every $x \in [a, b]$, $f_1(x), f_2(x) \geq 0$ and $f_1, f_2 \in RS(g)$.

Thus, we have $\int_a^b f_1\,dg \geq 0$, $\int_a^b f_2\,dg \geq 0$

Furthermore, we have

$$|f| = f_1 + f_2,\ f = f_1 - f_2$$

$\therefore$ $$\left| \int_a^b f\,dg \right| = \left| \int_a^b f_1\,dg - \int_a^b f_2 dg \right| \leq \left| \int_a^b f_1 dg \right| + \left| \int_a^b f_2 dg \right|$$

$$= \int_a^b f_1\,dg + \int_a^b f_2 dg = \int_a^b |f|\,dg$$ **Hence proved.**

Q 8. Let f and g be defined on [0, 2] by

$$f(x) = g(x) = \begin{cases} 0, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } 1 < x \le 2 \end{cases},$$ **then prove that both the integrals $\int_0^1 f\,dg$ and and $\int_1^2 f\,dg$ exist but the integral $\int_0^2 f\,dg$ does not exist.**

Sol. Since, $g = 0$ (constant) on [0, 1], then $\int_0^1 f\,dg = 1$, so the value exists.

Again, since f is constant on [1, 2], then $\int_1^2 f\,dg$ exists and has the value $g(2) - g(1) = 1 - 0 = 1$.

To investigate the existence of $\int_0^2 f\,dg$, let $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 2\}$ be a partition of [0, 2], which does not include the point 1.

Let for some r, $x_{r-1} < 1 < x_r$. Then, $\delta_{g_r} = g(x_r) - g(x_{r-1}) = 1 - 0 = 1$ and $\delta_{g_k} = 0$ for $k \ne r$

Now, $\sum_{k=1}^{n} M_k \delta_{g_k} = M_r \delta_{g_r} = M_r = 1$ and $\sum_{k=1}^{n} m_k \delta_{g_k} = m_r \delta_{g_r} = m_r = 0.$

$$\therefore \qquad \overline{\int_0^2} f\,dg = 1 \text{ and } \underline{\int_0^2} f\,dg = 0$$

Hence, $\int_0^2 f\,dg$ does not exist. **Hence proved.**

Long Answer Questions

Q 1. (i) Prove that, if $f_1, f_2 \in RS(g)$ for $[a, b]$, then $f_1 + f_2 \in RS(g)$ for $[a, b]$. **(2011)**

(ii) Prove that a continuous function f is RS-integrable with respect to g increasing on $[a, b]$. **(2011)**

Sol. (i) See the solution of Q. 6 of Short Answer Questions.

(i) See the solution of Q. 2 of Short Answer Questions.

Q 2. Define lower and upper RS-integrals. If f is RS-integrable on $[a, b]$ with respect to a increasing on $[a, b]$ and $m = \inf f(x)$ and $M = \sup f(x)$ on $[a, b]$, then prove that $m[\alpha(b) - \alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b) - \alpha(a)]$.

(2007)

Sol. **Part I Lower and Upper *RS*-intergrals** Let f be a bounded function on $[a, b]$ and g is a monotonically non-decreasing function on $[a, b]$.

Now, let P be the class of all partition of $[a, b]$. Then,

$$\underline{\int_a^b} f\, dg = \text{lub}\{L(f, g, P) : P \in \mathbf{P}\}$$

and $$\overline{\int_a^b} f\, dg = \text{glb}\{U(f, g, P) : P \in \mathbf{P}\}$$

These integrals are respectively called the lower and upper *RS*-integrals of relative to g over $[a, b]$.

Part II Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in the subinterval $[x_{r-1}, x_r]$. Then, we have $m \le m_r \le M_r \le M$

$$\Rightarrow \quad m\,\delta\alpha_r < m_r\delta\alpha_r \le M_r\delta\alpha_r \le M\,\delta\alpha_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} m\delta\alpha_r \le \sum_{r=1}^{n} m_r\,\delta\alpha_r \le \sum_{r=1}^{n} M_r\,\delta\alpha_r \le \sum_{r=1}^{n} M\,\delta\alpha_r$$

$$\Rightarrow \quad m\,[\alpha(b) - \alpha(a)] \le L\,(P, f, \alpha) \le U\,(P, f, \alpha) \le M\,[\alpha(b) - \alpha(a)] \quad \ldots\text{(i)}$$

Also, we know that,

$$L\,(P, f, \alpha) \le \underline{\int_a^b} f(\alpha)\, d\alpha \le \overline{\int_a^b} f(\alpha) d\alpha \le U(P, f, \alpha) \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we have

$$m\,[\alpha(b) - \alpha(a)] \le \underline{\int_a^b} f(\alpha)\, d\alpha \le \overline{\int_a^b} f(\alpha)\, dx \le M\,[\alpha(b) - \alpha(a)] \quad \ldots\text{(iii)}$$

and $$\underline{\int_a^b} f(\alpha)\, d\alpha \le \overline{\int_a^b} f(\alpha)\, d\alpha = \int_a^b f(\alpha)\, d\alpha \quad \ldots\text{(iv)}$$

From Eqs. (iii) and (iv), we get

$$m\,[\alpha(b) - \alpha(a)] \le \int_a^b f(\alpha)\, d\alpha \le M\,[\alpha(b) - \alpha(a)]$$

Hence proved