Chapter Nine

INFINITE SERIES

Important Points from the Chapter

1. **Series** A series is an expression of the form $u_1 + u_2 + u_3 + \dots + u_n + \dots$ in which every term is followed by another according to some definite law. If in a series the number of terms are infinite, then it is called an infinite series, otherwise it is called finite series.

   The series $u_1 + u_2 + u_3 + \dots + u_n + \dots$ is symbolically written as $\sum_{n \to 1}^{\infty} u_n$ or $\lim_{n \to \infty} s_n$.

2. **Types of Series** There are following three types of series.

   (i) **Convergent series** An infinite series Σu_n is said to be convergent, if S_n the sum of its first n terms, tends to a finite limit s as $n \to \infty$ and written as $s = \lim_{n \to \infty} s_n$.

   (ii) **Divergent series** If $s_n \to \pm \infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

   (iii) **Oscillatory series** If s_n does not tend to a finite limit or to $+ \infty$ or $- \infty$, the series is called oscillatory.

3. **Some Important Results**

   (i) $\lim_{n \to \infty} n^{1/n} = 1$

   (ii) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ or $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

   (iii) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n+p} = e^x$, if p is finite

   (iv) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^p = 1$, where p is a finite number.

   (v) $\lim_{n \to \infty} \frac{\log n}{n} = 0$

   (vi) $\lim_{n \to \infty} \frac{x^n}{n!} = 0$, where x is any number.

4. **Comparison Test**

   (i) If Σv_n and Σu_n are two series of position terms and Σv_n is convergent, then Σu_n is convergent if $u_n \le v_n, \forall\, n \in N$ and Σu_n is divergent if Σv_n is divergent, and $u_n \ge v_n, \forall\, n \in N$.

(ii) If Σu_n and Σv_n are two series of positive terms, then

(a) Σu_n will be convergent, if from and after some fixed terms $\lim_{n \to \infty} \frac{u_n}{v_n}$ is finite (including zero). Then, Σv_n is convergent.

(b) Σu_n will be divergent, if from and after some fixed terms, $\lim_{n \to \infty} \frac{u_n}{v_n}$ is some finite number greater than zero or infinite. Then, Σv_n is divergent. **(1995, 93)**

5. **Auxiliary Series** The infinite series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ is convergent, if $p > 1$ and divergent if $p \le 1$ **(2006, 1995)**

6. **D' Alembert's Ratio Test** (Ratio Test) Suppose Σu_n be a positive term series such that

(i) $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$, then the series is

(a) convergent, if $l < 1$. (b) divergent, if $l > 1$.

(c) test fail if, $l = 1$.

(ii) If $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \infty$, then Σu_n is divergent. **(2002, 01, 1998, 92)**

7. **Raabe's Test** Let Σu_n be the series of positive terms and

$$\lim_{n \to \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = l. \text{ Then,}$$

(i) the series is convergent, if $l > 1$.

(ii) the series is divergent, if $l < 1$.

(iii) test fail, if $l = 1$. **(2014)**

■ **Note** This test is used when D' Alembert's ratio test fails.

8. **Cauchy's Root Test** Suppose Σu_n be series of positive terms such that $\lim u_n^{1/n} = l$, then

(i) Σu_n is convergent, if $l < 1$. (ii) Σu_n is divergent, if $l > 1$.

(iii) Test fails, if $l = 1$. **(1998, 96, 94)**

9. **Logarithmic Test** Let Σu_n be the series of the positive terms and

$$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = l. \text{ Then,}$$

(i) the series is convergent, if $l > 1$. (ii) the series is divergent, if $l < 1$.

(iii) test fails, if $l = 1$. **(2007, 1994)**

10. **Cauchy's Condensation Test** If the function $f(x)$ is positive for all positive integral values of n and monotonically decreasing as n increasing, then the two infinite series $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together, 'a' being a positive integer greater than unity.

11. **De-Morgan's and Bertrand's Test** Suppose Σu_n is the series of positive terms and $\lim_{n \to \infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} \log n\right] = l$. Then,

(i) the series is convergent, if $l > 1$. (ii) the series is divergent, if $l < 1$.
(iii) test fails, if $l = 1$.

12. **Second Logarithmic Ratio Test** (Alternative to Bertrand's Test) Let Σu_n be the series of positive terms and $\lim_{n \to \infty} \left\{\left(n \log \frac{u_n}{u_{n+1}}\right) \log n\right\} = l$.

Then,

(i) the series is convergent, if $l > 1$. (ii) the series is divergent, if < 1.
(iii) test fails, if $l = 1$.

13. **Absolute Convergence**

(i) **Absolute convergent series** Let the series be $\Sigma u_n = u_1 + u_2 + \ldots + \ldots + u_n + \ldots$, in which any term may be either positive or negative. Let $| u_n |$ denote the absolute value of u_n, i.e. $| u_n | = u_n$ if u_n is positive and $| u_n | = - u_n$ if u_n is negative.

Then, $\Sigma | u_n | = | u_1 | + | u_2 | + | u_3 | + \ldots + | u_n | + \ldots$

The series Σu_n is said to be absolutely convergent, if the series $\Sigma | u_n |$ is convergent.

(ii) **Semi-convergent or conditionally convergent series** If the series Σu_n is convergent but the series $\Sigma | u_n |$ is divergent, then the series Σu_n is said to be semi-convergent.

Very Short Answer Questions

Q 1. Test the convergence of the series whose general term is $\frac{n^{n^2}}{(1+n)^{n^2}}$. **(2016)**

Sol. Let $u_n = \frac{n^{n^2}}{(1+n)^{n^2}} = \left(\frac{1}{1 + 1/n}\right)^{n^2}$

$$\therefore \lim_{n \to \infty} (u_n)^{1/n} = \lim \left(\frac{1}{1 + 1/n}\right)^n = \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Hence the series is convergent.

Q 2. Test the convergence of the series $\sum_{n\to 1}^{\infty}\left(\frac{\sqrt{n}}{n^2+1}\right)$. **(2005)**

Sol. We have,

$$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{1}{n^{3/2}}\left(1+\frac{1}{n^2}\right)^{-1}$$

Let we choose $v_n = \frac{1}{n^{3/2}}$

Then, $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{3/2}}\left(1+\frac{1}{n^2}\right)^{-1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{-1}$$

$$= 1 \text{ (Finite non-zero number)}$$

Since, v_n is convergent by the auxiliary series.
Hence, Σu_n is also convergent.

Q 3. Test the convergent of the series whose nth term is

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}.$$

(2003)

Sol. We have, $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$

Now, $$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n/n}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1} = \lim_{n\to\infty}\frac{1}{\left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)} = \frac{1}{e-1} = (e-1)^{-1} < 1$$

Hence, the given series is convergent.

Q 4. Discuss the nature of the series $\sum_{n=1}^{\infty}\sin\frac{1}{n}$. **(2014, 1990)**

Sol. Let $u_n = \sin\frac{1}{n} = \frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + \frac{1}{5!}\frac{1}{n^5} - \dots$

Suppose $v_n = \frac{1}{n}$, then the auxiliary series $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent.

$$\text{Now, } \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left[\frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + \frac{1}{5!}\cdot\frac{1}{n^5} - \dots\right]}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[1 - \frac{1}{3!}\cdot\frac{1}{n^2} + \frac{1}{5!}\frac{1}{n^4} - \dots\right] = 1$$

which is finite and non-zero.
Hence, by comparison test, the series is divergent.

Q 5. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then show that $\lim_{n\to\infty} a_n = 0$, but its converse is not true. **(2013)**

Sol. Let $s_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$ be a series of non-negative terms. Then, $s_n = \Sigma u_n$.

$\therefore \quad s_{n-1} = a_1 + a_2 + a_3 + \dots + a_{n-1}$

$\therefore \quad s_n - s_{n-1} = a_n$

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$

Since, Σa_n is convergent, therefore $\lim s_n$ and $\lim s_{n-1}$ both have same finite value as $n \to \infty$ and hence, $\lim_{n\to\infty} a_n = 0$.

To prove that its converse is not true, we consider the example

$$\Sigma a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \infty$$

Here, $a_n = \dfrac{1}{n}$

$$\therefore \quad \lim_{n\to\infty} a_n = 0$$

But the series Σa_n is not convergent.

Q 6. If Σu_n is convergent series, show that $\lim_{n\to\infty} u_n = 0$. **(2010)**

Sol. See the solution of Q. 5.
No need to show that the converse part in this question.

Q 7. Show that the series $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$ is absolutely convergent. **(2017)**

Sol. The term of the given series are alternately positive and negative.
Each term is numerically less than preceding term

$$1)^n \frac{1}{\ } = 0$$

$$\therefore \quad \lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \left| (-1)^n \frac{1}{n!} \right| = \lim_{n\to\infty} \frac{1}{n!} = 0$$

Hence, the given series is absolutely convergent.

Short Answer Questions

Q 1. **Test the convergence of the series** $\Sigma\left\{\frac{\log n}{\log(n+1)}\right\}^{n^2 \log n}$.

(2014, 1994)

Sol. We have, $u_n = \left\{\frac{\log n}{\log(n+1)}\right\}^{n^2 \log n}$

Therefore, $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left\{\frac{\log n}{\log(n+1)}\right\}^{n \log n}$

$$= \lim_{n\to\infty} \left\{\frac{\log n}{\log n + \log\left(1+\frac{1}{n}\right)}\right\}^{n\log n} = \lim_{n\to\infty} \left\{\frac{\log n}{\log n + \frac{1}{n} - \frac{1}{2n^2} + \dots}\right\}^{n\log n}$$

$$= \lim_{n\to\infty} \left\{\frac{\log n}{\log n\left(1 + \frac{1}{n\log n} - \frac{1}{2n^2 \log n} + \dots\right)}\right\}^{n\log n}$$

$$= \lim_{n\to\infty} \left\{\frac{1}{1 + \frac{1}{n\log n} - \frac{1}{2n^2 \log n} + \dots}\right\}^{n\log n}$$

Let $\quad y = \lim_{n\to\infty} \left\{\frac{1}{1 + \frac{1}{n\log n} - \frac{1}{2n^2 \log n} + \dots}\right\}^{n\log n}$

Taking log on both sides, we get

$$\log y = \lim_{n\to\infty} (-n\log n) \log\left[1 + \frac{1}{n\log n} - \frac{1}{2n^2 \log n} + \dots\right]$$

$$\Rightarrow \quad \log y = \lim_{n\to\infty}\left[-1 + \frac{1}{2n}\right]$$

$$\Rightarrow \quad \log y = -1 \Rightarrow y = e^{-1} = \frac{1}{e} < 1$$

Hence, the given series is convergent.

Q 2. **Test the convergence of the series $\Sigma \frac{1}{n} \sin \frac{1}{n}$.**

Sol. We have, $u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{n^3}\cdot\frac{1}{3!} + \frac{1}{n^5}\cdot\frac{1}{5!} - \ldots\right]$

$$= \frac{1}{n^2}\left[1 - \frac{1}{n^2}\cdot\frac{1}{3!} + \frac{1}{n^4}\cdot\frac{1}{5!} - \ldots\right]$$

Now, we choose $v_n = \frac{1}{n^2}$

Then, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[1 - \frac{1}{n^2}\cdot\frac{1}{3!} + \frac{1}{n^4}\cdot\frac{1}{5!} - \ldots\right]$

$= 1$ (Finite non-zero number)

Since, v_n is convergent by auxiliary series.
Hence, Σu_n is convergent series.

Q 3. **Test the convergence of the series**

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \ldots + \frac{(2n-1)}{n(n+1)(n+2)} + \ldots.$$ **(2007)**

Sol. We have, $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{2\left(1 - \frac{1}{2n}\right)}{n^2\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$

Now, let $v_n = \frac{1}{n^2}$

Then, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{2\left(1 - \frac{1}{2n}\right)}{n^2\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2\left(1 - \frac{1}{2n}\right)}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$

$= 2$ (Finite non-zero number)

Since, v_n is convergent by the auxiliary series.
Hence, Σu_n is convergent series.

Q 4. **Test the convergent of infinite series $\sum_{n=1}^{\infty} \frac{4n+1}{4n+3} \cdot x^n$.** **(2001)**

Sol. We have, $u_n = \frac{4n+1}{4n+3}\cdot x^n$

$\therefore$ $u_{n+1} = \frac{4n+5}{4n+7}\cdot x^{n+1}$

Then, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(4n+1)(4n+7)}{(4n+3)(4n+5)} \cdot \frac{1}{x}$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{4n}\right)\left(1+\frac{7}{4n}\right)}{\left(1+\frac{3}{4n}\right)\left(1+\frac{5}{4n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence, the given series is convergent, if $\frac{1}{x} > 1$ or $x < 1$ and divergent, if $x > 1$.

If $x = 1$, then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$.

So, the test fails.

In this case, $u_n = \frac{4n+1}{4n+3}$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(4n+1)}{(4n+3)} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{4n}\right)}{\left(1+\frac{3}{4n}\right)} = 1$$

Hence, the given series is divergent at $x = 1$.

Q 5. Test the convergence of the series $1 + \frac{2x}{2!} + \frac{3^2x^2}{3!} + \dots$. **(2005, 01)**

Sol. We have, $u_n = \frac{(nx)^{n-1}}{n!}$

$$\therefore \quad u_{n+1} = \frac{(n+1)^n x^n}{n+1!}$$

Then, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^{n-1}}{(n+1)^n} \cdot \frac{1}{x}(n+1)$

$$= \lim_{n\to\infty} \frac{n^{n-1}}{(n+1)^{n-1}} \cdot \frac{1}{x} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n-1}} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{-1}} \cdot \frac{1}{x} = \frac{1}{e(1+0)} \cdot \frac{1}{x} = \frac{1}{ex}$$

Hence, the given series is convergent, if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ and divergent, if $x > \frac{1}{e}$.

If $x = \frac{1}{e}$, then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$, test fails

Then, $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log \left[\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}} \right]$

$$= \lim_{n\to\infty} n \left[\log e - (n-1) \log \left(1+\frac{1}{n}\right) \right]$$

$$= \lim_{n\to\infty} n \left[1 - \frac{n-1}{n} + \frac{n-1}{2n^2} - \frac{n-1}{3n^3} + \dots \right]$$

$$= \lim_{n\to\infty} \left[n - (n-1) + \frac{n-1}{2n} - \frac{n-1}{3n^2} + \dots \right]$$

$$= \lim_{n\to\infty} \left[1 + \left(\frac{1}{2} - \frac{1}{2n}\right) - \left(\frac{1}{3n} - \frac{1}{3n^2}\right) + \dots \right]$$

$$= 1 + \frac{1}{2} = \frac{3}{2} > 0$$

Hence, the given series is convergent at $x = \frac{1}{e}$.

Q 6. Test the convergence of the series

$$\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \frac{(\log 4)^2}{4^2} + \dots + \frac{(\log n)^2}{n^2} + \dots .$$

Sol. Since, $\log 1 = 0$, so we may take the first term of the given series as $\frac{(\log 1)^2}{1}$ without affecting the sum.

Hence, the nth term of the given series is $f(n) = \frac{(\log n)^2}{n^2}$

Now, $m^n f(m^n) = m^n \frac{(\log m^n)^2}{(m^n)^2} = \frac{(n \log m)^2}{m^n} = \frac{n^2 (\log m)}{m^n}$, $m > 1$

i.e. $\Sigma u_n = \Sigma \frac{n^2 (\log m)^2}{m^n}$, then

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2 (\log m)^2}{m^n} \times \frac{m^{n+1}}{(n+1)(\log m)^2}$$

$$= \lim_{n\to\infty} m \left(\frac{n}{n+1} \right)^2$$

$$= \lim_{n\to\infty} \frac{m}{\left(1+\frac{1}{n}\right)^2} = m > 1$$

Hence, the given series is convergent.

Q 7. Prove that the series $\sum_{n=-1}^{\infty}(-1)^n [\sqrt{n^2+1}-n]$ is convergent. **(2001)**

Sol. We have, $u_n = \sqrt{n^2+1} - n$

$$\Rightarrow \quad u_n = n\sqrt{1+\frac{1}{n^2}} - n$$

$$\Rightarrow \quad u_n = n\left\{1+\frac{1}{2n^2}-\frac{1}{8n^4}+\ldots\right\} - n$$

$$\Rightarrow \quad u_n = \frac{1}{2n} - \frac{1}{8n^3} + \ldots$$

and
$$u_{n+1} = \frac{1}{2(n+1)} - \frac{1}{8(n+1)^3} + \ldots$$

Now,
$$u_{n+1} - u_n = \frac{1}{2}\left[\frac{1}{(n+1)} - \frac{1}{n}\right] - \frac{1}{8}\left[\frac{1}{(n+1)^3} - \frac{1}{n^3}\right] + \ldots$$

$$\Rightarrow \quad u_{n+1} - u_n = \frac{1}{2}\left[\frac{-1}{n(n+1)}\right] - \frac{1}{8}\left[\frac{n^3-(n+1)^3}{n^3(n+1)^3}\right] + \ldots$$

$\Rightarrow \quad u_{n+1} - u_n < 0$

$\Rightarrow \quad u_{n+1} < u_n$ and $\lim_{n\to\infty} u_n = 0$

Hence, by Leibnitz's theorem, $\sum_{n=1}^{\infty}(-1)^n [\sqrt{n^2+1}-n]$ is convergent.

Hence proved.

Q 8. Show that the series whose *n*th term is $[\sqrt{n^2+1}-n]$ is divergent. **(1992)**

Sol. We have, $u_n = \sqrt{n^2+1} - n = n\left(1+\frac{1}{n^2}\right)^{1/2} - n$

$$= n\left\{1 + \frac{1}{2}\cdot\frac{1}{n^2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{1}{n^4} + \ldots\right\} - n$$

$$= \frac{1}{2n} - \frac{1}{8n^3} + \ldots$$

Therefore, the approximate value of $u_n = \frac{1}{2n}$.

Now, let $v_n = \frac{1}{n}$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{8n^2} + \ldots\right] = \frac{1}{2} \text{ (A finite non-zero number)}$$

Hence, Σu_n is divergent, because Σv_n is divergent by auxiliary series.

Q 9. Discuss the convergence of the series $\Sigma \frac{1}{n^p}$. **(2003)**

Or **Prove that $\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ is convergent when $p > 1$ and divergent when $p < 1$.** **(2006)**

***Sol. Case* I** When $p > 1$

$$\Sigma \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} < 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} < 1 + \frac{2}{2^p} + \frac{4}{4^p} + \ldots \Rightarrow \Sigma \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} < \text{A}$$

geometric series whose common ratio $\frac{1}{2^{p-1}} < 1$.

Hence, $\Sigma \frac{1}{n^p}$ is convergent.

***Case* II** When $p = 1$

$$\Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} > \text{A divergent series}$$

Hence, $\Sigma \frac{1}{n^p}$ is divergent.

***Case* III** When $p < 1$

$$\Sigma \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots. \Rightarrow \Sigma \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} > \text{A divergent series}$$

Hence, $\Sigma \frac{1}{n^p}$ is a divergent series.

Thus, for $p < 1$, $\Sigma \frac{1}{n^p}$ is convergent and for $p \geq 1$, $\Sigma \frac{1}{n^p}$ is divergent.

Hence proved.

Q 10. Discuss the nature of the series $\sum_{n=\infty}^{\infty} \frac{n}{n^2+1} \cdot x^n$. (2016, 15, 1995)

Sol. We have, $u_n = \frac{n}{n^2+1} \cdot x^n$

$$\therefore \qquad u_{n+1} = \frac{n+1}{(n+1)^2+1} x^{n+1}$$

Now,
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot x^n \times \frac{(n+1)^2+1}{(n+1)} \cdot \frac{1}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n(n^2+2n+2)}{(n^2+1)(n+1)} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{n^3\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right) n\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

If $\frac{1}{x} > 1$ or $x < 1$, then the series is convergent.

If $\frac{1}{x} < 1$ or $x > 1$, then the series is divergent.

If $x = 1$, then $u_n = \frac{n}{n^2+1} = \frac{1}{n\left(1+\frac{1}{n^2}\right)}$

Let $v_n = \frac{1}{n}$, then
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n\left(1+\frac{1}{n^2}\right)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)} = 1$$

But the auxiliary series Σv_n is divergent.
Hence, Σu_n is divergent when $n = 1$.

Q 11. Discuss the nature of the geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots$. (2015)

Sol.

(i) When $x < 1$, then, $s_n = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$

Therefore, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x^n}{1-x}\right)$

But as $n \to \infty$, $x^n \to 0$ for $x < 1$.
Hence, the series is convergent.

(ii) When $x > 1$, then $s_n = \dfrac{x^n - 1}{x - 1} = \dfrac{x^n}{x-1} - \dfrac{1}{x-1}$

Therefore, $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\dfrac{x^n}{x-1} - \dfrac{1}{x-1} \right)$

But as $n \to \infty$, $x^n \to \infty$, for $x > 1$

$\therefore$ $$\lim_{n \to \infty} s_n = \infty$$

Hence, the series is divergent.

(iii) When $x = 1$, $s_n = 1 + 1 + 1 + \ldots + 1$ to n times $= n$

Therefore, $\lim_{n \to \infty} s_n = \infty$

Hence, the series is divergent.

(iv) When $x = -1$, then series become $1 - 1 + 1 - 1 + \ldots$, which is oscillates, hence the series is oscillating or periodic.

Q 12. Define Raabe's test and investigate convergence of the series

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots.$$

(2014)

Sol. **Part I Raabe's Test** Let Σu_n be the series of positive terms and $\lim_{n \to \infty} \left\{ n \left(\dfrac{u_n}{u_{n+1}} - 1 \right) \right\} = l$. Then,

(i) the series is convergent, if $l > 1$.

(ii) the series is divergent, if $l < 1$.

(iii) test fail, if $l = 1$.

Part II We have, $u_n = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2 \cdot 4 \cdot 6 \ldots (2n-2)} \cdot \dfrac{x^{2n-1}}{2n-1}$

$\therefore$ $$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n-2) \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

Now, $$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{2n(2n+1)}{(2n-1)\,(2n-1)} \cdot \frac{1}{x^2}$$

$$= \lim_{n \to \infty} \frac{2n \cdot 2n \left(1 + \frac{1}{2n}\right)}{2n\left(1 - \frac{1}{2n}\right) 2n \left(1 - \frac{1}{2n}\right)} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

Hence, if $\dfrac{1}{x^2} > 1$, i.e. $x^2 < 1$, the series is convergent and if $x^2 > 1$, then the series is divergent.

If $x^2 = 1$, then $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left[\frac{2n \cdot 2n \cdot \left(1 + \frac{1}{2n}\right)}{2n\left(1 - \frac{1}{2n}\right) \cdot 2n\left(1 - \frac{1}{2n}\right)} \right] = 1$

So, the test fails.

Now, applying Raabe's test,

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n \to \infty} n\left\{\frac{2n(2n+1)}{(2n-1)^2} - 1\right\}$$

$$= \lim_{n \to \infty} n\left\{\frac{4n^2 + 2n}{4n^2 - 4n + 1} - 1\right\}$$

$$= \lim_{n \to \infty} n\left\{\frac{6n - 1}{4n^2 - 4n + 1}\right\}$$

$$= \lim_{n \to \infty} \frac{6n^2\left(1 - \frac{1}{6n}\right)}{4n^2\left(1 - \frac{1}{n} + \frac{1}{4n^2}\right)} = \frac{6}{4} = \frac{3}{2} > 1$$

Hence, the series is convergent.

***Q* 13. State and prove Cauchy's *n*th root test for the convergence of a series of positive terms.** **(2013, 03, 01)**

***Sol.* Statement** Suppose Σu_n be series of positive terms such that $\lim (u_n)^{1/n} = l$, then

(i) Σu_n is convergent, if $l < 1$. (ii) Σu_n is divergent, if $l > 1$.

(iii) test fails, if $l = 1$.

Proof Let us suppose $\lim_{n \to \infty} (u_n)^{1/n} = l$.

Let $0 < \varepsilon < l$, so that $l - \varepsilon$ is positive.

Hence, there exists a natural number m such that

$$|(u_n)^{1/n} - l| < \varepsilon, \text{ for all } n \geq m$$

i.e. $l - \varepsilon < (u_n)^{1/n} < l + \varepsilon$, for all $n \geq m$

i.e. $(l - \varepsilon)^n < u_n < (l + \varepsilon)^n$, for all $n \geq m$ ...(i)

Now, we consider the following cases

***Case* I** When $l > 1$, then we choose $\varepsilon > 0$ such that $l - \varepsilon > 1$, set $l - \varepsilon = p$, so that $p > 1$.

Hence, from the Eq. (i), we have $p^n < u_n$, for all $n \geq m$,

i.e.
$$p^m < u_m$$
$$p^{m+1} < u_{m+1}$$
$$p^{m+2} < u_{m+2}$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$

On adding above inequalities, we get

$$p^m + p^{m+1} + p^{m+2} + \dots < u_m + u_{m+1} + u_{m+2} + \dots$$

i.e. $$u_m + u_{m+1} + u_{m+2} + \dots > p^m + p^{m+1} + p^{m+2} + \dots$$

Since, the common ratio p is greater than 1

$\therefore$ $$\Sigma u_n > 1$$

Hence, Σu_n is divergent, when $l > 1$.

***Case* II** When $l < 1$, then we choose $\varepsilon > 0$ such that $l < 1 + \varepsilon < 1$ set $l + \varepsilon = s$, so that $0 < s < 1$.

Hence, from Eq. (i), we have

$$u_n < s^n, \text{ for all } n \geq m$$

i.e.
$$u_m < s^m$$
$$u_{m+1} < s^{m+1}$$
$$u_{m+2} < s^{m+2}$$
....

....

On adding above inequalities, we get

$$u_m + u_{m+1} + u_{m+2} + \dots < s^m + s^{m+1} + < s^{m+2} + \dots$$

Since, the common ratio s is less than 1.

$\therefore$ $$\Sigma u_n < 1$$

Hence, Σu_n is convergent, when $l < 1$. **Hence proved.**

***Q* 14. State and prove Logarithmic test.** **(2007, 03, 02, 2000)**

***Sol.* Statement** Let Σu_n be the series of the positive terms and

$$\lim_{n \to \infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l.$$

Then,

(i) the series is convergent, if $l > 1$.

(ii) the series is divergent, if $l < 1$.

(iii) test fails, if $= 1$.

Proof *Case* I When $l > 1$, Σu_n is convergent, if

$$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p, \forall\, p > 1$$

if $$\log\left(\frac{u_n}{u_{n+1}}\right) > p \log\left(\frac{n+1}{n}\right)$$

$$> p \log\left(1 + \frac{1}{n}\right)$$

$$> p\left(\frac{1}{n} - \frac{1}{2n^2} + \dots\right)$$

if $$n \log\left(\frac{u_n}{u_{n+1}}\right) \geq p\left(1 - \frac{1}{2n} + \dots\right)$$

$$\Rightarrow \quad \lim_{n \to \infty}\left(n \log \frac{u_n}{u_{n+1}}\right) \geq p > 1$$

So, Σu_n is convergent.

***Case* II** When $l < 1$, then the series is divergent, if

$$\frac{u_n}{u_{n+1}} < \left(\frac{n+1}{n}\right)^p, \forall\ p < 1$$

$$\text{if} \quad \log\left(\frac{u_n}{u_{n+1}}\right) < p \log\left(\frac{n+1}{n}\right)$$

$$< p \log\left(1 + \frac{1}{n}\right)$$

$$< p\left(\frac{1}{n} - \frac{1}{2n^2} + \ldots\right)$$

$$\text{if} \quad \lim_{n \to \infty} n \log\left(\frac{u_n}{u_{n+1}}\right) \leq p < 1$$

So, Σu_n is divergent. **Hence proved.**

Long Answer Questions

Q 1. Investigate the convergence and divergence of the series $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots +$ and also write the statements of all the test you are using in it. **(2016)**

Sol. We have, $u_n = \dfrac{1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \ldots (2n)^2}$

$$\therefore \quad u_{n+1} \frac{1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \ldots (2n)^2 (2n+2)^2}$$

$$\text{Now,} \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{(2n+2)^2}{(2n+1)^2} = \lim_{n \to \infty} \frac{4n^2 + 8n + 4}{4n^2 + 4n + 1}$$

$$= \lim_{n \to \infty} \frac{4n^2\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)}{4n^2\left(1 + \frac{2}{n} + \frac{1}{4n^2}\right)} = 1$$

So, the ratio test fails.

Now, applying Raabe's test, $\lim_{n \to \infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$

$$= \lim_{n \to \infty} n\left[\frac{4n^2 + 8n + 4}{4n^2 + 4n + 1} - 1\right]$$

$$= \lim_{n \to \infty} n\left[\frac{4n + 3}{4n^2 + 4n + 1}\right] = \lim_{n \to \infty}\left[\frac{1 + \frac{3}{4n}}{1 + \frac{1}{n} + \frac{1}{4n^2}}\right] = 1$$

So, Raabe's test fails. Now, applying De-Morgan's and Bertrand's test,

$$\lim_{n \to \infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right]\log n$$

$$= \lim_{n \to \infty}\left[n\left(\frac{4n + 3}{4n^2 + 4n + 1}\right) - 1\right]\log n$$

$$= \lim_{n \to \infty}\left[\frac{-(n + 1)}{4n^2 + 4n + 1}\right]\log n$$

$$= \lim_{n \to \infty}\left[\frac{-(n^2 + n)}{4n^2 + 4n + 1}\right]\cdot\frac{\log n}{n}$$

$$= \lim_{n \to \infty} -\frac{1}{4}\left[\frac{\left(1 + \frac{1}{n}\right)}{\left(1 + \frac{1}{n} + \frac{1}{4n^2}\right)}\right]\cdot\lim_{n \to \infty}\frac{\log n}{n}$$

$$= -\frac{1}{4} \times 0 = 0 < 1$$

Hence, the series is divergent.

Here, we apply three tests, such as Ratio test, Raabe's test and De-Morgan's and Bertrand's test. For the statements of these tests, see the synopsis part.

Q 2. Investigate the convergence of the following series.

(i) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots$ **(2013)**

(ii) $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$ **(2013, 06, 1997)**

Sol. (i) On neglecting the first term, we have

$$u_n = \frac{x^{2n}}{(n + 2)\sqrt{(n + 1)}}$$

$$\therefore \quad u_{n+1} = \frac{x^{2n+2}}{(n + 3)\sqrt{(n + 2)}}$$

Now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n\left[1+\frac{3}{n}\right] n^{1/2}\left[1+\frac{2}{n}\right]^{1/2}}{n\left[1+\frac{2}{n}\right] n^{1/2}\left[1+\frac{1}{n}\right]^{1/2}} \cdot \frac{1}{x^2} = \frac{1}{x^2}$

Hence, the series is convergent, if $\frac{1}{x^2} > 1$ or $x^2 < 1$ and divergent, if $x^2 > 1$.

If $x^2 = 1$, then series become $\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \ldots$

Therefore, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Let $v_n = \frac{1}{n^{3/2}}$

Then, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1$ (Non-zero and finite value)

But the auxiliary series $\Sigma u_n = \frac{1}{\Sigma n^{3/2}}$ is convergent.

Hence, Σu_n is convergent, if $x^2 = 1$.

(ii) We have, $u_n = \frac{n^n x^n}{n!}$

$\therefore \quad u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

Now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{e} \cdot \frac{1}{x}$

Hence, the series is convergent, if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ and divergent, if $x > \frac{1}{e}$.

If $x = \frac{1}{e}$, then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$, test fails.

Now, we apply Logarithmic test, we have $\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\therefore \quad \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} n \left\{\log e - n \log\left(1+\frac{1}{n}\right)\right\}$$

$$= \lim_{n \to \infty} n\left\{1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right)\right\}$$

$$= \lim_{n \to \infty} n\left\{1 - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} + \ldots\right)\right\}$$

$$= \lim_{n \to \infty} n\left\{\frac{1}{2n} - \frac{1}{3n^2} - \ldots\right\}$$

$$= \lim_{n \to \infty}\left\{\frac{1}{2} - \frac{1}{3n} - \ldots\right\} = \frac{1}{2} < 1$$

Hence, the given series is divergent.

Q 3. Investigate the convergence of the following series.

(i) $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots$

(ii) $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \ldots$ **(2012)**

Sol. (i) See the solution of Q. 12 of Short Answer Questions.

(ii) We have, $u_n = \frac{x^n}{n(n+1)}$

$\therefore \qquad u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

Now, $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty}\left(1 + \frac{2}{n}\right) \cdot \frac{1}{x} = \frac{1}{x}$

Hence, the given series is convergent, if $\frac{1}{x} > 1$ or $x < 1$ and divergent, if $x > 1$.

If $x = 1$, then $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty}\left(1 + \frac{2}{n}\right) = 1$, test fails.

Therefore, $u_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1 + \frac{1}{n}\right)} = \frac{1}{n^2}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \ldots\right)$

Taking $\qquad v_n = \frac{1}{n^2}$

Now, $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \ldots\right)$

$= 1$ (Non-zero finite value)

But Σv_n is convergent by auxiliary series.

Hence, Σu_n is convergent.

Q 4. Discuss the convergence of the following series.

(i) $\Sigma(\sqrt{n^4+1}-\sqrt{n^4-1})$ **(ii)** $\Sigma\dfrac{\left(1+\dfrac{1}{n}\right)^{2n}}{e^n}$ **(2017)**

Sol.

(i) We have, $u_n = n^2\left(1+\dfrac{1}{n^4}\right)^{1/2} - n^2\left(1-\dfrac{1}{n^4}\right)^{1/2}$

$$= n^2\left\{1+\frac{1}{2}\cdot\frac{1}{n^4}-\frac{1}{8}\cdot\frac{1}{n^8}+\frac{1}{16n^{12}}-\ldots\right\}$$
$$-n^2\left\{1-\frac{1}{2}\cdot\frac{1}{n^4}-\frac{1}{8}\cdot\frac{1}{n^8}-\frac{1}{16n^{12}}+\ldots\right\}$$
$$= n^2\left\{\frac{1}{n^4}+\frac{1}{8}\cdot\frac{1}{n^{12}}+\ldots\right\}$$
$$= \frac{1}{n^2}+\frac{1}{8}\cdot\frac{1}{n^{10}}+\ldots$$

Let $v_n = \dfrac{1}{n^2}$, then $\Sigma v_n = \Sigma\dfrac{1}{n^2}$ is convergent.

Now, $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = 1$ (Non-zero finite quantity)

Hence, Σu_n is convergent.

(ii) We have, $u_n = \dfrac{\left(1+\dfrac{1}{n}\right)^{2n}}{e^n} \Rightarrow \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n}\right)^{2n}}{e^n} = \dfrac{e^2}{e^\infty} = 0\times e^2 = 0$

$[\because e^\infty = \infty]$

Hence, $\Sigma u_n = \Sigma\dfrac{\left(1+\dfrac{1}{n}\right)^{2n}}{e^n}$ is convergent.

Q 5. (i) State and prove the comparison test for the convergence of a series. **(2010, 1995)**

(ii) Test the convergence of the series $\sum\limits_{n\to\infty}^{\infty}\left(x^{2n}+\dfrac{1}{x^{2n}}\right)$. **(2010)**

Sol.

(i) **Statement**

(a) If Σv_n and Σu_n are two series of positive terms and Σv_n is convergent, then Σu_n is convergent, if $u_n \le v_n$, $\forall\, n \in N$ and Σu_n is divergent, if Σv_n is divergent, when $u_n \ge v_n$, $\forall\, n \in N$

(b) If Σu_n and Σv_n are two series of positive terms, then

- Σu_n will be convergent, if from and after some fixed terms $\lim_{n\to\infty} \frac{u_n}{v_n}$ is finite (including zero). Then, Σv_n is convergent.
- Σu_n will be divergent, if from and after some fixed terms, $\lim_{n\to\infty} \frac{u_n}{v_n}$ is some finite number greater than zero or infinite, then Σv_n is divergent.

Proof

(a) If Σv_n is convergent, we have $\lim_{n\to\infty} \Sigma v_n = S$, where S is a finite number, since $u_n \le v_n$, we have $\Sigma u_n \le \Sigma v_n$.

Therefore, $\lim_{n\to\infty} \Sigma u_n \le \lim_{n\to\infty} \Sigma v_n$

But Σv_n is convergent.

Hence, Σu_n is convergent.

Again, if Σv_n is divergent, we have $\lim_{n\to\infty} \Sigma v_n = \infty$

Since, $u_n \ge v_n$

Therefore, $\lim_{n\to\infty} \Sigma u_n \ge \lim_{n\to\infty} \Sigma v_n$

Hence, Σu_n is divergent. **Hence proved.**

(b) • Let $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (say), then for a given $\varepsilon > 0$, there exists m such that $(l - \varepsilon)v_n < u_n < (l + \varepsilon)v_n$, for $n \ge m$.

Now, $u_n < (l + \varepsilon)\, v_n$ for $n \ge m$ and Σv_n is convergent, hence Σu_m is also convergent.

If $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$, then for a given $\varepsilon > 0$, there exists m such that

$0 - \varepsilon < \frac{u_n}{v_n} < 0 + \varepsilon$, for $n \ge m \Rightarrow -\varepsilon v_n < u_n < \varepsilon v_n$, for $n \ge m$

Since, Σv_n is convergent, hence Σu_n is convergent.

• Let $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (A finite number > 0)

Then for given $\varepsilon > 0$, there exists m such that $l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon$, for $n \ge m$

or $\qquad u_n > (l - \varepsilon)\, v_n$, for $n \ge m$.

Since, Σv_n is divergent, therefore Σu_n is divergent.

If $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$, then $u_n > kv_n$, for any number k.

Hence, Σu_n is divergent. **Hence proved.**

(ii) We have, $u_n = x^{2n} + \frac{1}{x^{2n}} = \frac{x^{4n}+1}{x^{2n}}$

and $u_{n+1} = \frac{x^{4(n+1)}+1}{x^{2(n+1)}} = \frac{x^{4n+4}+1}{x^{2n+2}}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{4n}+1}{x^{2n}} \cdot \frac{x^{2n+2}}{x^{4n+4}+1} = \lim_{n\to\infty} \frac{(x^{4n}+1)x^2}{(x^{4n+4}+1)} = \frac{1}{x^2} > 1$$

Hence, the given series is convergent for $x^2 < 1$ and divergent for $x^2 > 1$.

If $x^2 = 1$, then $\frac{u_n}{u_{n+1}} = 1$, ratio test fails.

Now, for $x = 1$, $u_n = 1 + 1 = 2$

Therefore, $\sum_{n\to 0}^{\infty} u_n = \sum_{n=0}^{\infty} \left(x^{2n} + \frac{1}{x^{2n}}\right)$ is convergent.

Q 6. (i) State and prove the D' Alembert's ratio test for the convergence of the series. **(2011, 09, 08, 02)**

(ii) Test the convergence of the series

$$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \ldots .$$ **(2011, 1995)**

Sol.

(i) **Statement** Suppose Σu_n be a positive terms series such that

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$, then

(a) the series is convergent, if $l > 1$.

(b) the series is divergent, if $l < 1$.

(c) test fails, if $l = 1$.

Proof Let the series beginning from the fixed term be

$$u_1 + u_2 + u_3 + u_4 + \ldots + \infty$$

and let $\frac{u_1}{u_2} > l$ or $\frac{u_2}{u_1} < \frac{1}{l} = k$ (say), where $k < 1$

So that, $\frac{u_2}{u_1} < k, \frac{u_3}{u_2} < k, \frac{u_4}{u_3} < k, \ldots$

$$\text{Now,} \quad \Sigma u_n = u_1 + u_2 + u_3 + \ldots + \infty$$

$$= u_1\left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots + \infty\right]$$

$$= u_1\left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots + \infty\right]$$

$$< u_1[1 + k + k^2 + k^3 + \ldots \infty]$$

$$< \frac{u_1}{1-k} \text{ (A fixed number)}$$

Hence, the given series is convergent.

$$\text{If } \frac{u_1}{u_2} \le 1,\ \frac{u_2}{u_3} \le 1,\ \frac{u_3}{u_4} \le 1, \ldots$$

$$\text{i.e. if } \frac{u_2}{u_1} > 1,\ \frac{u_3}{u_2} \ge 1,\ \frac{u_4}{u_3} \ge 1, \ldots$$

$$\text{if } u_2 \ge u_1,\ u_3 \ge u_2,\ u_4 \ge u_3, \ldots$$

Then, the sum of nth terms is given by

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$\ge u_1 + u_1 + u_1 + u_1 + \ldots \text{ (}n\text{ terms)}$$

$$\ge nu_1$$

$$\therefore \quad \lim_{n \to \infty} s_n = \infty$$

Hence, the given series is divergent. **Hence proved.**

(ii) We have, $u_n = \dfrac{x^n}{(2n-1)(2n)}$

$$\therefore \quad u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\text{Now, } \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{(2n+1)(2n+2)}{(2n-1)(2n)} \cdot \frac{1}{x}$$

$$= \lim_{n \to \infty} \frac{\left(1 + \frac{1}{2n}\right)\left(1 + \frac{1}{n}\right)}{\left(1 - \frac{1}{2n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence, the given series is convergent, if $\frac{1}{x} > 1$ or $x < 1$ and divergent, if $x > 1$.

If $x = 1$, then $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = 1$, test fails.

$$\text{Let } u_n = \frac{1}{(2n-1)2n} = \frac{1}{4n^2\left(1 - \frac{1}{n}\right)}$$

and we choose $v_n = \dfrac{1}{n^2}$, then

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{4\left(1 - \frac{1}{n}\right)} = \frac{1}{4} \text{ (A non-zero finite value)}$$