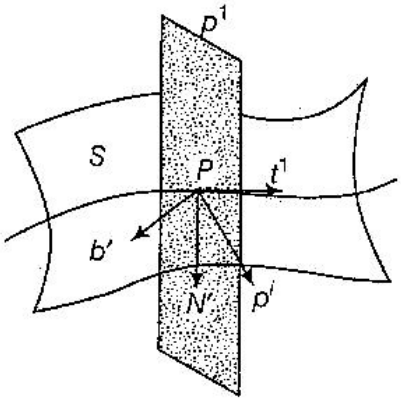
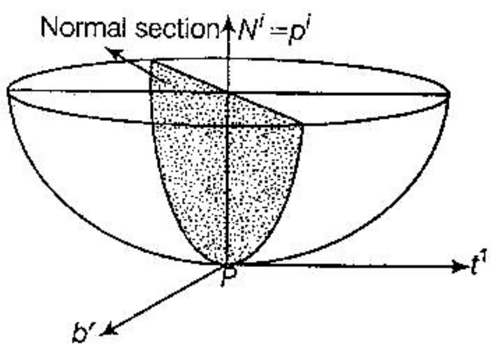
Chapter Nine

SECOND FUNDAMENTAL FORM AND CURVATURE OF SURFACES

Important Points from the Chapter

1. **Second Fundamental Form** A tensor $d_{\alpha\beta}$ on the surface is defined as $d_{\alpha\beta} = N^i x^i_{,\alpha\beta}$ which is called the second fundamental tensor of the surface and the quadratic form $d_{\alpha\beta}\, du^\alpha du^\beta$ is called the second fundamental form of the surface.
2. Gauss equation is $x^i_{\alpha\beta} = d_{\alpha\beta} N^i$.
3. Weingarten equation is $N^i_{,\alpha} = -\, d_{\alpha\gamma} g^{\gamma\beta} x^i_{,\beta}$.
4. **Normal Curvatures** The section of surface by the normal plane p^1 is called '**normal section**' and curvature of normal section at a point is called the '**normal curvature**' at that point and it is denoted by κ_n.





The section of surface other than the normal plane is called the **oblique** section, otherwise the curve is called **normal section.** The principal normal p^i to the normal section is parallel to the surface normal N^i. **(2016)**

5. **Principal Directions and Principal Curvatures**

(i) **Principal directions** The normal section of a surface through a given point having maximum or minimum curvatures at the point are called **principal sections** of the surface at that point and the tangent to these sections are called principal directions at the point.

(ii) **Principal curvatures** The curvature of the principal sections of a surface through a given point, i.e the maximum and minimum curvatures at that point are called **principal curvatures** at that point and their corresponding radius of curvatures are called principal radius of curvatures.

6. **Mean Curvature** The arithmetic mean of the principal curvature κ_1 and κ_2 at a point is called the mean curvature at the point and is denoted by M,

i.e. $$M = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} g^{\alpha\beta} d_{\alpha\beta}.$$ **(2014)**

7. **Gaussian Curvature** The product of the principal curvatures κ_1 and κ_2 at a point is called Gaussian curvature at the point and is denoted by κ,

i.e. $$\kappa = \kappa_1\kappa_2 = \frac{d}{g} = \frac{d_{11}d_{22} - d_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$ **(2014)**

8. **Umbilic Point** A point on a surface is called an umbilic point, if at each point, we have $d_{\alpha\beta} = \lambda g_{\alpha\beta}$.

9. **Minimal Surface** If the mean curvature of a surface is zero at all points, then the surface is called a minimal surface.

   Hence, the surface will be minimal, if
   $$M = 0 \Rightarrow \kappa_1 + \kappa_2 = 0$$
   $$\Rightarrow \quad g^{\alpha\beta} d_{\alpha\beta} = 0, \text{ at every point of the surface.}$$ **(2014, 12)**

10. **Developable Surface** The surface, for which the Gaussian curvature κ is zero, called the developable surface.

    Hence, the surface will be developable, if
    $$\kappa = 0 \Rightarrow d = 0 \Rightarrow d_{11}d_{22} - (d_{12})^2 = 0.$$

11. **Conjugate Directions** Let P be a point on surface $x^i = x^i(u^\alpha)$ and P and Q are two neighbouring points and PR be a live parallel to the line of intersection L. The limiting position of the directions PQ and PR as Q tends to P, are called conjugate directions at P.

12. **Lines of Curvature** A curve on a surface is called a line of curvature, if the tangent at any point of it is along the principal direction at that point.

    The equation of the line of curvature is
    $$\varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta} du^\gamma du^\delta = 0 \text{ or } (d_{\alpha\beta} - k_n g_{\alpha\beta})\, du^\beta = 0$$
    where, k_n is one of the principal curvature.

13. **Asymptotic Lines** The directions which are self-conjugate, are called the asymptotic directions and the curves whose tangents are along asymptotic directions, are called the asymptotic lines. **(2012)**

14. **Null (Minimal) Lines and Isometric Lines** A curve on a surface of zero length is called null lines or minimal lines. Therefore, the differential equation of the null lines is $g_{\alpha\beta} du^\alpha du^\beta = 0$ which is obtained by equating to zero the square of the line element.

15. **Some Important Theorems**

    (i) The necessary and sufficient condition that a surface be plane is that $d_{\alpha\beta} = 0$.

    (ii) The necessary and sufficient condition that a surface be sphere is that $d_{\alpha\beta} = c g_{\alpha\beta}$.

(iii) The surface, for which Gaussian curvature k is zero, is called developable surface.

(iv) The necessary and sufficient condition that the parametric curves at a point be along the lines of curvature are that

$$d_{12} = 0,\ g_{12} = 0 \text{ and } \frac{d_{11}}{g_{11}} \neq \frac{d_{22}}{g_{22}}.$$

(v) The normals to any surface at consecutive points of one of its line of curvature intersect.

(vi) **Rodrigue's formula** A necessary and sufficient condition that a curve on a surface be line of curvature is that $dN^i + \kappa_n dx^i = 0$ at each of its points, where κ_n denotes the normal curvature.

which is known as Rodrigue's formula.

(vii) **Dupin's theorem** The sum of the normal curvature in two orthogonal directions is equal to the sum of the principal curvatures at that point.

(viii) **Joachimsthal's theorem** If the curve of intersection of two surfaces is a line of curvature on both the surfaces, then the surfaces cut at a constant angle.

(ix) If the directions given by $P_{\alpha\beta} du^\alpha du^\beta = 0$ are conjugate, then $d^{\alpha\beta} P_{\alpha\beta} = 0$.

(x) The necessary and sufficient condition that the parametric curves at a point of a surface be conjugate is that $d_{12} = 0$.

(xi) The lines of curvature at a point of a surface form an orthogonal conjugate system.

(xii) The necessary and sufficient condition that two asymptotic directions be orthogonal at every point of the surface is that the surface be minimal.

(xiii) The osculating plane at any point A an asymptotic line is the tangent plane to the surface.

(xiv) At a point on a surface where the Gaussian curvature is negative and equal to k, the torsion of the asymptotic line is $\pm\sqrt{-k}$.

(xv) At a given point of a surface there are two null lines and both are imaginary.

(xvi) The necessary and sufficient conditions that the parametric curves are null lines are $g_{11} = g_{22} = 0,\ g_{12} \neq 0$.

(xvii) When the parametric curves are null lines, then the principal curvature are given by $g_{12}^2 k_n^2 - 2g_{12} d_{12} k_n - (d_{11} d_{22} - d_{12}^2) = 0$ and the lines of curvature are given by $d_{11}(du^1)^2 - d_{22}(du^2)^2 = 0$.

(xviii) The null lines on a minimal surface are conjugate.

(xix) The metric has the form $ds^2 = v^1 (du^1)^2 + v^2 (du^2)^2$, where v^α are functions of u^α, the parameters are isometric.

(xx) $d_{\alpha\beta,\gamma} - d_{\alpha\gamma,\beta} = 0$ is known as Mainardi-Codazzi equation.

Very Short Answer Questions

Q 1. Find the formula for normal curvature in terms of fundamental magnitudes. **(2012)**

Or **Find the equation for normal curvature in terms of fundamental magnitudes.**

Sol. Let $x^i = x^i(u^\delta)$ $(i = 1, 2, 3, \alpha = 1, 2)$ be the equation of surface and p be any point on the surface whose parametric coordinates are u^α. Let κ_n represents the curvature of the normal section, it will be positive when the curve is concave on the side towards which N^i points out.

Then, $$x^{1^i} = \kappa_n p^i = \kappa_n N^i \qquad [\because p^i = N^i]$$

$\therefore$ $$\kappa_n = N^i x''^i$$

Again, we have

$$x''^i = \frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial u^\alpha}{ds} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\cdot\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \frac{\partial x^i}{\partial u^\alpha}\cdot\frac{d^2u^\alpha}{ds^2}$$

$\therefore$ $$\kappa_n = N^i x''^i = N^i \frac{\partial^2 x^i}{\partial u^\alpha du^\beta}\cdot\frac{du^\alpha du^\beta}{ds\, ds} + N^i \frac{\partial x^i}{\partial u^\alpha}\cdot\frac{d^2u^\alpha}{ds^2}$$

$$= d_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = \frac{d_{\alpha\beta}\, du^\alpha du^\beta}{ds^2}$$

$\Rightarrow$ $$\kappa_n = \frac{d_{\alpha\beta}\, du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}$$

which is the required equation for normal curvature.

Q 2. Prove that the normal curvature in a direction perpendicular to an asymptotic line is twice the mean normal curvature.

Sol. Let du^α be an asymptotic line, then the normal curvature in the direction du^α is given by $\kappa_n = \dfrac{d_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta} = 0$

Again, by Euler's theorem, we have

$$\kappa_n = 0 = \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi \qquad \text{...(i)}$$

Now, let κ be the curvature along direction perpendicular to the asymptotic line, then Euler's theorem gives

$$\kappa = \kappa_1 \cos^2(90°+\psi) + \kappa_2 \sin^2(90°+\psi)$$
$$= \kappa_1 \sin^2\psi + \kappa_2 \cos^2\psi \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\kappa = \kappa_1 + \kappa_2 = 2\left(\frac{\kappa_1 + \kappa_2}{2}\right)$$

$\therefore$ $\kappa = 2$ (mean normal curvature at P)

Short Answer Questions

Q.1. Calculate the fundamental magnitude of the surface $x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2,\ x^3 = f(u^1) + cu^2$. **(2017)**

Sol. We have, $x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2,\ x^3 = f(u^1) + cu^2$

$\therefore \quad \partial_1 x^1 = \cos u^2,\ \partial_1 x^2 = \sin u^2,\ \partial_1 x^3 = f'$

$$\partial_2 x^1 = -u^1 \sin u^2,\ \partial_2 x^2 = u^1 \cdot \cos u^2,\ \partial_2 x^3 = c$$

$$\partial_1\partial_1 x^1 = 0,\ \partial_1\partial_1 x^2 = 0,\ \partial_1\partial_1 x^3 = f''$$

$$\partial_1\partial_2 x^1 = -\sin u^2,\ \partial_1\partial_2 x^2 = \cos u^2,\ \partial_1\partial_2 x^3 = 0$$

$$\partial_2\partial_2 x^1 = -u^1 \cos u^2,\ \partial_2\partial_2 x^2 = -u^1 \sin u^2,\ \partial_2\partial_2 x^3 = 0$$

$\therefore \quad g_{11} = (\cos^2 u^2) + \sin^2 u^2 + (f')^2 = 1 + (f')^2 \quad [\because \sin^2 x + \cos^2 x = 1]$

$$g_{12} = -u^1 \sin u^2 \cos u^2 + u^1 \sin u^2 \cos u^2 + cf' = cf'$$

and $\quad g_{22} = (u^1)^2 \sin u^2 + (u^1)^2 \cos^2 u^2 + (c)^2 = (u^1)^2 + c^2$

$$g = g_{11}g_{22} - g_{12}^2 = [1 + (f')^2]\,[(u^1)^2 + c^2] - (cf')^2$$

$$= (u^1)^2 + c^2 + (f')^2 (u^1)^2 + c^2(f')^2 - c^2(f')^2$$

$$= (u^1)^2\,[c^2(f')^2]$$

$$N = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{\sqrt{g}} = \frac{u^1(-f' \cos u^2, -f' \sin u^2, c)}{u^1 \sqrt{c^2 + (f')^2}}$$

$\therefore \quad N_1 = -\dfrac{f' \cos u^2}{\sqrt{c^2 + (f')^2}},\ N_2 = -\dfrac{f' \sin u^2}{\sqrt{c^2 + (f')^2}},\ N_3 = \dfrac{c}{\sqrt{c^2 + (f')^2}}$

Thus, $\quad d_{11} = \dfrac{uf''}{\sqrt{c^2 + (f')^2}}$

$$d_{12} = \frac{f' \sin u^2 \cos u^2 - f' \sin u^2 \cos u^2 - c}{\sqrt{c^2 + (f')^2}} = -\frac{c}{\sqrt{c^2 + (f')^2}}$$

$$d_{22} = \frac{u^1 f' \cos^2 u^2 + u^1 f' \sin^2 u^2 + 0}{\sqrt{c^2 + (f')^2}} = \frac{u^1 f'}{\sqrt{c^2 + (f')^2}}$$

Q 2. Calculate the fundamental magnitudes and normal to the surface $2z = ax^2 + 2bxy + by^2$ taking x, y as parameters.

(2018, 16, 14)

Sol. Given that, $z = \dfrac{ax^2 + 2bxy + by^2}{2}$

The parametric equations are $x = x,\ y = y,\ z = ax^2 + 2bxy + by^2$.

Here, $u^1 = x, u^2 = y$

$\therefore \quad \mathbf{X}_1 = \left(\dfrac{\partial x}{\partial u^1}, \dfrac{\partial y}{\partial u^1}, \dfrac{\partial z}{\partial u^1}\right)\left(1, 0, \dfrac{2ax + 2by}{2}\right) = (1, 0, ax + by)$

and $\mathbf{X}_2 = \left(\frac{\partial x}{\partial u^2}, \frac{\partial y}{\partial u^2}, \frac{\partial z}{\partial u^2}\right) = \left(0, 1, \frac{2bx + 2by}{2}\right) = (1, 0, bx + by)$

Now, $(\partial_1\partial_1 x, \partial_1\partial_1 y, \partial_1\partial_1 z) = (0, 0, a)$,

$(\partial_1\partial_2 x, \partial_1\partial_2 y, \partial_1\partial_2 z) = (0, 0, b)$

and $(\partial_2\partial_2 x, \partial_2\partial_2 y, \partial_2\partial_2 z) = (0, 0, b)$

$\therefore$ $\mathbf{X}_1 \times \mathbf{X}_2 = \{-(ax + by), -(bx + by), 1\}$

Also,
$$g_{11} = \mathbf{X}_1 \times \mathbf{X}_1 = 1 + (ax + by)^2$$
$$g_{12} = \mathbf{X}_1 \times \mathbf{X}_2 = 0 + 0 + (ax + by)(bx + by)$$
$$g_{22} = \mathbf{X}_2 \times \mathbf{X}_2 = 1 + (bx + by)^2$$

Now,
$$g = g_{11}g_{22} - (g_{12})^2$$
$$= [1 + (ax + by)^2][1 + (bx + by)^2] - (ax + by)^2 (bx + by)^2$$
$$= [1 + (ax + by)^2 + (bx + by)^2]$$

$\therefore$
$$\sqrt{g} = \sqrt{[1 + (ax + by)^2 + (bx + by)^2]}$$

$\therefore$
$$N^i = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \left[-\frac{(ax + by)}{\sqrt{g}}, -\frac{(bx + by)}{\sqrt{g}}, \frac{1}{\sqrt{g}}\right]$$

Also,
$$N^1 = -\frac{(ax + by)}{\sqrt{g}}, N^2 = -\frac{(bx + by)}{\sqrt{g}}, N^3 = \frac{1}{\sqrt{g}}$$

Now,
$$d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i \qquad [\because \alpha, \beta = 1, 2, \ldots]$$

$\therefore$
$$d_{11} = N^1 \partial_1\partial_1 x + N^2 \partial_1\partial_1 y + N^3 \partial_1\partial_1 z = 0 + 0 + \frac{a}{\sqrt{g}} = \frac{a}{\sqrt{g}}$$

$$d_{12} = d_{21} = N^1\partial_1\partial_2 x + N^2\partial_1\partial_2 y + N^3\partial_1\partial_2 z = 0 + 0 + \frac{b}{\sqrt{g}} = \frac{b}{\sqrt{g}}$$

and
$$d_{22} = N^1\partial_2\partial_2 x + N^2\partial_2\partial_2 + N^3\partial_2\partial_2 z = 0 + 0 + \frac{b}{\sqrt{g}} = \frac{b}{\sqrt{g}}$$

Hence, $d_{\alpha\beta} = [d_{11}, d_{12}, d_{22}] = \left[\frac{a}{\sqrt{g}}, \frac{b}{\sqrt{g}}, \frac{b}{\sqrt{g}}\right]$.

Q 3. **Prove that Gauss formula $x^i_{,\alpha\beta} = d_{\alpha\beta}N^i$.** **(2009, 2000, 1993)**

Sol. We have, $\frac{\partial x^i}{\partial \mu^\alpha} \cdot \frac{\partial x^i}{\partial \mu^\beta} = g_{\alpha\beta} \Rightarrow x^i_{,\alpha} \cdot x^i_{,\beta} = g_{\alpha\beta}$

Taking covariant derivative w.r.t. u^y, we get

$$x^i_{\alpha\gamma} \cdot x^i_\beta + x^i_\alpha x^i_{\beta\gamma} = g_{\alpha\beta,\gamma} = 0 \qquad [\because g_{\alpha\beta\gamma} = 0] \quad \ldots\text{(i)}$$

On taking cyclic permutations α, β and γ, we get

$$x^i_{\beta\alpha} \cdot x^i_\gamma + x^i_\beta x^i_{\gamma\alpha} = 0 \qquad \ldots\text{(ii)}$$

and
$$x^i_{,\gamma\beta} \cdot x^i_{,\alpha} + x^i_{,\gamma} x^i_{,\alpha\beta} = 0 \qquad \ldots\text{(iii)}$$

Now, adding Eqs. (ii) and (iii) and then subtracting Eq. (i), we get

$$2\, x^i_{\alpha\beta} \cdot x^i_{\gamma} = 0 \qquad [\because x^i_{\alpha\beta} = x^i_{\beta\alpha}]$$

$$\Rightarrow \quad x^i_{\alpha\beta} \cdot x^i_{\gamma} = 0$$

$$\Rightarrow \quad x^i_{\alpha\beta} \perp x^i_{\gamma}$$

$$\Rightarrow \quad x^i_{\alpha\beta} \parallel N^i$$

$$\Rightarrow \quad x^i_{\alpha\beta} = A_{\alpha\beta}\, N^i \qquad \text{...(iv)}$$

where, $A_{\alpha\beta}$ is to be determined.

Now, $\quad N^i \cdot x^i_{\alpha\beta} = A_{\alpha\beta}\, N^i N^i \qquad [\because N^i \cdot N^i = 1\ d_{\alpha\beta} = N^i \cdot x^i_{\alpha\beta}]$

On putting this value in Eq. (iv), we find that

$$x^i_{\alpha\beta} = d_{\alpha\beta}\, N^i$$

Hence proved.

Q 4. State and prove that Weingarten equation or formula.

(2008, 06, 04, 1999, 92)

Sol. **Statement** If N^i are components of unit normal vector N to the surface, then the derivatives of the normal vector N is given by

$$N^i_{,\alpha} = -\, d_{\alpha\gamma}\, g^{\gamma\beta} x^i_{,\beta}$$

Proof Since, N^i are components of unit normal vector N to the surface, we have $N^i N^i = 1$, taking covariant derivative of this relation with respect to u^α, we get

$$N^i_{,\alpha} N^i + N^i N^i_{,\alpha} = 0, \text{ i.e. } N^i_{,\alpha} N^i = 0$$

which relation shows that $N^i_{,\alpha}$ is orthogonal to N^i. Hence, it lies on the tangent plane to the surface, therefore $N^i_{,\alpha}$ can be expressed on linear combination of $x^i_{,\beta}$ $(\beta = 1, 2)$.

Thus, we may write $\quad N^i_{,\alpha} = B^\beta_\alpha\, x^i_{,\beta} \qquad$ ...(i)

where, the expression for B^β_α is to be determine. Taking inner product on both sides of Eq. (i) with $x^i_{,\gamma}$ and using the fact that

$$N^i_{,\alpha} x^i_{,\gamma} = -\, d_{\alpha\gamma} \text{ and } x^i_{,\beta}\, x^i_{,\gamma} = g_{\beta\gamma}, \text{ we get}$$

$$-\, d_{\alpha\gamma} = B^\beta_\alpha\, g_{\beta\gamma}$$

Contracting above equation by $g^{\gamma\delta}$, we find that

$$-\, d_{\alpha\gamma}\, g^{\gamma\delta} = B^\beta_\alpha\, \delta^\delta_\beta$$

$$\Rightarrow \quad B^\delta_\alpha = -\, d_{\alpha\gamma}\, g^{\gamma\delta} \Rightarrow B^\beta_\alpha = -\, d_{\alpha\gamma}\, g^{\gamma\beta}$$

Substituting this value of B^β_α in Eq. (i), we get

$$N^i_{,\alpha} = -\, d_{\alpha\gamma}\, g^{\gamma\beta}\, x^i_{,\beta}$$

which is known as Weingarten equation.

Q 5. Prove that, if at any point of the surface, there exists two principal directions, they are orthogonal.

(2003)

Sol. The differential equation of the principal directions at a point P of a surface is given by

$$\varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta}\, du^\gamma du^\delta = 0 \qquad \text{...(i)}$$

whose equation is quadratic, so there are, in general two principal directions at each point of the surface. But there may be points on the surface for which normal curvature is independent of the direction of the normal sections. In such case of Eq. (i) does not hold.

Excluding this particular case, we may write Eq. (i) as

$$P_{\gamma\delta}\, du^{\gamma} du^{\delta} = 0 \qquad \text{...(ii)}$$

where, $$P_{\gamma\delta} = \varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta}$$

Now, $$P_{\gamma\delta} g^{\gamma\delta} = \varepsilon^{\alpha\beta} g_{\alpha\gamma} d_{\beta\delta} g^{\gamma\delta} = \varepsilon^{\alpha\beta} d_{\beta\delta} \delta^{\delta}_{\alpha}$$
$$= \varepsilon^{\alpha\beta} d_{\beta\alpha} = d_{21} - d_{12} = 0$$

Thus, condition for orthogonality of two directions given by Eq. (ii) is satisfies.

Hence, the two principal directions are orthogonal.

Q 6. Find the principal directions and the principal curvature of the surfaces $x^1 = a(u+v)$, $x^2 = b(u-v)$, $x^3 = uv$. **(2010)**

Sol. Here, $x^1 = a(u+v), x^2 = b(u-v), x^3 = uv$

$$\Rightarrow \quad \mathbf{X}_1 = \frac{\partial x^i}{\partial u} = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right) = (a, b, v),$$

$$\mathbf{X}_2 = \frac{\partial x^i}{\partial v} = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v}\right) = (a, -b, u)$$

and
$$\frac{\partial^2 x^i}{\partial u \partial u} = \partial_1 \partial_1 x^i = (\partial_1\partial_1 x^1, \partial_1\partial_1 x^2, \partial_1\partial_2 x^3) = (0,0,0)$$
$$\frac{\partial^2 x^i}{\partial u \partial v} = \partial_1 \partial_2 x^i = (\partial_1\partial_2 x^1, \partial_1\partial_2 x^2, \partial_1\partial_2 x^3) = (0,0,1)$$
$$\frac{\partial^2 x^i}{\partial v \partial v} = \partial_2 \partial_2 x^i = (\partial_2\partial_2 x^1, \partial_2\partial_2 x^2, \partial_2\partial_2 x^3) = (0,0,0)$$

$\therefore$
$$g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = a^2 + b^2 + v^2$$
$$g_{12} = \mathbf{X}_1 \cdot \mathbf{X}_2 = a^2 - b^2 + uv$$

and
$$g_{22} = \mathbf{X}_2 \cdot \mathbf{X}_2 = a^2 + b^2 + u^2$$

Now,
$$g_{11} g_{22} - (g_{12})^2 = (a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 - b^2 + uv)$$

$$\Rightarrow \quad \sqrt{g} = |\mathbf{X}_1 \cdot \mathbf{X}_2| = \sqrt{(a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 + b^2 + uv)}$$

Also,
$$g^{11} = \frac{g_{22}}{g} = \frac{(a^2 + b^2 + u^2)}{g}$$

$$g^{12} = g^{21} = -\frac{g_{12}}{g} = -\left(\frac{a^2 - b^2 + uv}{g}\right)$$

and
$$g^{22} = \frac{g_{11}}{g} = \frac{(a^2 + b^2 + v^2)}{g}$$

The components of unit normal vector is

$$N^i = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \left[\frac{b\,(u+v)}{\sqrt{g}}, \frac{a\,(v-u)}{\sqrt{g}}, \frac{-2ab}{\sqrt{g}}\right]$$

Now, $$d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i$$

$$d_{11} = N^1 \partial_1 \partial_1 x^1 + N^2 \partial_1 \partial_1 x^2 + N^3 \partial_1 \partial_1 x^3 = 0$$

$$d_{12} = d_{21} = N^1 \partial_1 \partial_2 x^1 + N^2 \partial_1 \partial_2 x^2 + N^3 \partial_1 \partial_2 x^3 = -\frac{2ab}{\sqrt{g}}$$

and $$d_{22} = N^1 \partial_2 \partial_2 x^1 + N^2 \partial_2 \partial_2 x^2 + N^3 \partial_2 \partial_2 x^3 = 0$$

$\therefore$ $$d = d_{11} d_{12} - (d_{12})^2 = -\frac{4a^2 b^2}{g}$$

Then, the principal curvature κ_n of the surface are given by

$$\kappa_n^2 - \kappa_n g^{\alpha\beta} d_{\alpha\beta} + \frac{d}{g} = 0 \qquad [\because \alpha, \beta = 1, 2]$$

$$\Rightarrow \quad \kappa_n^2 - \kappa_n\,(g^{11} d_{11} + 2g^{12} d_{12} + g^{22} d_{22}) + \frac{d}{g} = 0$$

$$\Rightarrow \quad \kappa_n^2 - \kappa_n \frac{4ab}{\sqrt{g}} \left(\frac{a^2 - b^2 + uv}{g}\right) - \frac{4a^2 b^2}{g^2} = 0$$

$$\Rightarrow \quad g^2 \kappa_n^2 - 4ab\sqrt{g}\,(a^2 - b^2 + uv) \Rightarrow \kappa_n - 4a^2 b^2 = 0$$

Hence, the principal curvatures are the roots of

$$g^2 \kappa_n^2 - 4ab\,\sqrt{g}\,(a^2 - b^2 + uv)\,\kappa_n - 4a^2 b^2 = 0$$

where, $g = (a^2 + b^2 + v^2)\,(a^2 + b^2 + u^2) - (a^2 - b^2 + uv)$

Also, the principal directions are given by

$$\varepsilon_{\alpha\beta}\, g_{\alpha\gamma}\, d_{\beta\delta}\, du^\gamma du^\delta = 0$$

$$(g_{1\gamma} d_{2\delta} - g_{2\gamma} d_{1\delta})\, du^\gamma du^\delta = 0 \qquad [\because \varepsilon^{11} = \varepsilon^{22} = 0, \varepsilon^{12} = -\varepsilon^{21} = 1]$$

$$\Rightarrow \quad (g_{11} d_{21} - g_{21} d_{11})\,(du^1)^2 + (g_{11} d_{22} - g_{21} d_{11})\, du^1 du^2 + (g_{12} d_{12} - g_{22} d_{12})\,(du^2)^2 = 0$$

$$\Rightarrow \quad (g_{11} d_{21} - g_{21} d_{11})\,(du)^2 + (g_{11} d_{22} - g_{21} d_{11})\, du\, dv + (g_{12} d_{22} - g_{22} d_{12})\,(dv)^2 = 0$$

$$\Rightarrow \quad (a^2 + b^2 + v^2) \left(\frac{-2ab}{\sqrt{g}}\right) (du)^2 + 0 + (a^2 + b^2 + u^2) \left(\frac{2ab}{\sqrt{g}}\right) (dv)^2 = 0$$

$$\Rightarrow \quad (a^2 + b^2 + v^2)^{1/2}\, du = \pm\,(a^2 + b^2 + u^2)^{1/2} dv$$

$$\frac{du}{(\sqrt{a^2 + b^2 + u^2})^{1/2}} = \pm \frac{du}{(\sqrt{a^2 + b^2 + v^2})^{1/2}}$$

On integrating both the sides, we get

$$\sinh^{-1}\left(\frac{u}{\sqrt{a^2 + b^2}}\right) = \pm \sinh^{-1}\left(\frac{v}{\sqrt{a^2 + b^2}}\right) + C$$

where, C is an arbitrary constant.

Q 7. Calculate the fundamental magnitudes of the surfaces
$x^1 = a(u+v), x^2 = b(u-v), x^3 = uv.$

Or **Calculate the fundamental magnitudes of the surfaces**
$x^1 = a(u^1+u^2), x^2 = b(u^1-u^2), x^3 = u^1u^2.$ **(2016)**

Sol. From Q. 6, the fundamental magnitudes of the surface are
$g_{11} = a^2 + b^2 + v^2$ and $g_{12} = a^2 - b^2 + uv$

Q 8. State and prove Meusnier's theorem for a surface of three dimensional space. **(2010, 06, 2000)**

Or **Establish a relation between normal curvature and oblique curvature or sections.**

Or **State and prove Meusnier's theorem.** **(2008)**

Sol. **Statement** If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ is the angle between those sections, then $\kappa_n = \kappa \cos\theta$

Proof Let p be a point u^α on the surface $x^i = x^i(u^\alpha)$ and x''^i be the components of the curvature vector at P of the oblique section through P, containing the direction du^α.

Then, $$x'' = \kappa p^i \qquad \text{...(i)}$$

where, p^i are components of the unit principal normal vector to the oblique section at P. Again, the unit normal vector to the surface at P is the unit principal normal vector of the normal section at P parallel to the direction du^α.

Since, θ is the angle between oblique and normal sections at P through the same tangent line, therefore θ is the angle between the vectors p^i and N^i, i.e. $p^iN^i = \cos\theta$.

Now, taking inner product on both sides of Eq. (i) by N^i, we have

$$x''^iN^i = \kappa p^iN^i = \kappa\cos\theta$$

Again, $x''^i N^i$ = Normal curvature at P in the direction du^α

= Curvature of the normal section at P parallel to the direction du^α

$= \kappa_n$

Therefore, $\kappa_n = \kappa\cos\theta$.

Q 9. Find the radius of normal curvature of the section $x = y$ of the paraboloid $2z = 5x^2 + 4xy + 2y^2$ at the origin. Also, find its principal radii at the origin. **(2012)**

Sol. It is given that, $2z = 5x^2 + 4xy + 2y^2 \Rightarrow z = \dfrac{5x^2 + 4xy + 2y^2}{2}$

So, the parametric equations are

$$x = x,\ y = y \text{ and } z = \frac{5x^2 + 4xy + 2y^2}{2}$$

where, x and y are parameters.

Here, $u^1 = x,\ u^2 = y$

$\therefore$ $$\mathbf{X}_1 = \left(\frac{\partial x}{\partial u^1}, \frac{\partial y}{\partial u^1}, \frac{\partial z}{\partial u^1}\right) = \left(1, 0, \frac{10x+4y}{2}\right) = (1, 0, 5x+2y)$$

and $$\mathbf{X}_2 = \left(\frac{\partial x}{\partial u^2}, \frac{\partial y}{\partial u^2}, \frac{\partial z}{\partial u^2}\right) = \left(0, 1, \frac{4x+4y}{2}\right) = (0, 1, 2x+2y)$$

Also, $(\partial_1\partial_1 x, \partial_1\partial_1 y, \partial_1\partial_1 z) = (0,0,5)$,

$(\partial_1\partial_2 x, \partial_1\partial_2 y, \partial_1\partial_2 z) = (0,0,2)$

and $(\partial_2\partial_2 x, \partial_2\partial_2 y, \partial_2\partial_2 z) = (0,0,2)$

$\therefore$ $\mathbf{X}_1 \times \mathbf{X}_2 = (-(5x+2y), -(2x+2y), 1)$

Now, $g_{11} = X_1 \cdot X_2 = 1 + 0 + (5x+2y)^2$

$g_{12} = X_1 \cdot X_2 = 0 + 0 + (2x+2y)\cdot(5x+2y)$

$g_{22} = X_2 \cdot X_2 = 0 + 1 + (2x+2y)^2$

At origin, $x = 0,\ y = 0$ i.e. $u^1 = u^2 = 0$

$g_{11} = 1,\ g_{12} = 0,\ g_{22} = 1$

$\mathbf{X}_1 \cdot \mathbf{X}_2 = (0, 0, 1)$

Now, $g = g_{11}g_{22} - (g_{12})^2 = 1\times 1 - 0 = 1$

$\Rightarrow$ $g = 1 \Rightarrow \sqrt{g} = \sqrt{1} = 1$

$\therefore$ $$N^i = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = (0,0,1)$$

$\Rightarrow$ $N^1 = 0, N^2 = 0, N^3 = 1$

Now, $d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i$ $\quad [\because \alpha, \beta = 1, 2, \ldots]$

$d_{11} = N^1\partial_1\partial_1 x + N^2\partial_1\partial_1 y + N^3\partial_1\partial_1 z = 0 + 0 + 1\times 5 = 5$

$d_{12} = d_{21} = 0 + 0 + 2 = 2$

$d_{22} = N^1\partial_2\partial_2 x + N^2\partial_2\partial_2 y + N^3\partial_2\partial_2 z$

$d_{22} = 2$

At origin, $d_{11} = 5,\ d_{12} = d_{21} = 2,\ d_{22} = 2$

Now, $d = d_{11}d_{22} - (d_{12})^2 = 5\times 2 - (2)^2 = 10 - 4 = 6$

Part I The normal curvature is

$$\kappa_n = \frac{d_{\alpha\beta}du^\alpha du^\beta}{g_{\alpha\beta}du^\alpha du^\beta} = \frac{d_{\alpha\beta}\cdot \dfrac{du^\alpha}{dx}\cdot\dfrac{du^\beta}{dx}}{g^{\alpha\beta}\cdot\dfrac{du^\alpha}{dx}\cdot\dfrac{du^\beta}{dx}}$$

$\because$ $u^1 = x,\ u^2 = y\ (x = y)$

$\therefore$ $$\frac{du^1}{dx} = 1, \frac{du^2}{dx} = 1$$

Now, $$\kappa_n = d_{11}\left(\frac{du^1}{dx}\right) + 2d_{12}\frac{du^1}{dx}\cdot\frac{du^2}{dx} + d_{22}\left(\frac{du^2}{dx}\right)^2$$

$$= g_{11}\left(\frac{du^1}{dx}\right)^2 + 2g_{12}\frac{du^1}{dx}\cdot\frac{du^2}{dx} + g_{22}\left(\frac{du^2}{dx}\right)^2$$

$$= \frac{5\times1+2\times2\times1\times1+2\times1}{1\times1+2\times0\times1\times1+1\times1} = \frac{11}{2}$$

Hence, radius of normal curvature is

$$P_n = \frac{1}{\kappa_n} = \frac{2}{11}$$

Part II The principal curvature κ_n is

$$\kappa_n^2 - \kappa_n g^{\alpha\beta} d_{\alpha\beta} + \frac{d}{g} = 0$$

$$\Rightarrow \quad \kappa_n^2 - \kappa_n\,(g^{11}d_{11} + zy^{12}d_{12} + y^{22}d_{22}) + \frac{d}{g} = 0$$

$$\Rightarrow \quad \kappa_n^2 - \kappa_n(1\times5+2\times0\times2+1\times2) + \frac{6}{1} = 0$$

$$\Rightarrow \quad \kappa_n^2 - \kappa_n\,(5+2) + 6 = 0$$

$$\Rightarrow \quad \kappa_n^2 - 7\kappa_n + 6 = 0$$

$$\Rightarrow \quad \kappa_n^2 - 6\kappa_n - \kappa_n + 6 = 0$$

$$\Rightarrow \quad \kappa_n(\kappa_n - 6) - 1\,(\kappa_n - 6) = 0$$

$$\Rightarrow \quad (\kappa_n - 6)\,(\kappa_n - 1) = 0$$

$$\Rightarrow \quad \kappa_1 = 1,\ \kappa_2 = 6$$

Hence, its principal radii at origin are $P_1 = 1$ and $P_2 = \frac{1}{6}$.

Q 10. Define normal curvature and establish the relation between the curvature of normal and oblique sections.

Sol. **Part I Normal Curvature** The section of surface by the normal plane p^1 is called 'normal section' and curvature of 'normal section, at a point is called the 'normal curvature' at that point and it is denoted by κ_n.

Part II See the solution of Q. 8.

Q 11. Define normal curvature and deduce the formula for normal curvature in terms of fundamental magnitudes. **(2016)**

Sol. **Part I Normal Curvature** The section of surface by the normal plane p^1 is called **'normal section'** and curvature of normal section at a point is called the **'normal curvature'** at that point and it is denoted by κ_n.

Part II See the solution of Q. 1 of Very Short Answer Questions.

Q 12. Prove that at any point of the surface, the sum of the radii of normal curvatures in conjugate directions is constant.

Sol. Let the lines of curvature be taken as parametric curves.

Then, $\quad g_{12} = 0,\ d_{12} = 0.$

Again, let du^α and δu^α be two conjugate directions through any point $P(u^\alpha)$.

Then, $$d_{\alpha\beta} du^\alpha \delta u^\beta = 0$$

$$\Rightarrow \quad d_{11} du^1 \delta u^1 + d_{22} du^2 \delta u^2 = 0$$

$$\Rightarrow \quad \frac{\delta u^1}{d_{22} du^2} = \frac{\delta u^2}{d_{11} du^1} = \lambda \text{ (say)} \qquad \text{...(i)}$$

The normal curvature κ_n at P in the direction du^α is given by

$$\kappa_n = \frac{d_{\alpha\beta}\, du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta} = \frac{d_{11}(du^1)^2 + d_{22}(du^2)^2}{g_{11}(du^1)^2 + g_{22}(du^2)^2}$$

Suppose that ρ_1 and ρ_2 are the radii of normal curvatures at P in the conjugate directions du^α and δu^α, respectively.

$$\Rightarrow \quad \rho_1 = \frac{g_{11}(du^1)^2 + g_{22}(du^2)^2}{d_{11}(du^1)^2 + d_{22}(du^2)^2} \qquad \text{...(ii)}$$

and $$\rho_2 = \frac{g_{11}(\delta u^1)^2 + g_{22}(\delta u^2)^2}{d_{11}(\delta u^1)^2 + d_{22}(\delta u^2)^2} \qquad \text{...(iii)}$$

On putting the values of $\delta u^1, \delta u^2$ from Eq. (i) in Eq. (iii), we get

$$\rho_2 = \frac{g_{11} d_{22}^2 (du^2)^2 + g_{22}\, d_{11}^2 (du^1)^2}{d_{11} d_{22}^2 (du^2)^2 + d_{22}\, d_{11}^2 (du^1)^2}$$

$$= \frac{g_{11} d_{22}^2 (du^2)^2 + g_{22} d_{11}^2 (du^1)^2}{d_{11} d_{22}[d_{22}(du^2)^2 + d_{11}(du^1)^2]} \qquad \text{...(iv)}$$

On adding Eqs. (ii) and (iv), we get

$$\rho_1 + \rho_2 = \frac{g_{11} d_{22} + g_{22} d_{11}}{d_{11} d_{22}}$$

which is independent of the conjugate directions du^α and δu^α.

$\therefore$ $\rho_1 + \rho_2 =$ Constant.

Hence proved.

Q 13. **Find the normal curvature of the curves $u^1 = a \sin\theta, u^2 = a\cos\theta$ on the surface $x^1 = u^1, x^2 = u^2, x^3 = (u^1)^2 - (u^2)^2$ at the origin for $\theta = \frac{\pi}{6}$.** **(2011)**

Sol. For the given equation of surface, we have

$$\partial_1 x^1 = 1, \partial_1 x^2 = 0, \ \partial_1 x^3 = 2u^1$$
$$\partial_2 x^1 = 0, \ \partial_2 x^2 = 1, \ \partial_2 x^3 = -2u^2$$
$$\partial_1\partial_1 x^1 = 0, \ \partial_1\partial_1 x^2 = 0, \ \partial_1\partial_1 x^3 = 2$$
$$\partial_1\partial_2 x^1 = 0, \ \partial_1\partial_2 x^2 = 0, \ \partial_1\partial_2 x^3 = 0$$
$$\partial_2\partial_2 x^1 = 0, \ \partial_2\partial_2 x^2 = 0, \ \partial_2\partial_2 x^3 = -2$$

$$\therefore \quad g_{11} = 1 + 4(u^1)^2, \ g_{12} = -4u^1u^2, \ g_{22} = 1 + 4(u^2)^2$$

At origin, $u^1=0$, $u^2=0$, $g_{11}=1$, $g_{12}=0$, $g_{22}=1$

$\therefore$ $$g=g_{11}g_{22}-g_{12}^2=1$$

Now, $$N^1=\frac{-2u^1}{\sqrt{g}},\ N^2=\frac{2u^2}{\sqrt{g}},\ N^3=\frac{1}{\sqrt{g}}$$

$\therefore$ At origin $N^1=0$, $N^2=0$, $N^3=1$

Hence, $d_{11}=2$, $d_{12}=0$, $d_{22}=-2$

From the given equation of curve on the surface, we have

$$\frac{du^1}{d\theta}=a\cos\theta,\ \frac{du^2}{d\theta}=-a\sin\theta$$

Therefore, at $\theta=\frac{\pi}{6}$, $\frac{du^1}{d\theta}=\frac{\sqrt{3}a}{2}$, $\frac{du^2}{d\theta}=-\frac{a}{2}$

The normal curvature of the curve is given by

$$\kappa_n=\frac{d_{\alpha\beta}\,du^\alpha du^\beta}{g_{\alpha\beta}du^\alpha du^\beta}=\frac{d_{\alpha\beta}\frac{du^\alpha}{d\theta}\cdot\frac{du^\beta}{d\theta}}{g_{\alpha\beta}\frac{du^\alpha}{d\theta}\cdot\frac{du^\beta}{d\theta}}$$

At origin and in direction $\theta=\frac{\pi}{6}$, we have

$$\kappa_n=\frac{d_{11}\left(\frac{du^1}{d\theta}\right)^2+d_{22}\left(\frac{du^2}{d\theta}\right)^2}{g_{11}\left(\frac{du^1}{d\theta}\right)^2+g_{22}\left(\frac{du^2}{d\theta}\right)^2}\qquad[\because d_{12}=0, g_{12}=0]$$

$$=\frac{2\cdot\frac{3a^2}{4}-\frac{2a^2}{4}}{\frac{3a^2}{4}+\frac{a^2}{4}}=\frac{a^2}{a^2}=1$$

***Q* 14. Prove that the origin is the umbilic point of the surface $x^1=u$, $x^2=v$ and $x^3=u^2+v^2$.** **(2007)**

Sol. We have, $\mathbf{X}_1=\frac{\partial x^i}{\partial u}=(1,0,2u)$

and $$\mathbf{X}_2=\frac{\partial x^i}{\partial v}=(0,1,2v)\text{ and }\frac{\partial^2 x^i}{\partial u\partial v}=\partial_1\partial_1 x^i=(0,0,2)$$

Also, $$\frac{\partial^2 x^i}{\partial u\partial v}=\partial_1\partial_2 x^i=(0,0,0)\text{ and }\frac{\partial^2 x^i}{\partial u\partial v}=\partial_2\partial_2 x^i=(0,0,2)$$

$\therefore$ $$g_{11}=\mathbf{X}_1\cdot\mathbf{X}_1=1+4u^2\text{ and }g_{12}=\mathbf{X}_1\cdot\mathbf{X}_2=4uv$$
$$g_{22}=\mathbf{X}_2\cdot\mathbf{X}_2=1+4v^2$$

and $$g=g_{11}g_{22}-(g_{12})^2=1+4u^2+4v^2$$

$$\Rightarrow \qquad \sqrt{g} = |\mathbf{X}_1 \times \mathbf{X}_2| = \sqrt{1+4u^2+4v^2}$$

$$\Rightarrow \qquad N^i = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \left(\frac{-2u, -2v, 1}{\sqrt{1+4u^2+4v^2}}\right)$$

Now, $\qquad d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i$

$$\therefore \qquad d_{11} = N^1 \partial_1 \partial_1 x^1 + N^2 \partial_1 \partial_1 x^2 + N^3 \partial_1 \partial_1 x^3 = \frac{2}{\sqrt{1+4u^2+4v^2}}$$

$$d_{12} = d_{21} = N^1 \partial_1 \partial_2 x^1 + N^2 \partial_1 \partial_2 y x^2 + N^3 \partial_1 \partial_2 x^3 = 0$$

$$d_{22} = N^2 \partial_2 \partial_2 x^1 + N^2 \partial_2 \partial_2 x^2 + N^3 \partial_2 \partial_2 x^3 = \frac{2}{\sqrt{1+4u^2+4v^2}}$$

At origin, $u = v = 0$, $g_{11} = 1$, $g_{12} = 0$, $g_{22} = 0$ and at is clear that at origin, $d_{\alpha\beta}$ is proportional to $g_{\alpha\beta}$.

Hence, the origin is the umbilic point of the given surface.

***Q* 15. Define minimal surface and show that the surface $e^z \cos x = \cos y$ is minimal.** **(2012)**

Or **Define minimal surface and show that surface $z = \log \cos y - \log \cos x$ is minimal.** **(2012)**

Sol. **Part I Minimal Surface** If the mean curvature of a surface is zero at all points, then the surface is called a minimal surface.

Part II The surface will be minimal, if

$$M = 0 \Rightarrow \kappa_1 + \kappa_2 = 0$$

$$\Rightarrow \qquad g^{\alpha\beta} d_{\alpha\beta} = 0, \text{ at every point of the surface.}$$

The given surface is $e^z \cos x = \cos y$

$$\Rightarrow \qquad e^z = \frac{\cos y}{\cos x} \Rightarrow z = \log \cos y - \log \cos x$$

The parametric equation of the given surface is

$$X = x, Y = y, z = \log \cos y - \log \cos x$$

where x and y are parameters

Here, $\qquad u^1 = x, u^2 = y$

$$\therefore \qquad \mathbf{X}_1 = \left(\frac{\partial x}{\partial u^1}, \frac{\partial y}{\partial u^1}, \frac{\partial z}{\partial u^1}\right) = (1, 0, \tan x)$$

$$\text{and} \qquad \mathbf{X}_2 = \left(\frac{\partial x}{\partial u^2}, \frac{\partial y}{\partial u^2}, \frac{\partial z}{\partial u^2}\right) = (1, 0, -\tan y)$$

and $\qquad (\partial_1 \partial_1 x, \partial_1 \partial_1 y, \partial_1 \partial_1 z) = (0, 0, \sec^2 x)$

$$(\partial_1 \partial_2 x, \partial_1 \partial_2 y, \partial_1 \partial_2 z) = (0, 0, 0)$$

$$(\partial_2 \partial_2 x, \partial_2 \partial_2 y, \partial_2 \partial_2 z) = (0, 0 - \sec^2 y)$$

$$\therefore \qquad \mathbf{X}_1 \times \mathbf{X}_2 = (-\tan x, \tan y, 1)$$

Now, $g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = 1 + 0 + \tan^2 x = \sec^2 x$

$g_{12} = g_{21} = \mathbf{X}_1 \cdot \mathbf{X}_2 = -\tan x \tan y$

and $g_{22} = \mathbf{X}_2 \cdot \mathbf{X}_2 = 0 + 1 + \tan^2 y = \sec^2 y$

Also, $$N^i = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \frac{(-\tan x, \tan y, 1)}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

$\therefore$ $d_{\alpha\beta} = N^i \partial_\alpha \partial_\beta x^i \; (\alpha, \beta = 1, 2)$

$\Rightarrow$ $d_{11} = N^1 \partial_1 \partial_1 x + N^2 \partial_1 \partial_1 y + N^3 \partial_1 \partial_1 z$

$$\Rightarrow \quad = 0 + 0 + \frac{\sec^2 x}{\sqrt{\tan^2 x + \tan^2 y + 1}} = \frac{\sec^2 x}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

$\Rightarrow$ $d_{12} = d_{21} = N^1 \partial_1 \partial_2 x + N^2 \partial_1 \partial_2 y + N^3 \partial_1 \partial_2 z = 0 + 0 + 0 = 0$

$\Rightarrow$ $d_{22} = N^1 \partial_2 \partial_2 x + N^2 \partial_2 \partial_2 y + N^3 \partial_2 \partial_2 z$

$$= 0 + 0 - \frac{\sec^2 y}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

$$= \frac{-\sec^2 y}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

Hence, the given surface to be minimal, if $g^{\alpha\beta} d_{\alpha\beta} = 0$.

$$\Rightarrow \quad g^{11} d_{11} + 2g^{12} d_{12} + g^{22} d_{22} = 0 \quad \left[\because g^{11} = \frac{g_{22}}{g}, g^{12} = -\frac{g_{12}}{g}, g^{22} = \frac{g_{11}}{g}\right]$$

$$\Rightarrow \quad g_{22} d_{11} - 2g_{12} d_{12} + g_{22} d_{22} = 0$$

$$\text{LHS} = g_{22} d_{11} - 2g_{12} d_{12} + g_{22} d_{22}$$

$$= \frac{\sec^2 x \sec^2 y}{\sqrt{\tan^2 x + \tan^2 y + 1}} - 0 - \frac{\sec^2 x \sec^2 y}{\sqrt{\tan^2 x + \tan^2 y + 1}} = 0 = \text{RHS}$$

Therefore, the given surface is minimal surface.

***Q* 16. Define asymptotic lines and prove that the asymptotic lines on the paraboloid $2z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ are $\dfrac{x}{a} \pm \dfrac{y}{b} = \lambda$, where λ is an arbitrary constant.** **(2012)**

Sol. **Part I Asymptotic Lines** The directions which are self-conjugate, are called the asymptotic directions and the curves whose tangents are along asymptotic directions, are called the asymptotic lines.

Part II Consider, x, y as parameters, then the parametric equations are

$$x = x, \; y = y, \; z = \frac{x^2}{2a^2} - \frac{y^2}{2b^2} \qquad [\text{where, } x^1 = x, x^2 = y, x^3 = z]$$

$$\mathbf{X}_1 = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right) = \left(1, 0, \frac{x}{a^2}\right)$$

and $\quad \mathbf{X}_2 = \left(\frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}\right) = \left(0, 1, \frac{-y}{b^2}\right)$

Also, $\dfrac{\partial^2 x^i}{\partial x \partial x} = \partial_1 \partial_1 x^i = (\partial_1\partial_1 x, \partial_1\partial_1 y, \partial_1\partial_1 z) = \left(0, 0, \frac{1}{a^2}\right)$

$$\frac{\partial^2 x^i}{\partial y \partial y} = \partial_2 \partial_2 x^i = \left(0, 0, -\frac{1}{b^2}\right)$$

$\therefore \quad g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = 1 + \dfrac{x^2}{a^4}, \; g_{12} = g_{21} = \mathbf{X}_1 \cdot \mathbf{X}_2 = -\dfrac{xy}{a^2 b^2}$

$\therefore \quad g_{22} = \mathbf{X}_2 \cdot \mathbf{X}_2 = 1 + \dfrac{y^2}{b^4}$

Now, $\quad g = g_{11} g_{22} - (g_{12})^2$

$$= \left(1 + \frac{x^2}{a^4}\right)\left(1 + \frac{y^2}{b^4}\right) - \left(\frac{xy}{a^2 b^2}\right)^2 = \frac{x^2}{a^4} + \frac{y^2}{b^4} + 1$$

$\therefore \quad \sqrt{g} = |\mathbf{X}_1 \times \mathbf{X}_2| = \sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}$

Also, $N^i = \dfrac{\mathbf{X}_1 \times \mathbf{X}_2}{|\mathbf{X}_1 \times \mathbf{X}_2|} = \left[\dfrac{-x}{a^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}, \dfrac{y}{b^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}, \dfrac{1}{\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}\right]$

Now, $d_{\alpha\beta} = N^i \partial_\alpha \cdot \partial_\beta x^i$

$\therefore \quad d_{11} = N^1 \partial_1\partial_1 x + N^2 \partial_1 \partial_1 y + N^3 \partial_1 \partial_1 z = \dfrac{1}{a^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}$

$\Rightarrow \quad d_{12} = d_{12} = 0 \;\Rightarrow\; d_{22} = \dfrac{-1}{b^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}$

The differential equation of asymptotic is

$$d_{11}(du^1)^2 + 2d_{12} du^1 du^2 + d_{22}(du^2)^2 = 0$$

$$\Rightarrow \quad d_{11}(dx)^2 + 2d_{12} dx\, dy + d_{22}(dy)^2 = 0$$

$$\Rightarrow \quad \frac{1}{a^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}(dx)^2 + 0 - \frac{1}{b^2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + 1}}(dy)^2 = 0$$

$$\Rightarrow \quad \frac{1}{a^2}(dx)^2 - \frac{1}{b^2}(dy)^2 = 0 \Rightarrow \frac{dx}{a} = \pm \frac{dy}{b} \Rightarrow \frac{dx}{a} \pm \frac{dy}{b} = 0$$

On integrating both the sides, we get

$$\frac{x}{a} \pm \frac{y}{b} = \lambda$$

Hence, $\dfrac{x}{a} \pm \dfrac{y}{b} = \lambda$, where λ is an arbitrary constant.

Q 17 **Prove that the necessary and sufficient condition that a surface be**

(i) plane is that $d_{\alpha\beta} = 0$.

(ii) sphere is that $d_{\alpha\beta} = C\, g_{\alpha\beta}$. **(2015, 13, 12, 11, 09, 03)**

Sol.

(i) **Necessary condition** Suppose that the surface is plane.

Then, unit normal at each point of the surface is constant.

$$\Rightarrow \qquad N^i = C^i \text{ (say)}$$

Taking covariant differential w.r.t. μ^d, we get

$$N^i_{,\alpha} = 0$$

$$- d_{\alpha\gamma} g^{\gamma\delta} \cdot x^i_{,\delta} = 0 \qquad \text{[by Weingarten equation]}$$

$$\Rightarrow \qquad d_{\alpha\gamma} g^{\gamma\delta} \cdot x^i_{,\delta} = 0$$

$$\Rightarrow \qquad d_{\alpha\gamma} \cdot g^{\gamma\delta} \cdot x^i_{,\delta} \cdot x^i_{,\beta} = 0 \cdot x^i_{,\beta}$$

$$\Rightarrow \qquad d_{\alpha\gamma} g^{\gamma\delta} \cdot g_{\beta\delta} = 0 \qquad [\because x^i_{,\delta} \cdot x^i_{,\beta} = g_{\delta\beta} = g_{\beta\delta}]$$

$$\Rightarrow \qquad d_{\alpha\gamma} \delta^\gamma_\beta = 0$$

$$\Rightarrow \qquad d_{\alpha\beta} = 0$$

Sufficient condition Suppose $d_{\delta\beta} = 0$

By Weingarten equation, we have

$$N^i_{,\alpha} = - d_{\alpha\gamma}\, g^{\gamma\beta}\, x^i_{,\beta} \qquad [\because d_{\alpha\gamma} = 0]$$

$$\Rightarrow \qquad N^i_{,\alpha} = 0$$

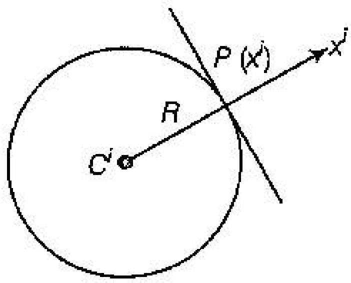
On integrating, we get

$N^i = C^i$, where C^i is a constant.

i.e. unit normal at each point of surface is constant.

So, the surface is plane, i.e. $d_{\alpha\beta} = 0$

(ii) **Necessary condition** Suppose that the surface be a sphere of radius R and centre C^i. From the figure, it is clear that



$$(C^i - x^i) \,||\, N^i$$

$$\Rightarrow \qquad C^i - x^i = \lambda N^i \qquad \ldots\text{(i)}$$

$$\Rightarrow \qquad |C^i - x^i| = |\lambda|\,|N^i| \qquad [\because |\lambda| = \lambda]$$

$$\Rightarrow \qquad |C^i - x^i| = \lambda \qquad [\because |N^i| = 1]$$

$$\Rightarrow \qquad \lambda = R \qquad [\because |C^i - x^i| = R]$$

On putting $\lambda = R$ in Eq. (i), we get

$$C^i - x^i = RN^i$$

$$x^i = C^i - RN^i$$

Taking covariant derivative w.r.t. μ^α, we get

$$x^i_{,\alpha} = 0 - RN^i_{,\alpha}$$

$$\Rightarrow \qquad x^i_{,\alpha} = - RN^i_{,\alpha}$$

$$\Rightarrow \qquad x^i_{,\alpha} = R d_{\alpha\gamma} g^{\gamma\beta} x^i_{,\beta} \qquad \text{[by Weingarten equation]}$$

$$\Rightarrow \qquad x^i_{,\alpha} \cdot x^i_{,\beta} = R\, d_{\alpha\gamma}\, g^{\gamma\beta}\, x^i_{,\beta}\, x^i_{,\delta}$$

$$g_{\alpha\delta} = R\, d_{\alpha\gamma} \delta^{\gamma}_{\delta}$$

$$\Rightarrow \qquad g_{\alpha\delta} = R\, d_{\alpha\delta}$$

$$\Rightarrow \qquad d_{\alpha\delta} = \frac{1}{R}\, g_{\alpha\beta}$$

$$\Rightarrow \qquad d_{\alpha\delta} = C\, g_{\alpha\beta} \qquad \left[\text{where, } C = \frac{1}{R} = \text{constant}\right]$$

i.e. $\qquad d_{\alpha\delta} \propto g_{\alpha\beta}$

Sufficient condition Suppose that $d_{\alpha\beta} = C\, g_{\alpha\beta}$

By Weingarten equation, we have

$$N^i_{,\alpha} = -\, d_{\alpha\gamma}\, g^{\gamma\delta} x^i_{,\delta} \quad \Rightarrow \quad N^i_{,\alpha} = -\, C\, g_{\alpha\gamma}\, g^{\gamma\delta} x^i_{,\delta}$$

$$\Rightarrow \qquad N^i_{,\alpha} = -\, C\, g^{\delta}\alpha\, x^i_{,\delta} \quad \Rightarrow \quad N^i_{,\alpha} = -\, C\, x^i\alpha$$

On integrating both the sides, we get

$$N^i = -\, C\, x^i + a^i \qquad \text{[where, } a^i \text{ is a constant]}$$

which show that the surface is sphere.

i.e. $\qquad d_{\alpha\beta} = C\, g_{\alpha\beta}$ **Hence proved.**

Long Answer Questions

Q. 1. Derive Gauss and Weingarten equations and hence show that $x^i_{,\sigma} = -\, d^{\alpha p} g_{p\sigma} N^i_{,\alpha}$. **(2015)**

Sol. **Part I** See the solution of Q. 3 of Short Answer Questions.

Part II See the solution of Q. 4 of Short Answer Questions.

Part III We know that Weingarten equation is

$$N^i_{,\alpha} = -\, d_{\alpha\gamma} g^{\gamma\beta} x^i_{,\beta} \qquad \text{...(i)}$$

On multiplying Eq. (i) by $d^{\alpha p} g_{\rho\sigma}$, we get

$$d^{\alpha p} g_{\rho\sigma}\, N^i_{,\alpha} = -\, d_{\alpha\gamma} d^{\alpha p}\, g^{\gamma\beta} g_{\rho\sigma} x^i_{,\beta}$$

$$\Rightarrow \qquad d^{\alpha p} g_{\rho\sigma}\, N^i_{,\alpha} = -\, d^p_{\gamma}\, g^{\gamma\beta} g_{\rho\sigma} x^i_{,\beta}$$

$$\Rightarrow \qquad d^{\alpha p} g_{\rho\sigma}\, N^i_{,\alpha} = -\, g^{\rho\beta} g_{\rho\sigma} x^i_{,\beta}$$

$$\Rightarrow \qquad d^{\alpha p} g_{\rho\sigma}\, N^i_{,\alpha} = -\, d^{\beta}_{\sigma}\, x^i_{,\beta} = -\, x^i_{,\sigma}$$

Q 2. Prove that

(i) $x^i_{,\alpha\beta} = d_{\alpha\beta} N^i$ (ii) $N^i_{,\alpha} = -d_{\alpha\gamma} g^{\gamma\beta} x^i_{,\beta}$ **(2017)**

Sol. (i) See the solution of Q. 3 of Short Answer Questions.

(ii) Do same as Q. 1 (part III).

Q 3. If the asymptotic lines are the parametric curves, then show that Mainardi-Codazzi equations are

$$\frac{\partial}{\partial u^1}(\log d_{12}) - \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = 0,$$

$$\frac{\partial}{\partial u^2}(\log d_{12}) - \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = 0$$

and Gauss characteristic equation is

$$d_{12}^2 = g_{2\delta}\left[\frac{\partial}{\partial u^1}\begin{Bmatrix} \delta \\ 12 \end{Bmatrix} - \frac{\partial}{\partial u^2}\begin{Bmatrix} \delta \\ 11 \end{Bmatrix} + \begin{Bmatrix} \theta \\ 12 \end{Bmatrix}\begin{Bmatrix} \delta \\ \theta 1 \end{Bmatrix} - \begin{Bmatrix} \theta \\ 11 \end{Bmatrix}\begin{Bmatrix} \delta \\ \theta 2 \end{Bmatrix}\right].$$

Sol. When the asymptotic lines are parametric curves, then $d_{11} = 0$, $d_{22} = 0$.
Now, Mainardi-Codazzi equation is $d_{\alpha\beta,\gamma} = d_{\alpha\gamma,\beta}$
On putting $\beta = 1$ and $\gamma = 2$, it becomes $d_{\alpha 1,2} = d_{\alpha 2,1}$

$$\Rightarrow \quad \frac{\partial}{\partial u^2} d_{\alpha 1} = d_{\theta 1}\begin{Bmatrix} \theta \\ \alpha 2 \end{Bmatrix} - d_{\alpha\theta}\begin{Bmatrix} \theta \\ 12 \end{Bmatrix}$$

$$= \frac{\partial}{\partial u^1} d_{\alpha 2} - d_{\theta 2}\begin{Bmatrix} \theta \\ \alpha 1 \end{Bmatrix} - d_{\alpha\theta}\begin{Bmatrix} \theta \\ 21 \end{Bmatrix}$$

Since, $\begin{Bmatrix} \theta \\ 12 \end{Bmatrix} = \begin{Bmatrix} \theta \\ 21 \end{Bmatrix}$ and $d_{11} = d_{22} = 0$, then we have

$$\frac{\partial}{\partial u^2} d_{\alpha 1} - d_{21}\begin{Bmatrix} 2 \\ \alpha 2 \end{Bmatrix} = \frac{\partial}{\partial u^1} d_{\alpha 2} - d_{12}\begin{Bmatrix} 1 \\ \alpha 1 \end{Bmatrix} \qquad \text{...(i)}$$

Now, putting $\alpha = 1$, we get

$$-d_{12}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial u^1} d_{12} - d_{12}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix}$$

$$\Rightarrow \quad \frac{\partial}{\partial u^1}[\log d_{12}] - \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = 0$$

Similarly, putting $\alpha = 2$ in Eq. (i), we get

$$\frac{\partial}{\partial u^2} d_{21} - d_{21} \cdot \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = - d_{12} \begin{Bmatrix} 1 \\ 21 \end{Bmatrix}$$

$$\Rightarrow \quad \frac{\partial}{\partial u^2} (\log d_{12}) - d_{21} \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = 0 \quad \text{...(ii)}$$

Now, Gauss characteristic equation is

$$d_{\alpha\gamma} d_{\beta\sigma} - d_{\alpha\beta} d_{\gamma\sigma} = g_{\sigma\delta} R^{\delta}_{\alpha\beta\gamma}$$

$$\Rightarrow \quad d_{\alpha\gamma} d_{\beta\sigma} - d_{\alpha\beta} d_{\gamma\sigma} = g_{\alpha\delta} \left[\frac{\partial}{\partial u^{\beta}} \cdot \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} - \frac{\partial}{\partial u^{\gamma}} \begin{Bmatrix} \delta \\ \alpha\beta \end{Bmatrix} \right.$$

$$\left. + \begin{Bmatrix} \theta \\ \alpha\gamma \end{Bmatrix} \begin{Bmatrix} \delta \\ \theta\beta \end{Bmatrix} - \begin{Bmatrix} \theta \\ \alpha\beta \end{Bmatrix} \begin{Bmatrix} \delta \\ \theta\gamma \end{Bmatrix} \right] \quad \text{...(iii)}$$

On putting $\alpha = 1, \beta = 1, \gamma = 2, \sigma = 2$ and using $d_{11} = d_{22} = 0$, we get

$$d_{12}^2 = g_{1\delta} \left[\frac{\partial}{\partial u^1} \begin{Bmatrix} \delta \\ 12 \end{Bmatrix} - \frac{\delta}{\partial u^2} \begin{Bmatrix} \delta \\ 11 \end{Bmatrix} + \begin{Bmatrix} \theta \\ 12 \end{Bmatrix} \begin{Bmatrix} \delta \\ \theta 1 \end{Bmatrix} - \begin{Bmatrix} \theta \\ 11 \end{Bmatrix} \begin{Bmatrix} \delta \\ \theta 2 \end{Bmatrix} \right]$$

Q 4. If the lines of curvature are parametric curves, then show that $\frac{\partial}{\partial u^2} \left(\frac{1}{\rho_1} \right) + \frac{1}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \frac{\partial}{\partial u^2} (\log g_{11}) = 0$

and $\frac{\partial}{\partial u^1} \left(\frac{1}{\rho_2} \right) + \frac{1}{2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{\partial}{\partial u^1} (\log g_{22}) = 0$**,**

where ρ_1 and ρ_2 are the radii of principal curvatures.

Sol. When the parametric curves are lines of curvature, then $d_{12} = 0$ and $g_{12} = 0$.

In this case, the principal radii of curvatures are given by

$$\frac{1}{\rho_1} = \frac{d_{11}}{g_{11}}, \frac{1}{\rho_2} = \frac{d_{22}}{g_{22}} \quad \text{...(i)}$$

On putting $\beta = 1, \gamma = 2$ in Mainardi-Codazzi equation $d_{\alpha\beta,\gamma} - d_{\alpha\gamma,\beta} = 0$, we get

$$d_{\alpha 1, 2} = d_{\alpha 2, 1}$$

$$\Rightarrow \quad \frac{\partial}{\partial u^2} d_{\alpha 1} - d_{\theta 1} \begin{Bmatrix} \theta \\ \alpha 2 \end{Bmatrix} - d_{\alpha\theta} \begin{Bmatrix} \theta \\ 12 \end{Bmatrix} = \frac{\partial}{\partial u^1} d_{\alpha 2} - d_{\theta 2} \begin{Bmatrix} \theta \\ \alpha 1 \end{Bmatrix} - d_{\alpha\theta} \begin{Bmatrix} \theta \\ 21 \end{Bmatrix}$$

$$\Rightarrow \quad \frac{\partial}{\partial u^2} d_{\alpha 1} - \frac{\partial}{\partial u^1} d_{\alpha 2} - d_{11} \begin{Bmatrix} 1 \\ \alpha 2 \end{Bmatrix} + d_{22} \begin{Bmatrix} 2 \\ \alpha 1 \end{Bmatrix} = 0 \quad [\because d_{12} = 0] \text{ ...(ii)}$$

On putting $\alpha = 1, 2$ in Eq. (ii) and using $d_{12} = 0$, we get

$$\frac{\partial}{\partial u^2} d_{11} - d_{11}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} + d_{22}\begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = 0 \qquad \text{...(iii)}$$

and $$\frac{\partial}{\partial u^1} d_{22} + d_{11}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} - d_{22}\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = 0 \qquad \text{...(iv)}$$

Since, $g_{12} = 0$, then we have

$$g^{11} = \frac{1}{g_{11}},\ g^{22} = \frac{1}{g_{22}} \text{ and } g_{12} = 0$$

$\therefore$ $$\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = g^{1\delta}[\delta, 12] = g^{11}[1, 12] = \frac{1}{2g_{11}} \cdot \frac{\partial}{\partial u^2} g_{11} \qquad \text{...(v)}$$

and $$\begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = g^{2\delta}[\delta, 11] = g^{22}[2, 11] = -\frac{1}{2g_{22}} \cdot \frac{\partial}{\partial u^2} g_{11} \qquad \text{...(vi)}$$

On putting the values of $\begin{Bmatrix} 1 \\ 12 \end{Bmatrix}$ and $\begin{Bmatrix} 2 \\ 11 \end{Bmatrix}$ from Eqs. (v) and (vi) in Eq. (iii), we get

$$\frac{\partial d_{11}}{\partial u^2} - \frac{1}{2}\left(\frac{d_{11}}{g_{11}} + \frac{d_{22}}{g_{22}}\right)\frac{\partial g_{11}}{\partial u^2} = 0$$

$\Rightarrow$ $$\frac{\partial d_{11}}{\partial u^2} - \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\frac{\partial g_{11}}{\partial u^2} = 0 \qquad \text{[from Eq. (i)] ...(vii)}$$

Now, differentiating Eq. (i) w.r.t. u^2, we get

$$\frac{\partial}{\partial u^2}\left(\frac{1}{\rho_1}\right) = \frac{1}{g_{11}} \cdot \frac{\partial}{\partial u^2} d_{11} - \frac{d_{11}}{g_{11}^2} \cdot \frac{\partial}{\partial u^2} g_{11}$$

$$= \frac{1}{g_{11}}\left[\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\frac{\partial g_{11}}{\partial u^2} - \frac{1}{\rho_1} \cdot \frac{\partial g_{11}}{\partial u^2}\right] \qquad \text{[from Eq. (vii)]}$$

$$= \frac{1}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\frac{\partial}{\partial u^2} \log g_{11}$$

$\therefore$ $$\frac{\partial}{\partial u^2}\left(\frac{1}{\rho_1}\right) + \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial}{\partial u^2} \log g_{11} = 0$$

Similarly, the other result can be obtained.