Chapter Eight

PLANETARY MOTION

Important Points from the Chapter

1. **Newton's Law of Gravitation** Every particle of universe attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of their distance apart.

   Suppose, two particles of mass m_1, m_2 are distance r apart, the force of attraction F between them is given by $F = \frac{G\, m_1\, m_2}{r^2}$, where G is called **gravitational constant** or **universal constant.**

   This law holds in the case of the motion of all planets in the solar system.

   The motion of the Earth about Sun or that of satellites about the planets are all take place under the influence of inverse square law and the paths described are known as **planetary orbits.**

2. **Orbits Described under Inverse Square Law** Suppose, a particle is moving under central acceleration μ/r^2, then the planetary orbit will be a conic section.

   For this, there are three cases arise

   If $v^2 < \frac{2\mu}{r}$, then orbit is an ellipse.

   If $v^2 = \frac{2\mu}{r}$, then orbit is parabola.

   and if $v^2 > \frac{2\mu}{r}$, then it is a hyperbola.

   where, v is the velocity of the particle, r is the distance of the particle from the centre of force and μ is a constant (absolute acceleration).

   (2010, 08, 06, 05, 02, 01)

3. **Kepler's Laws of Planetary Motion** Kepler deduced the following laws of planetary motion

   (i) Each planet describes an ellipse having the Sun in one of its foci.

   (ii) The radius vector drawn from the Sun to the planet describes equal areas in equal time.

   (iii) The square of the periodic times of the planets are proportional to the cubes of the semi-major axes of their orbits, i.e. $T^2 = \frac{4\pi^2}{\mu} a^3$.

   (2017, 13, 06, 05)

4. **Disturbed Orbits** When a particle is describing an elliptical orbit, it may happen that at some points of the path it receives an impulse so that it describes another path. An alternation in the strength of the centre of force will also give an alternation in the path.

   This change in path is called disturbed orbit. To obtain the new orbit, we will have to know that how the major axis has been altered in magnitude and position, what is the new eccentricity.

Very Short Answer Questions

***Q* 1. A particle describes an ellipse under a force $\frac{\mu}{r^2}$ and has a velocity v at a distance r from the centre of force, show that its time period is $\frac{2\pi}{\sqrt{\mu}}\cdot\left(\frac{2}{r}-\frac{v^2}{\mu}\right)^{-3/2}$.** **(2003, 1999, 96)**

Sol. As we know that, $v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$

$$\Rightarrow \qquad \frac{v^2}{\mu} = \frac{2}{r} - \frac{1}{a} \Rightarrow \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu} \Rightarrow a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1}$$

If T be the time for describing the ellipse by the particle, then

$$T = \frac{\text{Area of ellipse}}{h/2} = \frac{2\pi ab}{\sqrt{\mu b^2/a}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$= \frac{2\pi}{\sqrt{\mu}}\left[\left(\frac{2}{r}-\frac{v^2}{\mu}\right)^{-1}\right]^{3/2}$$

$$= \frac{2\pi}{\sqrt{\mu}}\left[\left(\frac{2}{r}-\frac{v^2}{\mu}\right)\right]^{-3/2}$$

Hence proved.

***Q* 2. Find the periodic time of a satellite moving around a planet.** **(2006)**

Sol. As the laws similar to those of Kepler's laws hold for the planets and their satellites.

$\therefore$ Periodic time of a satellite moving around a planet,

$$T = \frac{2\pi ab}{\sqrt{\mu l}} = \frac{2\pi ab}{\sqrt{\mu\frac{b^2}{a}}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \qquad \left[\because l = \frac{b^2}{a}\right]$$

Short Answer Questions

***Q* 1.** **A particle moves with a central acceleration $= \dfrac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R, show that its path is a rectangular hyperbola, if the angle of projection is $\sin^{-1} \dfrac{\mu}{VR\sqrt{(V^2 - 2\mu/R)^{1/2}}}$.** **(2005, 02, 1990)**

Or **If the central force varies inversely as the square of the distance from a fixed point, find the orbit.** **(2001)**

Sol. When the path is rectangular hyperbola, we have

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

Given that, $v = V$ and $r = R$

$$\therefore \quad V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right) \Rightarrow V^2 - \frac{2\mu}{R} = \frac{\mu}{a} \quad \text{...(i)}$$

As $h = vp$ and $h = \sqrt{\mu \times \text{Semi-latus rectum}}$

We have, $$\sqrt{\frac{\mu b^2}{a}} = Vp$$

$$\Rightarrow \quad \frac{\mu b^2}{a} = V^2p^2 = V^2R^2\sin^2\phi \quad \text{...(ii)}$$

where, ϕ is the angle that the direction of projection makes with radius R.
In the case of rectangular hyperbola, $b = a$.

$\therefore$ Eq. (ii) becomes $\mu a = V^2R^2\sin^2\phi$

$$\Rightarrow \quad \sin^2\phi = \frac{\mu a}{V^2R^2} = \frac{\mu^2}{V^2R^2\left(V^2 - \dfrac{2\mu}{R}\right)}$$

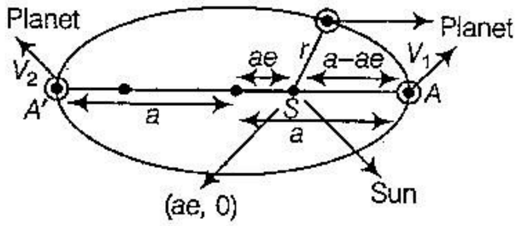
$$\Rightarrow \quad \sin\phi = \frac{\mu}{VR\left(V^2 - \dfrac{2\mu}{R}\right)^{1/2}}$$

$$\Rightarrow \quad \phi = \sin^{-1}\frac{\mu}{VR\left(V^2 - \dfrac{2\mu}{R}\right)^{1/2}}$$

***Q* 2.** **Let v_1 and v_2 be the linear velocities of a planet when it is respectively nearest and farthest from the Sun, prove that $(1 - e)\,v_1 = (1 + e)\,v_2$.** **(2002, 1999, 96)**

Sol. By Kepler's law of planetary motion, each planet describes an ellipse having the sun in one of its foci.

Let AA', major axis $= 2a$ and C is the centre of ellipse.



We know that, $CS = ae$ and $CA = CA' = a$

Here, A and A' are the nearest and farthest positions of the planet.

Now, $\quad SA = CA - CS = a - ae = a\,(1 - e)$

and $\quad SA' = CS + CA' = ae + a = a\,(1 + e)$

As we know for the ellipse, $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

when the planet is at A, we have

$$v_1^2 = \mu\left(\frac{2}{SA} - \frac{1}{a}\right) = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right] = \frac{\mu}{a}\cdot\left(\frac{1+e}{1-e}\right)$$

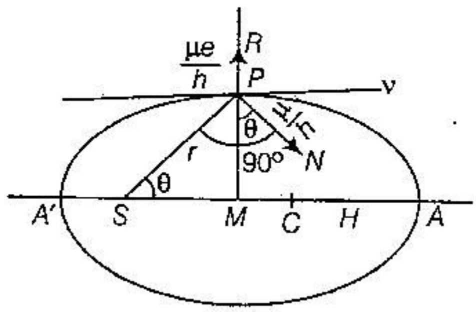
and when the planet is at A' we will have

$$v_2^2 = \mu\left[\frac{2}{SA'} - \frac{1}{a}\right] = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a}\right] = \frac{\mu}{a}\cdot\frac{1-e}{1+e}$$

$\therefore \quad \frac{v_1^2}{v_2^2} = \frac{(1+e)^2}{(1-e)^2} \Rightarrow (1-e)\,v_1 = (1+e)\,v_2$ **Hence proved.**

Q 3. Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities, $\frac{\mu}{h}$ perpendicular to the radius and $\frac{\mu e}{h}$ perpendicular to the major axis. **(2005)**

Sol. Let v be the velocity of the particle at the point $P(r, \theta)$ moving in an ellipse about a centre of force in the focus S.



$$\therefore \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \ldots\text{(i)}$$

where, $SP = r$ and a is the semi-major axis of ellipse.

Also, $\quad h^2 = \mu l = \frac{\mu b^2}{a} = \mu a\,(1 - e^2)$

$$\Rightarrow \quad \frac{1}{a} = \frac{\mu}{h^2}(1 - e^2) \quad \ldots\text{(ii)}$$

Now, the equation of the ellipse taking S as the pole is

$$\frac{l}{r} = 1 - e\cos\theta$$

$$\Rightarrow \qquad \frac{1}{r} = \frac{\mu}{h^2}(1 - e\cos\theta) \qquad [\because h^2 = \mu l] \quad \ldots\text{(iii)}$$

On putting the values of $\frac{1}{a}$ and $\frac{1}{r}$ from Eqs. (ii) and (iii) in Eq. (i), we get

$$v^2 = \frac{\mu^2}{h^2}[2(1 - e\cos\theta) - (1 - e^2)]$$

$$\Rightarrow \qquad v = \frac{\mu}{h}\sqrt{(1 - 2e\cos\theta + e^2)} \qquad \ldots\text{(iv)}$$

The resultant of two velocities, one $\frac{\mu}{h}$ along PN (perpendicular to the radius vector) and the other $\frac{\mu e}{h}$ along PR (perpendicular to the major axis)

$$= \sqrt{\frac{\mu^2}{h^2} + \frac{\mu^2 e^2}{h^2} + 2\frac{\mu}{h}\cdot\frac{\mu e}{h}\cos RPN} = \frac{\mu}{h}\sqrt{1 + e^2 + 2e\cos(\pi - \theta)}$$

$$= \frac{\mu}{h}\sqrt{1 + e^2 - 2e\cos\theta} = v \qquad \text{[from Eq. (iv)]}$$

Therefore, v can be compounded of two constant velocities $\frac{\mu}{h}$ perpendicular to the radius and $\frac{\mu e}{h}$ perpendicular to the major axis. **Hence proved.**

***Q* 4. Describe Kepler's third law of planetary motion.** **(2018)**

Sol. See the Part II of Q. 3 of Long Answer Question.

Long Answer Questions

***Q* 1. A particle is moving under central acceleration $\frac{\mu}{r^2}$, show that its orbit is a conic section and differentiate between three cases that arise.** **(2006, 05, 02, 01)**

Or A particle moves in a path so that its acceleration $\frac{\mu}{r^2}$ is always directed towards a fixed point. Show that path is a conic section and distinguish between the three cases. **(2017)**

Sol. Given, $P = \frac{\mu}{r^2}$, where P is acceleration at a distance r from the centre of force.

Now, differential equation of the central orbit in the pedal form is

$$\frac{h^2}{p^3}\cdot\frac{dp}{dr} = P \Rightarrow \frac{h^2}{p^3}\cdot\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\Rightarrow \qquad \frac{h^2}{p^3} \cdot dp = \frac{\mu}{r^2}\, dr$$

On integrating, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + A \qquad \left[\because v = \frac{h}{p}\right] \quad \text{...(i)}$$

Now, there are three cases arise

***Case* I** On comparing Eq. (i) with the pedal equation of ellipse

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \text{ we get}$$

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{A}{-1} \Rightarrow A = \frac{-\mu}{a} \qquad \text{[from Eq. (i)]}$$

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \Rightarrow v^2 < \frac{2\mu}{r} \qquad \text{...(ii)}$$

***Case* II** On comparing Eq. (i) with pedal equation of parabola, we get

$$p^2 = ar$$

$$\Rightarrow \qquad \frac{1}{p^2} = \frac{1}{ar} \Rightarrow \frac{h^2}{1} = \frac{2\mu}{1/a} = \frac{A}{0}$$

$$\Rightarrow \qquad A = 0$$

$\therefore$ From Eq. (i), we get $v^2 = \dfrac{2\mu}{r}$ ...(iii)

***Case* III** On comparing Eq. (i) with pedal equation of hyperbola

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1, \text{ we have } \frac{h^2}{b^2} = \frac{\mu}{a} = \frac{A}{1} \Rightarrow A = \frac{\mu}{a}$$

Therefore, from Eq. (i),

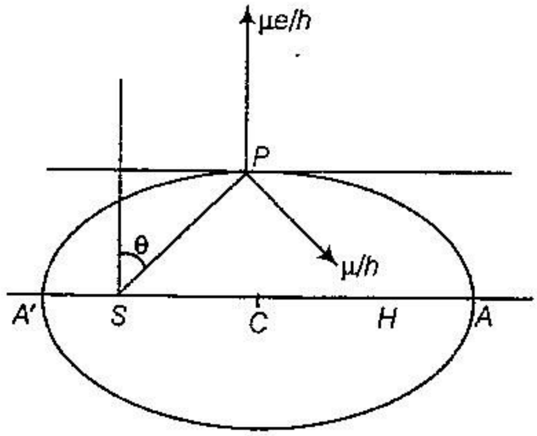
$$v^2 = \frac{2\mu}{r} + \frac{\mu}{a} \Rightarrow v^2 > \frac{2\mu}{r} \qquad \text{...(iv)}$$

Hence, from the above relations given by Eqs. (ii), (iii) and (iv), we conclude that if $v^2 < \dfrac{2\mu}{r}$, orbit is an ellipse.

If $v^2 = \dfrac{2\mu}{r}$, then orbit is parabola and if $v^2 > \dfrac{2\mu}{r}$, then orbit is hyperbola.

***Q* 2. A planet is describing an ellipse about the Sun as focus, show that its velocity away from the Sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is $\dfrac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, e is eccentricity and T the periodic time.** **(2001)**

Sol. The velocity of a planet P describing an ellipse about the Sun S as a focus is resultant of two constant velocities, $\frac{\mu}{h}$ perpendicular to the radius vector SP and $\frac{\mu e}{h}$ (perpendicular to the major axis). Therefore, the velocity of the planet away from the Sun, i.e. along SP is the sum of resolved parts of the two components of velocities mentioned above but the component velocity $\frac{\mu}{h}$ perpendicular to SP has no resolved part along SP.



Hence, the velocity say V of the planet away from the Sun is given by $V = \frac{e\mu}{h}\cos\theta$, where θ is the angle that SP makes with the perpendicular to the major axis.

Now, V is maximum, when $\theta = 0$, i.e. When the radius vector SP is perpendicular to the major axis.

$$\therefore \text{ Maximum value of } V = \frac{e\mu}{h} = \left(\frac{e^2\mu^2}{h^2}\right)^{1/2} = \left(\frac{e^2\mu a}{b^2}\right)^{1/2} \qquad \left[\because h^2 = \frac{\mu b^2}{a}\right]$$

$$= e\sqrt{\frac{\mu}{a(1-e^2)}} = \frac{e\sqrt{\mu}}{\sqrt{a(1-e^2)}} = e\cdot\frac{2\pi a}{T\sqrt{1-e^2}} \qquad \left[\because T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}\right]$$

***Q 3.* State the Kepler's law of planetary motion and obtain the more accurate form of third law.** **(2013)**

Sol. **Part I Statement** Kepler deduced the following laws of planetary motion

(i) Each planet describes an ellipse having the Sun in one of its foci.

(ii) The radius vector drawn from the Sun to the planet describes equal areas in equal time.

(iii) The square of the periodic times of the planets are proportional to the cubes of the semi-major axes of their orbits, i.e. $T^2 = \frac{4\pi^2}{\mu}a^3$.

Part II Kepler's third law is based on the supposition that Sun is fixed or that the mass of the planet is neglected in comparison with that of Sun. A more accurate form of third law can be obtained as follows

Proof Let S and P be the mass of Sun and any of its planet respectively and γ be the gravitational constant. Then, by Newton's law of gravitation the force of attraction between Sun and planet will be $\gamma\cdot\frac{S\cdot P}{r^2}$, where r is the distance between the Sun and planet.

$\therefore$ Acceleration of the planet will be $\alpha = \gamma \cdot \dfrac{S}{r^2}$ (towards Sun)

and acceleration of the Sun will be $\beta = \dfrac{\gamma P}{r^2}$ (towards planet)

$\therefore$ Acceleration of the planet relative of Sun is

$$\alpha + \beta = \left(\frac{\gamma (S+P)}{r^2}\right) = \frac{\mu}{r^2} \qquad [\text{where, } \mu = \gamma (S+P)]$$

Thus, the periodic time of the planet is given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{\gamma(S+P)}} \Rightarrow T^2 = \frac{4\pi^2 a^3}{\gamma (S+P)} \qquad \text{...(i)}$$

Similarly, for the planet of mass P_1 with semi-major axis a,
The periodic time

$$T_1^2 = \frac{4\pi^2 a_1^3}{\gamma(S+P_1)} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{T^2}{T_1^2} = \left(\frac{S+P_1}{S+P}\right)\frac{a^3}{a_1^3} \Rightarrow \left(\frac{S+P}{S+P_1}\right)\frac{T^2}{T_1^2} = \frac{a^3}{a_1^3}$$

Similar relation holds for the planet of mass P and its satellite of mass p, if d be the mean distance of satellite from planet.
The periodic time is

$$t = \frac{2\pi d^{3/2}}{\sqrt{\gamma(P+p)}} \qquad \text{...(iii)}$$

From Eqs. (i) and (iii), we get

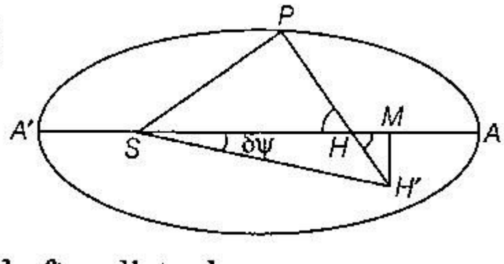
$$\frac{T}{t} = \frac{a^{3/2}}{d^{3/2}} \cdot \sqrt{\frac{P+p}{S+P}} \Rightarrow \left(\frac{S+P}{P+p}\right)\frac{T^2}{t^2} = \frac{a^3}{d^3}$$

which is more accurate from of Kepler's third law.

Q 4. Discuss the tangential disturbing force. (2013)

Or What are the effects of tangential disturbing forces on the elliptical orbit? (2015, 07, 1992)

Sol. Let APA' be the path of a particle moving about a centre of force at focus S and let H be the other focus.



Now, when particle reaches at point P on its path, its velocity v is changed to $v + \delta v$, the direction being unaltered.

Let $2a$ and $2a'$ be the major axes before and after disturbance

$$\therefore \qquad v^2 = \mu \left\{\frac{2}{SP} - \frac{1}{a}\right\} \qquad \text{...(i)}$$

and

$$(v + \delta v)^2 = \mu \left\{\frac{2}{SP} - \frac{1}{a'}\right\} \qquad \text{...(ii)}$$

Since, the direction of motion is not changed at P, the new focus lies on PH and let H' be its new position

$$\therefore \quad HH' = H'P - HP = (H'P + SP) - (HP + SP)$$
$$= 2a' - 2a - = 2(a' - a) \quad \text{...(iii)}$$

On differentiating Eq. (i), we get $2v\delta v = \frac{\mu}{a^2}\delta a$

Since, SP is constant for these instantaneous change.

Then, increment in semi-major axis,

$$\delta a = \frac{2v \cdot \delta v \cdot a^2}{\mu} \quad \text{...(iv)}$$

Now, $$\tan HSH' = \frac{H'M}{SM} = \frac{HH' \sin H}{SH + HM}$$
$$= \frac{2\delta a \sin H}{2ae + HH' \cos H} = \frac{2\delta a \sin H}{2ae}$$

Since, $HH' = 2(a' - a) = 2\delta a$ is small.

$$\Rightarrow \quad \tan \delta\Psi = \frac{\delta a \sin H}{ae}$$

So, the small angle $\delta\Psi$ through which the major axis moves is given by

$$\delta\Psi = \frac{\delta a \sin H}{ae} \quad \text{...(v)}$$

$$\Rightarrow \quad \delta\Psi = \frac{2va}{e\mu} \cdot \sin H \cdot \delta v \quad \text{[from Eq. (iv)] ...(vi)}$$

Since, the direction of motion at P is unaltered, the perpendicular p is unaltered at P. Therefore, taking logarithmic differential of the equation $h = pv$, we have

$$\frac{\delta h}{h} = \frac{\delta v}{v} \quad \text{...(vii)}$$

But $$h^2 = \mu a(1 - e^2) \quad \text{...(viii)}$$

Therefore, $$2h\delta h = \mu\delta a(1 - e^2) - \mu a \cdot 2e\delta e$$

$$\Rightarrow \quad \mu a \cdot 2e\delta e = 2v\delta v\, a^2(1 - e^2) - \frac{2\delta v}{v}h^2 \quad \text{[from Eqs. (iv) and (vii)]}$$
$$= 2v\delta v a^2(1 - e^2) - \frac{2\delta v}{v}\mu a(1 - e^2) \quad \text{[from Eq. (viii)]}$$

$$\Rightarrow \quad \delta e = \frac{\delta v}{v} \cdot \frac{1 - e^2}{e} \cdot \frac{av^2 - \mu}{\mu} \quad \text{...(ix)}$$

which gives the increase in the value of the eccentricity.

Since, the periodic time $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$, by taking logarithmic differential, we get

$$\frac{\delta T}{T} = \frac{3}{2}\frac{\delta a}{a} = \frac{3va\delta v}{\mu} \quad \text{[from Eq. (iv)] ...(x)}$$

From Eqs. (iv), (vi), (ix), (x), we can find various changes due to disturbance.