Chapter Eight

IMPROPER INTEGRALS

Important Points from the Chapter

1. **Improper Integrals** If the function f becomes unbounded on $[a, b]$ or the limits of integration become infinite, then the integral $\int_a^b f(x)\,dx$ is called improper integral.
2. **Singular Point** If $|f(x)| \to \infty$ for $x = c$. Then, c is called a singular point of the function $f(x)$. We say that f has infinite discontinuity at $x = c$.
3. **Integral with Finite Range** If a is the only singular point in $[a, b]$ the improper integral of $f(x)$ over (a, b) is defined by the equation
$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx$$
and b is the only singular point in $[a, b]$, then
$$\int_a^b f(x)\,dx = \lim_{\delta \to 0} \int_a^{b-\delta} f(x)\,dx$$
If the improper integral over $[a, b]$ exists, we say that the integral over $[a, b]$ is convergent.
4. **Principal and General Values of Improper Integrals** Let f is bounded at all points of $[a, b]$ except at c. Again, let the point c lies in the interval $(c - \varepsilon, c + \delta)$, where ε and δ are arbitrary positive numbers and independent to each other.
Then, $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta \to 0} \int_{c+\delta}^b f(x)\,dx$ provided both the limits exists. This value is called the general value of the integral.
   (i) If the general value exists, then we can say that the integral converges.
   (ii) If $\varepsilon = \delta$, the value of above limit is called the principal value of the integral.
5. **Integral with Infinite Limits** If the function f is bounded and integrable for $x \geq a$, then
$$\int_a^\infty f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{1/\varepsilon} f(x)\,dx$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \int_{-1/\delta}^{1/\varepsilon} f(x)\,dx.$$

If the above limits exists and has finite value, then the integrals are called convergent integral.

6. **Absolute Convergence of Infinite Integrals** If $\int_a^\infty |f(x)|\,dx$ converges, then the integral $\int_a^\infty f(x)\,dx$ is called absolutely convergent integral.

7. **Tests for Convergence of Integral $\int_a^\infty f(x)\,dx$**

   (i) **Comparison Test**

   (a) If $0 \le f(x) \le g(x)$ for all $x > a$ and $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is also convergent.

   (b) If $f(x) \ge g(x) \ge 0$ for all $x > a$ and $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is also divergent.

   (ii) **μ-Test**

   (a) If $x^\mu f(x)$ is bounded for $x > a$ and $\mu > 1$, then $\int_a^\infty f(x)\,dx$ is absolutely convergent. **(2000)**

   (b) If $x^\mu f(x)$ is always of the same sign (not zero) for $\mu \le 1$, then $\int_a^\infty f(x)\,dx$ does not converge.

   (iii) **Abel's Test** If $\int_a^\infty f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $x > a$, then $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent. **(2014)**

   (iv) **Dirichlet Test** If $\phi(x)$ is bounded and monotonic for $x \ge a$ and $\lim_{x \to \infty} \phi(x) = 0$ and $\int_a^b f(x)\,dx$ is bounded as b takes all finite values, then $\int_a^\infty f(x)\,\phi(x)\,dx$ converges. **(2016, 05)**

8. **Test for the Convergence of the Improper Integral $\int_a^b f(x)\,dx$**

   (i) **Comparison Test**

   (a) If $0 \le f(x) \le g(x)$ for $a < x \le b$ and $\int_a^b g(x)\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is also convergent.

(b) If $f(x) \geq g(x) \geq 0$ for $a < x \leq b$ and $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is also divergent.

(ii) **μ-Test** Suppose $f(x)$ be unbounded at a and integrable in the arbitrary interval $[a+\varepsilon, b]$, where $0 < \varepsilon < b - a$. If there is a number μ between 0 and 1 such that $\lim_{x \to a+0} (x-a)^\mu f(x)$ exists, then $\int_a^b f(x)\,dx$ converges absolutely. If there is a number $\mu \geq 1$ such that $\lim_{x \to a+0} (x-a)^\mu f(x)$ exists and is not zero, then $\int_a^b f(x)\,dx$ diverges and the same is true, if $\lim_{x \to a+0} (x-a)^\mu f(x) = \pm\infty$.

(iii) **Abel's Test** If $\int_a^b f(x)\,dx$ converges and $\phi(x)$ is bounded monotonic in (a, b), then $\int_a^b f(x)\,\phi(x)\,dx$ converges.

(iv) **Dirichlet Test** If $\int_{a+\varepsilon}^b f(x)\,dx$ is bounded and $\phi(x)$ is bounded and monotonic in (a, b) converges to zero as $x \to a$, then $\int_a^b f(x)\,\phi(x)\,dx$ converges.

Very Short Answer Questions

Q 1. Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. **(2011)**

Sol. We have,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{\substack{\varepsilon \to 0\\ \delta \to 0}} \int_{-1/\delta}^{1/\varepsilon} \frac{1}{1+x^2}\,dx = \lim_{\substack{\varepsilon \to 0\\ \delta \to 0}} [\tan^{-1} x]_{-1/\delta}^{1/\varepsilon}$$

$$= \lim_{\substack{\varepsilon \to 0\\ \delta \to 0}} \left[\tan^{-1}\frac{1}{\varepsilon} - \tan^{-1}\left(\frac{1}{-\delta}\right)\right]$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Q 2. Evaluate the integral $\int_0^1 \frac{dx}{\sqrt{x}}$. **(2010)**

Sol. Since, $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0$, then

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\delta \to 0} \int_\delta^1 \frac{dx}{\sqrt{x}} = \lim_{\delta \to 0} \left[\frac{x^{1/2}}{1/2}\right]_\delta^1 = \lim_{\delta \to 0} [2x^{1/2}]_\delta^1$$

$$= 2 \lim_{\delta \to 0} [1 - \delta^{1/2}] = 2\,[1 - 0] = 2$$

Q 3. Test the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. **(2006)**

Sol. Here, $f(x) = \frac{1}{x^{1/3}(1+x^2)}$, then $\mu = \frac{7}{3} - 0$

Now, $\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{7/3} \cdot \frac{1}{x^{1/3}(1+x^2)} = \lim_{x \to \infty} \frac{1}{\left(\frac{1}{x^2} + 1\right)} = \frac{1}{0+1} = 1$

$\because \qquad \mu = \frac{7}{3} > 1$

Hence, $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

Q 4. Test the convergence of the integral $\int_1^{\infty} \frac{dx}{\sqrt{x^3+1}}$. **(2016, 15, 12)**

Sol. Since, $f(x) = \frac{1}{\sqrt{x^3+1}}$, therefore we take $g(x) = \frac{1}{x^{3/2}}$

$$\therefore \qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^{3/2}}{\sqrt{x^3+1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x^3}}} = \frac{1}{\sqrt{1+0}} = 1$$

But $\int_1^{\infty} g(x)\, dx = \int_1^{\infty} \frac{1}{x^{3/2}}\, dx$ is convergent, therefore the given integral is convergent.

Q 5. Test the convergence of $\int_0^{\infty} \frac{\cos x}{1+x^2}\, dx$. **(2012, 09, 07, 06, 02)**

Sol. We have, $f(x) = \frac{\cos x}{1+x^2}$

Let $\qquad g(x) = \frac{1}{1+x^2}$, then $\left| \frac{\cos x}{1+x^2} \right| \le \frac{1}{1+x^2}$

Also, $\int_0^{\infty} \frac{dr}{1+x^2} = \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2} = \lim_{b \to \infty} [\tan^{-1} x] = \frac{\pi}{2}$

So, $\int_0^{\infty} \frac{dx}{1+x^2}$ is convergent.

Hence, by comparison test, $\int_0^{\infty} \frac{\cos x}{1+x^2}$ is convergent.

Q 6. Show that $\int_0^{\infty} e^{-x} dx$ convergent. **(2017)**

Sol. We have, $\int_0^{\infty} e^{-x} dx = \lim_{\delta \to 0} \int_0^{1/\delta} e^{-x} dx = \lim_{\delta \to 0} [1 - e^{-1/\delta}] = 1$

Hence, the integral is convergent and converge to 1.

Short Answer Questions

Q 1. Find the general and principal values of the integral $\int_0^3 \frac{dx}{(x-1)^3}$.

Sol. Since, the function $\frac{1}{(x-1)^3}$ is infinite at $x = 1$, then we have

$$\int_0^3 \frac{dx}{(x-1)^3} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{(x-1)^3} + \int_{1+\delta}^3 \frac{dx}{(x-1)^3}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{-2(x-1)^2}\right]_0^{1-\varepsilon} + \lim_{\delta \to 0} \left[\frac{1}{-2(x-1)^2}\right]_{1+\delta}^3$$

$$= \lim_{\varepsilon \to 0} \left(\frac{1}{2} - \frac{1}{2\varepsilon^2}\right) + \lim_{\delta \to 0} \left(\frac{1}{2\delta^2} - \frac{1}{8}\right)$$

$$= \frac{1}{2} - \frac{1}{8} - \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2} + \lim_{\delta \to 0} \frac{1}{2\delta^2}$$

This limit does not exist, because it is of the form $\infty - \infty$.
So, the general value does not exist.
But the principal value exists, which is

$$\frac{1}{2} - \frac{1}{8} + \lim_{\varepsilon \to 0} \left(\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon^2}\right) = \frac{3}{8}.$$

Q 2. Show that $\int_a^b \frac{1}{(x-a)^n} dx$ converges if and only if $n < 1$. **(2013)**

Sol. If $n = 1$, then

$$\int_{a+\varepsilon}^b \frac{dx}{x-a} = [\log (x-a)]_{a+\varepsilon}^b = \log \left(\frac{b-a}{\varepsilon}\right)$$

$$\therefore \qquad \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \infty$$

So, the integral is divergent for $n = 1$.

If $n \neq 1$, then $\int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \left[-\frac{1}{(n-1)} (x-a)^{-n+1}\right]_{a+\varepsilon}^b$

$$= \frac{1}{1-n} \left\{\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right\}$$

Now, if $n > 1$, then

$$\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \frac{1}{1-n} \left\{\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right\} = \infty$$

Thus, the given integral is also divergent for $n > 1$.

If $n < 1$, then

$$\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \frac{1}{(1-n)(b-a)^{n-1}}$$

Hence, $\int_a^b \frac{1}{(x-a)^n}\, dx$, converges for $n < 1$.

Q 3. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent. **(2017)**

Sol. Since, the integrand $\frac{1}{\sqrt{1-x^2}} \to \infty$ at $x = 1$.

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\delta \to 1} \int_0^{\delta} \frac{dx}{\sqrt{1-x^2}} = \lim_{\delta \to 1} [\sin^{-1} x]_0^{\delta}$$

$$= \lim_{\delta \to 1} [\sin^{-1} \delta - \sin^{-1} 0]$$

$$= \sin^{-1} 1 - 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Hence, the given integral is convergent and converge to $\frac{\pi}{2}$.

Q 4. Examine the convergence of $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$. **(2015, 13)**

Sol. We have, $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}} = \int_0^{a} \frac{x^{2m}}{1+x^{2n}}\, dx + \int_a^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$, where $a > 0$

Here, $f(x) = \frac{x^{2m}}{1+x^{2n}}$

Now, $\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} x^{2m}}{1+x^{2n}} = \lim_{x \to \infty} \frac{x^{\mu+2m}}{1+x^{2n}} = 1$, if $\mu + 2m = 2n$

$\therefore$ $\mu > 1$, if $n > m$ and $\mu \le 1$, if $n \le m$ for m and n are positive integers.

Then, if $\mu > 1$, i.e. $n > m$, then $\int_a^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$ is convergent and if $\mu \le 1$, i.e. $n \le m$ then the integral is divergent.

Also, $\int_0^{a} \frac{x^{2m}}{1+x^{2n}}$ is not an infinite integral and so convergent. Hence, the given integral is convergent if $n > m$ and divergent if $n \le m$.

Q 5. Test the convergence of the integral $\int_{-1}^{1} \frac{dx}{x^{2/3}}$. **(2011)**

Sol. Here, the integrand $\frac{1}{x^{2/3}} \to \infty$ as $x \to 0$

$$\therefore \quad \int_{-1}^{1} \frac{dx}{x^{2/3}} = \lim_{\varepsilon \to 0} \int_{-1}^{0-\varepsilon} \frac{dx}{x^{2/3}} + \lim_{\delta \to 0} \int_{0+\delta}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{\varepsilon \to 0} [3x^{1/3}]_{-1}^{-\varepsilon} + \lim_{\delta \to 0} [3x^{1/3}]_{\delta}^{1}$$

$$= \lim_{\varepsilon \to 0} (-3\varepsilon^{1/3} + 3) + \lim_{\delta \to 0} (3 - 3\delta^{1/3}) = 6$$

Hence, the given integral is convergent.

Q 6. Test the convergence of the following integrals.

(i) $\int_1^\infty \frac{dx}{\sqrt{x^5+1}}$ (ii) $\int_a^b x^{n-1} e^{-x}\, dx, n > 0$ **(2010)**

Sol.

(i) We have, $f(x) = \frac{1}{\sqrt{x^5+1}}$

Let $g(x) = \frac{1}{x^{5/2}}$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^{5/2}}{\sqrt{x^5+1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x^5}}} = \frac{1}{\sqrt{1+0}} = 1$$

Since, $\int_1^\infty g(x)\, dx = \int_1^\infty \frac{dx}{x^{5/2}}$ is convergent for $n = \frac{5}{2} > 1$.

Hence, $\int_1^\infty \frac{dx}{\sqrt{x^5+1}}$ is convergent.

(ii) We have, $x = 0$ is singular point, for $0 < n < 1$.

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} [x^{\mu+n-1} \cdot e^{-x}] = 1, \text{ if } \mu + n = 1$$

Hence, by the μ-test, the given integral is convergent if $1 - n < 1$, i.e. $n > 0$ and divergent if $n \le 0$.

Q 7. Discuss the convergence of $\int_0^1 x^{n-1} \log x\, dx$. **(2001)**

Sol. Since, $\lim_{x \to 0} x^r \log x = 0$, where $r > 0$. So, the integrand $f(x) = x^{n-1} \log x$ has no infinite discontinuity at $x = 0$ for $n - 1 > 0$, i.e. $n > 1$.

Thus, the given integral is convergent for $n > 1$.

If $n = 1$, then

$$\int_0^1 x^{n-1} \log x\, dx = \int_0^1 \log x\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \log x\, dx$$

$$= \lim_{\varepsilon \to 0} [x \log x - x]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} [-1 - \varepsilon(\log \varepsilon - 1)] = -1$$

Hence, the integral is convergent, if $n = 1$.

If $n < 1$, then $\lim_{x \to 0} [x^{\mu} f(x)] = \lim_{x \to 0} [x^{\mu + n - 1} \log x] = \begin{cases} 0, & \text{if } \mu > 1 - n \quad \ldots \text{(i)} \\ \infty, & \text{if } \mu \leq 1 - n \quad \ldots \text{(ii)} \end{cases}$

Hence, when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying Eq. (i). The integral is therefore convergent by μ-test when $0 < n < 1$.
Again, when $n \leq 0$, we can take $\mu = 1$ and satisfying Eq. (ii). Hence, by μ-test the integral is divergent, when $n \leq 0$.
Thus, the given integral is convergent if $n > 0$ and divergent if $n \leq 0$.

Q 8. Explain μ-test and hence test the convergence of integral $\int_1^{\infty} \frac{dx}{x^{1/3}(1 + x^{1/2})}$. **(2012)**

Sol.

(i) If $x^{\mu} f(x)$ is bounded for $x > a$ and $\mu > 1$, then $\int_a^{\infty} f(x)\, dx$ is absolutely convergent.

(ii) If $x^{\mu} f(x)$ is always of the same sign (not zero) for $\mu \leq 1$, then $\int_a^{\infty} f(x)\, dx$ is divergent.

Here, $\quad f(x) = \frac{1}{x^{1/3}(1 + x^{1/2})}$, then $\mu = \frac{5}{6} - 0 = \frac{5}{6}$

$\therefore \lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{5/6} \cdot \frac{1}{x^{1/3}(1 + x^{1/2})} = \lim_{x \to \infty} \frac{1}{(1/x^{1/2} + 1)} = \frac{1}{0 + 1} = 1$

Since, $\mu = \frac{5}{6} < 1$, then the given integral is divergent.

Q 9. State and prove Abel's test and hence test the convergence of $\int_a^{\infty} \frac{(1 - e^{-x})\cos x}{x^2}\, dx$, where $a > 0$. **(2015)**

***Or* If ϕ is bounded monotonic in $[a, \infty]$, $\phi(x)$ is convergent as $x \to \infty$ and $\int_0^{\infty} f(x)$ is convergent, $\forall x > a$, then prove that $\int_0^{\infty} f(x)\, \phi(x)\, dx$ is convergent.**

Sol. **Statement** $\int_a^{\infty} f(x)\, dx$ converges and $\phi(x)$ is bounded and monotonic for $x > a$, then $\int_a^{\infty} f(x)\phi(x)\, dx$ is convergent.

Proof Since, $\phi(x)$ is bounded and monotonic for $x > a$, therefore ϕ is integrable in $[a, b]$, where b is any number $\geq a$.
By second mean value theorem, we have

$$\int_{b_1}^{b_2} f(x)\, \phi(x)\, dx = \phi(b_1) \int_{b_1}^{\varepsilon} f(x)\, dx + \phi(b_2) \int_{\varepsilon}^{b_2} f(x)\, dx \qquad \ldots \text{(i)}$$

where, $a < b_1 \leq \varepsilon \leq b_2$.

Again, since $\phi(x)$ is bounded, there exists a number $A > 0$ such that $|\phi(b_1)| \le A$ and $|\phi(b_2)| \le A$.

Also, since $\int_{b_1}^{b_2} f(x)\,dx$ is convergent, then there exists a number b_0 such that

$$\left|\int_{b_1}^{b_2} f(x)\,dx\right| < K \text{ for } b_1, b_2 \ge b_0$$

where, $K > 0$ is only arbitrary number.

Since, $b_1 < \varepsilon < b_2$, therefore $\varepsilon \ge b_0$.

$$\therefore \quad \left|\int_{b_1}^{\varepsilon} f(x)\,dx\right| < K \text{ and } \left|\int_{\varepsilon}^{b_2} f(x)\,dx\right| < K$$

Hence, from Eq. (i), we get

$$\left|\int_{b_1}^{b_2} f(x)\,\phi(x)\,dx\right| \le |\phi(b_1)|\left|\int_{b_1}^{\varepsilon} f(x)\,dx\right| + |\phi(b_2)|\left|\int_{\varepsilon}^{b_2} f(x)\,dx\right|$$

$$< AK + AK$$

$$< 2AK$$

where $2AK$ is an arbitrary positive number.

Hence, $\int_a^{\infty} f(x)\,\phi(x)\,dx$ is convergent at ∞. **Hence proved.**

Let $f(x) = \dfrac{\cos x}{x^2}$ and $\phi(x) = 1 - e^{-x}$.

Then, $\phi(x)$ is bounded and monotonic increasing for $x > a$ and

$$\int_a^{\infty} \frac{\cos x}{x^2}\,dx \le \int_a^{\infty} \frac{dx}{x^2}$$

Since, $\int_a^{\infty} \dfrac{dx}{x^2}$ is convergent $(n = 2)$, by comparision test $\int_a^{\infty} \dfrac{\cos x}{x^2}\,dx$ is convergent.

Hence, by Abel's test, $\int_a^{\infty} (1 - e^{-x})\dfrac{\cos x}{x^2}\,dx$ is convergent.

Q 10. Define Abel's test for convergence of improper integral and hence test the convergence of $\int_a^{\infty} \dfrac{(1 - e^{-x})\sin x}{x^3}\,dx$, where $a > 0$. **(2014)**

Sol. **Part I Abel's Test** If $\int_a^b f(x)\,dx$ converges and $\phi(x)$ is bounded monotonic in (a, b), then $\int_a^b f(x)\,\phi(x)\,dx$ converges.

Part II Let $f(x) = \dfrac{\sin x}{x^3}$ and $\phi(x) = (1 - e^{-x})\,dx$, then $\phi(x)$ is bounded and monotonic increasing for $x > a$, and $\int_a^{\infty} \dfrac{\sin x}{x^3}\,dx \le \int_a^{\infty} \dfrac{dx}{x^3}$.

Since, $\int_a^\infty \frac{dx}{x^3}$ is convergent $(n = 3)$, therefore by comparison test, $\int_a^\infty \frac{\sin x}{x^3}\, dx$ is convergent. Hence, by Abel's test, $\int_a^\infty \frac{(1 - e^{-x})\sin x}{x^3}\, dx$ is convergent.

Q 11. Define Dirichlet test for the convergence of improper integral and show that $\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, a \geq 0$ is convergent. **(2016)**

Sol. **Part I Dirichlet Test** If $\int_{a+\varepsilon}^b f(x)\, dx$ is bounded and $\phi(x)$ is bounded and monotonic in (a, b) converges to zero as $x \to a$, then $\int_a^b f(x)\, \phi(x)\, dx$ converges.

Part II We have, $\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx = \int_0^a e^{-ax} \frac{\sin x}{x}\, dx + \int_a^\infty e^{-ax} \frac{\sin x}{x}\, dx,$ for $a > 0$.

Obviously, $\int_0^a e^{-ax} \frac{\sin x}{x}\, dx$ is proper integral for $\lim_{x \to 0} e^{-ax} \frac{\sin x}{x} = 1.$

Let $$\phi(x) = e^{-ax} \text{ and } f(x) = \frac{\sin x}{x}$$

Then, $\phi(x)$ is bounded and monotonic decreasing function for all positive values of x and for $a \geq 0$.

Also, $$\lim_{(x \to \infty)} f(x) = \lim_{(x \to \infty)} \frac{\sin x}{x} = 0$$

Hence, $\int_0^b f(x)\, dx$ is bounded, when $b \to \infty$.

Thus, by Dirichlet's theorem, the integral $\int_a^\infty f(x)\, \phi(x)\, dx = \int_a^\infty e^{-ax} \frac{\sin x}{x}\, dx$ is convergent.

Q 12. Prove that $\int_2^\infty \frac{dx}{\sqrt{x^2 - 1}}$ diverges. **(2017)**

Sol. Here, $f(x) = \frac{1}{\sqrt{x^2 - 1}} = \frac{1}{x\sqrt{1 - 1/x^2}}$

Take $$g(x) = \frac{1}{x}$$

We have, $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1}{\sqrt{1 - 1/x^2}} = 1$, which is finite and non-zero.

Therefore, $\int_2^\infty f(x) dx$ and $\int_2^\infty g(x) dx$ either both converge or both diverge.

But $\int_2^\infty g(x) dx = \int_2^\infty \frac{dx}{x}$ is divergent because here, $n = 1$. Hence, $\int_2^\infty f(x) dx,$ i.e. $\int_2^\infty \frac{1}{\sqrt{x^2 - 1}}\, dx$ is also divergent.

Long Answer Questions

Q 1. Test the convergence of following integrals.

(i) $\int_0^1 x^{m-1}(1-x^{n-1})dx$ (2002) (ii) $\int_0^\infty x^{n-1}e^{-x}\,dx$ (2005)

Sol.

(i) When $m, n \geq 1$, then the integrand is finite for each value of x, where $0 < x < 1$.

Hence, the given integral is convergent.

When $m, n < 1$ then the integrand has infinites at $x = 0$ and $x = 1$, these points are the points of infinite discontinuity.

Let $a \in (0, 1)$, then we have

$$\int_0^1 x^{m-1}(1-x^{n-1})\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx + \int_a^1 x^{m-1}(1-x)^{n-1}\,dx$$

At $x = 0$

First, we consider $\int_0^a x^{m-1}(1-x)^{n-1}\,dx$ has infinite discontinuity when $m < 1$.

Let $f(x) = x^{m-1}(1-x)^{n-1} = \dfrac{(1-x)^{n-1}}{x^{1-m}}$

and $\phi(x) = \dfrac{1}{x^{1-m}}$.

Now, $$\lim_{x\to 0}\frac{f(x)}{\phi(x)} = \lim_{x\to 0}\frac{\dfrac{(1-x)^{n-1}}{x^{1-m}}}{1/x^{1-m}} = \lim_{x\to 0}(1-x)^{n-1}$$

But $\int_0^a \phi(x)\,dx = \int_0^a \dfrac{1}{(1-x)^{1-m}}\,dx$ is convergent if $1-m < 1$, i.e. $m > 0$.

Hence, $\int_0^a f(x)\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx$ is convergent for $0 < m < 1$.

At $x = 1$

The integral $\int_a^1 x^{m-1}(1-x)^{n-1}\,dx$ has infinite discontinuity when $n < 1$.

Let $f(x) = x^{m-1}(1-x)^{n-1} = \dfrac{x^{m-1}}{(1-x)^{1-n}}$ and $\phi(x) = \dfrac{1}{(1-x)^{1-n}}$

Now, $$\lim_{x\to 1}\frac{f(x)}{\phi(x)} = \lim_{x\to 1}\frac{\dfrac{x^{m-1}}{(1-x)^{1-n}}}{\dfrac{1}{(1-x)^{1-n}}} = \lim_{x\to 1} x^{m-1} = 1$$

But $\int_a^1 \phi(x)\,dx = \int_a^1 \dfrac{1}{(1-x)^{1-n}}\,dx$ is convergent if $1-n < 1$, i.e. $n > 0$.

Hence, integral $\int_a^1 x^{m-1}(1-x)^{n-1}$ is convergent if $0 < n < 1$.

Therefore, the given integral is convergent for $m, n > 0$.

(ii) When $n < 1$, then the point $x = 0$ is a point of infinite discontinuity of $f(x) = e^{-x}x^{n-1}$.

So, $\int_0^\infty e^{-x}x^{n-1}dx = \int_0^a e^{-x}x^{n-1}dx + \int_a^\infty e^{-x}x^{n-1}dx$, where $a \in (0, \infty)$

At $x = 0$ First, we consider $\int_0^a e^{-x}x^{n-1}dx$.

Let $f(x) = e^{-x}x^{n-1}$ by μ-test,

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} x^\mu e^{-x}x^{n-1} = 1, \text{ for } \mu = 1 - n$$

Hence, $\int_0^a e^{-x}x^{n-1}dx$ is convergent if $\mu < 1$ and $n > 0$.

At $x = \infty$ Now, we consider $\int_a^\infty e^{-x}x^{n-1}dx$.

We know that, $e^x > x^{n+1}, \forall\ n > 0$

$$\therefore \qquad e^{-x}x^{n-1} > \frac{1}{x^2}$$

But $\int_0^\infty \frac{1}{x^2}dx$ is convergent.

Hence, $\int_0^\infty e^{-x}x^{n-1}dx$ is also convergent for all $n > 0$.

***Q* 2. (i) State and prove Dirichlet test.** **(2005)**

(ii) Show that the integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx, a \geq 0$ is convergent. **(2000)**

Sol.

(i) **Statement** If $\phi(x)$ is bounded and monotonic for $x \geq a$ and $\lim_{x \to \infty} \phi(x) = 0$ and $\int_a^b f(x)\,dx$ is bounded as b takes all finite values, then $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent.

Proof Since, $\phi(x)$ is bounded and monotonic for $x \geq a$, therefore $\phi(x)$ is integrable in (a, b), where b is any number greater than a.

Now, using second mean value theorem, we get

$$\int_{b_1}^{b_2} f(x)\,\phi(x)\,dx = \phi(b_1)\int_{b_1}^{\eta} f(x)\,dx + \phi(b_2)\int_{\eta}^{b_2} f(x)\,dx, \qquad \ldots(i)$$

where $\qquad a < b_1 \leq \eta \leq b_2 < b$

Also, $\left|\int_a^b f(x)\,dx\right| \leq A,\ \forall\ b \geq a$

where, A is any positive number, since $\int_a^b f(x)\,dx$ is bounded for $b \geq a$.

Thus, $\qquad \left|\int_{b_1}^{\eta} f(x)\,dx\right| = \left|\int_a^{\eta} f(x)\,dx - \int_a^{b_1} f(x)\,dx\right|$

$$\leq \left|\int_a^{\eta} f(x)\,dx\right| + \left|\int_a^{b_1} f(x)\,dx\right|$$
$$\leq A + A$$
$$= 2A$$

$\therefore \qquad \left|\int_{b_1}^{\eta} f(x)\,dx\right| \leq 2A$

In the similar way, $\left|\int_{\eta}^{b_2} f(x)\,dx\right| \leq 2A$

Again, $\lim_{x \to \infty} \phi(x) = 0$, there exists $b_0 : |\phi(x)| < K$ when $x \geq b_0$, where K is any positive number.

Now, if $b_1 \leq b_0$, $b_2 \leq b_0$, $|\phi(b_1)| \leq K$ and $|\phi(b_2)| \leq K$

Hence, from Eq. (i), we get

$$\left|\int_{b_1}^{b_2} f(x)\,\phi(x)\,dx\right| \leq |\phi(b_1)|\left|\int_{b_1}^{\eta} f(x)\,dx\right| + |\phi(b_2)|\left|\int_{\eta}^{b_2} f(x)\,dx\right|$$
$$\leq K\cdot 2A + K\cdot 2A$$
$$= 4KA$$

where, $4KA$ is arbitrary positive number.

Hence, $\int_a^{\infty} f(x)\,\phi(x)\,dx$ is convergent at ∞.

(ii) See the solution of Q. 11 of Short Answer Questions.